Extrapolation Methods of the Term Structure of Interest Rates under Solvency II

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A/A 2015/2016

Sessione invernale

Acknoledgments

I would first like to acknowledge my supervisor Prof. Barbara Rogo who was always willing to answer my questions, permitted my survey to be a unique experience and guided my writing through her unfailing counseling while still letting this thesis be my own work.

I would also like to express my profound gratitude to Prof. José Luis Vilar Zanòn, Professor at Universidad Complutense de Madrid that accepted to act as my tutor during two magnificent months. His passionate participation and original suggestions allowed for a pleasant research stay and a valuable lifetime experience.

My research would have never been the same without all the fantastic people I met during my short stay in Madrid. Thank you for making me enjoy the process of my writing this thesis.

Last, but certainly not least, I must thank my parents and lifelong friends for providing me with continuous and unconditional support throughout my years of study. My achievements would have never been possible without them.

Maria-Magdalena Magurean

Abstract

A term structure of interest rates typically attempts to design the smoothness possible curve while including the highest degree of market information. Several models could produce different fair values of long-term liabilities and thus lead to different solvency profiles of insurance companies. In the current Solvency II framework the European Insurance and Occupational Pensions Authority (EIOPA) privileges a particular inter- and extrapolation process of the term structure of interest rates: the Smith-Wilson method. This technique implies the pre-definition of a long term equilibrium rate. The current level of this asymptotic rate has raised concerns among long-term liabilities insurers and pension funds alike.
The primary purpose of this research is to achieve a solid empirical and theoretical understanding and highlight the limitations of the Smith-Wilson method by analysing its formal justification and experimental outcome. When the results achieved fail to provide an acceptable term structure, alternative methods and recommendations on further research areas will be proposed.

Keywords: Smith-Wilson, extrapolation, interpolation, interest rates, EIOPA, Solvency II, Nelson Siegel and Svensson

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Chapter 1

Introduction

An essential input in the economic evaluation of long term liabilities of insurance companies and pensions funds is undeniably the discount factor. The discount factor can be simply determined from market financial instruments. This appears an easy process when the market guarantees a high traceability of default-free bonds with maturities matching the cash flows originating from insurers' liabilities. But, this is rarely the case for most life insurance companies and pensions funds where commitments spread over several decades letting available market maturities fall behind the extinction periods of insurers' liabilities.

The technique used to determine yields and consequently discount factors for unavailable maturities has been long under discussion in the past few years. This topic entered into the debate around the fair evaluation of assets and liabilities among the international insurance community. An indicator of the quality of technical provision is a sufficiently high performance in terms of the Liability Adequacy Test (LAT) imposed by the international accounting standards. The LAT is based on an estimation of a "Best Estimate" of liabilities which requires the introduction of a term structure of interest rates to estimate the present value. Although, the LAT is currently still a raw procedure that only provides basic principles, the concept of a present value of future cash flows will have large impact on international insurers as soon as the IFRS4 phase 2 will be enacted. However, the European Union long-term liabilities companies already experience the necessity to deal with the evaluation of a present value of future commitments. On 1-1-2016 a new solvability regulation has been implemented all over European Union countries.

1.0.1 The European Union Solvability Regulation

The European Commission's project commonly referred to as Solvency II has its primary reference in the 2009/138/EC Directive. The 2007 financial crises acted as a textbook case in how a financial markets shock could undermine the long term liabilities market. Under such circumstances, the transition to a new regulatory framework was deemed necessary in the prevailing insurance business. In the current situation, the insurers' solvability profile could no longer be mapped through the simple methodology imposed by the previous legislation. The new supervisory directive focuses on many areas to achieve a final and complete quality evaluation of insurance companies and enhance protection against future financial shocks.

However, for the purpose of this thesis, we will mainly refer to legislation dealing with asset and liability valuation. The key factor in the asset-liability valuation is the "exchange" or "transfer" value. According to Article 75.1: "assets shall be valued at the amount for which they could be exchanged between knowledgeable willing parties in an arms length transaction" and "liabilities shall be valued at the amount for which they could be transferred, or settled, between knowledgeable willing parties in an arms length transaction". The concept of a transfer value is deeply connected with the fair value mentioned in the International Financial Reporting Standards (IFRS)[1]. The Solvency II liabilities evaluation is indeed a fair value. This is easily perceived when reading Article 76: "The value of technical provisions shall correspond to the current amount insurance and reinsurance undertakings would have to pay if they were to transfer their insurance and reinsurance obligations immediately to another insurance or reinsurance undertaking." Hence, in a Solvency II scheme market information plays a key role. This global accounting principle goes under the name of "market consistency" and it is directly cited in Article 76.3: "The calculation of technical provisions shall make use of and be consistent with information provided by the financial markets and generally available data on underwriting risks (market consistency)."

A first reference to the discount factor is found in Article 77 that concerns with the estimation of technical provisions: "The value of technical provisions shall be equal to the sum of a best estimate and a risk margin as set out in paragraphs 2 and 3. The best estimate shall correspond to the probability-weighted average of future cash-flows, taking account of the time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure." In line with Solvency II, the interest rates used to actualize future commitments is composed by three ingredients:

1. The relevant risk free interest rate structure that is to be determinate in accordance

[1]Quoting IFRS13: The fair value is "the price that would be received to sell an asset or paid to transfer a liability in an orderly transaction between market participants at the measurement date"

with Article 77a;
2. A volatility adjustment computed when the criteria established by Article 77d is met;
3. A matching adjustmenet calculated under the requirements provided by Article 77b and 77c.
In this thesis, we will only cope with the extrapolation of the basic curve. The primary features of this curve are claimed in Article 77a: "The determination of the relevant risk-free interest rate term structure referred to in Article 77(2) shall make use of, and be consistent with, information derived from relevant financial instruments. That determination shall take into account relevant financial instruments of those maturities where the markets for those financial instruments as well as for bonds are deep, liquid and transparent. For maturities where the markets for the relevant financial instruments or for bonds are no longer deep, liquid and transparent, the relevant risk-free interest rate term structure shall be extrapolated. The extrapolated part of the relevant risk-free interest rate term structure shall be based on forward rates converging smoothly from one or a set of forward rates in relation to the longest maturities for which the relevant financial instrument and the bonds can be observed in a deep, liquid and transparent market to an ultimate forward rate."
The two additional ingredients are included to mitigate pro-cyclicality that may be induced by high financial shocks. The volatility adjustment is a positive shift to the liquid part of the curve regularly computed by EIOPA. Whereas, the adjustment introduced in Article 77b is a shift of the entire curve and is directly determined by insurance companies. The application of the matching adjustment excludes the volatility adjustment, and vice versa.
We will conclude our insight in the Solvency II directive with Article 86. The norm claims that the European Commission shall adopt delegated acts to regulate "the methodologies, principles and techniques for the determination of the relevant risk-free interest rate term structure to be used to calculate the best estimate referred to in Article 77(2)". In 2014 the European Parliament published the Directive 2014/51/EU (also known as the Delegated Acts or Omnibus II). The Delegated Acts entirely devolve the risk-free interest rate term structure derivation task to the European Insurance and Occupational Pensions Authority (EIOPA). Moreover, the directive mentions the UFR-methodology and the necessity to exclude "artificial volatility" in the valuation of technical provisions (recital 30). In line with the Delegated Acts, EIOPA dicloses on a monthly basis technical information and documentation relative to the risk-free term structure for various currencies in the form of interest rates generated by zero coupon bonds with maturities varying from 1 to 150 years. The technical information includes the risk-free interest rates that are to be used in the technical provision estimation process. Whereas, the technical documentation specifies the methodology adopted to create the risk-free interest rate term structure.

This ensures transparency and enables insurers to easily understand the procedure and to possibly predict how the term structure could change as a consequence of the current market evolution. The risk-free term structure will continue changing over time in line with the interest rate market development.

Technical literature has largely been concerned with producing a term structure that best fits the observed financial instruments and generates coherent long term yields. The most notable application of term structure methods has been employed by central banks all over the globe, who routinely publish interest rates. However, while central banks regularly focus on an estimation that allows comparisons between countries and macroeconomic predictions, EIOPA needs to face different challenges that are going to be a topic of analysis in the present thesis.

Among the concerns of this survey, we find the downturns of the framework developed by EIOPA that still falls short from the long term insurers and pension funds needs.

The determination of the basic risk-free interest rate structure in the Solvency II framework has changed over time. The first term structure method ever to be applied as a consequence of the 2009 European directive is a technique called "Boostrap". Bootstaping was used to determine the risk-free interest rate structure for the fourth Quantitative Impact Study in 2008. However, it failed to produce a structure in line with Solvency II's requests. Hence, in 2010 with the implementation of the fifth Quantitative Impact Study the European Authority approached a different term structure method: the Smith-Wilson method. This procedure is still in use today to compute the basic curve.

What the European Authority has been seeking through both methods is to perfectly fit all liquid market interest rate while still maintaining a smooth movement between market points. Thus, a major point of concern includes the input data on which to implement the method. As for the the non-liquid sector the adopted process is driven by a plausible extrapolation approach based on forward rates. The forward rate better interprets long term market expectations. The Smith-Wilson method is indeed an attempt to introduce a market based fair evaluation of market interest rates combined with a certain long term stability. The stability factor is a long term asymptotic rate exogenously imposed. This parameter has prompted discussions in the current low interest environment. Quoting Mannix R., (2016) : "At its current 4.2% level, many in the industry see the UFR as unjustifiably high. But others see it as part of the political bargain struck to bring the directive into being". The row over the UFR lead EIOPA to reduce its current level. This change "may result in a significant impact on insurers with long-term liabilities and restricted use of transitional arrangements" (Chrisou E., 2016).

This thesis outlines the study to the above mentioned point of critique and other issues arising from the Smith-Wilson methodology and will provide different recommendations to improve the method.

1.0.2 Summary of Contents

The general objective of the research is to deliver a thorough characterization of the Smith-Wilson method in the present low rate market situation. This survey is organized this way: it is composed by four chapter, each one of them dealing with different aspects of the inter-and extrapolation method favored by EIOPA.
Chapter One initially consists of a generic outlook on the different definitions of interest rates and afterwards moves to the description of the optimization problem build and solved by Smith A. and Wilson T. It further proceeds with an analysis of the Smith-Wilson method as applied by EIOPA and closes with an outline of the model's parameters.
Chapter Two gives a characterization of two existing relationships between the Smith-Wilson method and the Ornstein-Uhlenbeck process and between the Smith-Wilson method and the Hull and White process. The concluding section of this chapters looks at the possible problematics emerging from EIOPA's term structure technique.
The third chapter concentrates on potential recommendations to improve the Smith-Wilson method and approaches different alternative methods that could have a positive impact on the term structure methodology.
In Chapter Four we provide an empirical analysis of the Smith-Wilson method using current rates as input data and compare the resulting Smith-Wilson structure with the results originated from the methods mentioned in Chapter Three.
The final Chapter of the survey is dedicated to conclusions drawn from the analysis given in all previous chapters and to further research suggestions.

Chapter 2

The Smith-Wilson Method

The Smith-Wilson technique is a macroeconomic approach originally designed to determine the interest rate curve. It was introduced to generate the optimal discount curve for long term liabilities and pension funds. In the European long term liability management business, the Smith-Wilson method is selected for both interpolation and extrapolation of the yield term structure. When seeking to interpolate the method researches a perfect fit to all present values delivered from liquid financial markets while maintaining the smoothest possible movement. Whereas, the extrapolation sector assumes a pre-determined long term forward equilibrium rate.

The request that all market data must be perfectly incorporated in the curve implies that a great number of parameters will be needed. In such a case, we face a typical interpolation problem where the key factor is smoothness. The resulting structure must have a certain degree of smoothness, otherwise we will obtain an undesired discontinuous curve. Spline based interpolation methods have been put forward in order to cope with this particular set of problems[1]. The Smith-Wilson method is indeed the result of an exponential spline based optimization problem build to obtain a smooth term structure of interest rates.

Still, to represent the term structure we have different alternatives all dependent on the definition of interest rates we use to derive the curve. Thus, before presenting the Smith-Wilson method, in the following section we will analyse different ways of expressing the yield produced by a generic financial instrument exchanged on the market.

[1]Spline functions are regularly used in engineering to create a smooth curve between discrete points that usually represent empirical data

2.0.1 On Interest Rates

When expressing interest rates we have two options ahead: annual compounding or continuous compounding. From these two regimes originate two different pricing functions. We can write the price at time 0 of a unitary zero coupon bond with maturity t as following:

$$v(0,t) = e^{-h(0,t)\cdot t}, \qquad \text{for continuous compouding} \tag{2.1a}$$

$$v(0,t) = [1+i(0,t)]^{-t}. \qquad \text{for annual compounding} \tag{2.1b}$$

Thus, creating two yield formulations that can be easily linked to one another. The equation below expresses the connection between the two types of interest rates:

$$h(0,t) = ln[1+i(0,t)] \tag{2.2}$$

Where:
h is the rate applied in a continuous compounding regime, also known as yield to maturity.
i is the rate applied in a discrete compounding regime. Henceforth, we will refer to i as the annual interest rate assuming that one time unity is equal to one year.
In terms of the zero coupon pricing function, we can calculate the yield to maturity and annual interest rate between two generic periods of time t and T, with $T > t$, as below:

$$h(t,T) = -\frac{\ln v(t,T)}{T-t} \tag{2.3}$$

and

$$i(t,T) = \frac{1}{v(t,T)^{\frac{1}{T-t}}} - 1. \tag{2.4}$$

In an infinitesimal period of time the istantaneous rate (also referred to as short rate or spot rate) can be determined as a limit of the annual interest rate:

$$r(t) = \lim_{T\to t} i(t,T) = \lim_{T\to t} \delta(t,T) = \lim_{T\to t} h(t,T). \tag{2.5}$$

The discount factor between periods t and T can be represented in terms of the short rate, only when assuming a deterministic evolution of interest rates, by using:

$$v(t,T) = e^{-\int_t^T r(u)du}. \tag{2.6}$$

In the hypothesis that the short rate is constant, it can be proven that:

$$r(t) = h(t,T). \tag{2.7}$$

We can furtherly define an additional typology of rate applicable for a transaction in a future period of time: the forward interest rate.

Moreover, a future rate for a generic unitary period $\{t+k, t+k+1\}$ is not observable in t but, given the hypothesis of deterministic interest rates and by applying an existing relationship between present interest rates and forward rates we can also calculate yields that reveal the interest rate legit in a future time span.
In an annual compounding system the following generic equation represents the connection between present annual interest rates and forward rates:

$$\begin{aligned} i(0,t) &= \sqrt[t]{(1+i_{0,t-1})^{t-1}\cdot(1+rf_{t-1,t})} - 1 = \\ &= \sqrt[t]{(1+i_{0,t-2})^{t-2}\cdot(1+rf_{t-2,t-1})\cdot(1+rf_{t-1,t})} - 1 = \\ &= \dots = \sqrt[t]{(1+rf_{0,1})\cdot(1+rf_{1,2})\cdot\dots} \\ &\sqrt{\dots\cdot(1+rf_{t-2,t-1})\cdot(1+rf_{t-2,t-1})\cdot(1+rf_{t-1,t})} - 1, \end{aligned} \tag{2.8}$$

with $rf(t-k, t-k-1)$ denoting the forward rate[2] for one period.
Therefore, the value of a generic forward rate in t can be effortlessly computed through:

$$rf_t(t+k, t+k+1) = \begin{cases} i(t,t+1), & \text{if } k=0 \\ \frac{[1+i(t,t+k+1))]^{k+1}}{[1+i(t,t+k)]^{k}} - 1, & \text{if } k>0. \end{cases} \tag{2.9}$$

Whereas in the continuous compounding case, the forward rate is the root in:

$$\begin{aligned} e^{h_t t} &= e^{(t-1)h_{t-1}}\cdot e^{rf_{t-1,t}} = e^{(t-2)h_{t-2}}\cdot e^{rf_{t-2,t-1}}\cdot e^{rf_{t-1,t}} = \\ &= \dots = e^{rf_{0,1}}\cdot e^{rf_{1,2}}\cdot\dots e^{rf_{t-2,t-1}}\cdot e^{rf_{t-1,t}} \end{aligned} \tag{2.10}$$

Obtaining the following solution:

$$rf_t(t+k, t+k+1) = \begin{cases} h_t, & \text{if } k=0 \\ (k+1)h_{k+1} - kh_k, & \text{if } k>0. \end{cases} \tag{2.11}$$

The outcome of the Smith-Wilson method is an estimation of the pricing function in 0 for several periods of time. In this particular situation, the representation of all yield functions in terms of pricing function appears to be much more useful.
To ease the formulation, the forward rate valued in 0 that will be applied between periods $\{t, T\}$ can be easily reduced to:

$$rf_0(t,T) = -\frac{\ln v(t,T)}{T-t} = -\frac{\ln v(0,T) - \ln v(0,t)}{T-t}, \tag{2.12}$$

given the existing no-arbitrage and deterministic evolutionary assumption the final result will necessarily be:

$$v(t,T) = \frac{v(0,T)}{v(0,t)}. \tag{2.13}$$

[2]The forward rate denotes the yield in a present contract but applicable between two future periods of time.

A final yield-expressing function can be defined: the forward intensity function (also known as instantaneous forward rate). The forward intensity function is a derivate of the exponential representation of the discount factor. Between two generic periods of time $\{t, T\}$, the forward intensity function is simply given by:

$$\delta(t,T) = -\frac{\partial \ln v(t,T)}{\partial T} = -\frac{1}{v(t,T)}\frac{\partial v(t,T)}{\partial T}, \tag{2.14}$$

Hence, the pricing function between time t and T in terms of the forward intensity function, can be expressed through the following exponential form:

$$v(t,T) = e^{-\int_t^T \delta(t,u)du}. \tag{2.15}$$

Obtaining an immediate link between the yield to maturity function and the forward intensity function:

$$h(t,T) = \frac{1}{T-t}\int_t^T \delta(t,u)du, \tag{2.16}$$

and in the limit $(T-t) \to \infty$:

$$h(\infty) = \delta(\infty). \tag{2.17}$$

Moreover, the forward rate can be expressed in terms of the forward intensity function:

$$\delta(t,T) = \lim_{k\to 0} rf_t(t+k, t+k+1) \tag{2.18}$$

We will conclude this section with a final relationship between the forward intensity function and the short rate that is legit only in case of deterministic behavior of the market:

$$\delta(t,u) = r(u). \tag{2.19}$$

2.0.2 The Optimization Problem

As discussed by J. de Kort and M.H. Vellekoop (2016)[3], an optimization problem con be constructed to obtain a model that best suits EIOPA's requests. To apply the procedure, at time 0 we need to possess the following values:

- the market value m_i of N financial instruments with fixed income, where $i \in \{1, 2, \ldots, N\}$;
- N maturities related to the observed financial instruments;
- the cash flow x_{ij} at time j of the i-th financial instrument, with $j \in \{1, 2, \ldots, M\}$.

[3]see: **Term structure extrapolation and asymptotic forward rates**

The interpolating pricing function $\bar{v}(0,t)$ as seen at time 0 is computed as a root of the following equation:

$$m_i = \sum_{j=1}^{M} x_{ij} v(0, t_j). \tag{2.20}$$

Once we obtain all these input data, we can build our optimization problem.
What we aim to achieve is a model that delivers a smooth curve which will match all observed market prices. To regulate the smoothness between all time points, t_j, a criterion on the first order derivative and on the second order derivative is chosen. When an interpolating function function g is selected and when reasoning in a quadratic sense, a minimization problem over all g-functions in a function space can be set. This can only happen if we first define a smoothness regulator, $\mathscr{L}_\alpha$[4]:

$$\mathscr{L}_\alpha[g] = \int_0^\infty [g''(u)^2 + \alpha^2 g'(u)^2] du \tag{2.21}$$

and thus, the problem is defined in the following terms:

$$\min_g \quad \mathscr{L}_\alpha[g], \tag{2.22}$$

where α is a parameter that guarantees that the structure's curvature and slope will stay moderate.
Solving this problem means obtaining a function g, dependent on the generic period of time t, that regulates the smoothness in the interest rate curve between all market points reached by the term structure.

2.1 EIOPA's Application of the Smith-Wilson Method

The European Insurance and Occupational Pensions Authority approach to extrapolation is founded on a central parameter: a forward rate to which all financial instruments converge in the very long run, f_∞. To structure the problem in terms of the long-term forward rate, it is required to redefine the existing relationship between the pricing function $v(0,t)$ and the interpolation function $g(t)$, by imposing:

$$v(0,t) = [1 + g(t)]e^{-f_\infty t}, \tag{2.23}$$

Furthermore, we need to define the following two different subsets of spaces:

$$\varepsilon = \{g \in L^2(\mathbb{R}^+) : \lim_{t\to\infty} g(t) = 0\}, \tag{2.24}$$

[4]Way of reasoning: **Term structure extrapolation and asymptotic forward rates**

where $L^2(\mathbb{R}^+)$ is the space of square integrable functions on the positive real space;

$$\mathcal{F}_a = \{g \in C^2(\mathbb{R}^+) : g''(0) = a, g' \in \varepsilon, g^{''} \in \varepsilon\}, \tag{2.25}$$

where $C^2(\mathbb{R}^+)$ is a space of twice differentiable functions on the positive real space and $a \in \mathbb{R}$. $\mathcal{F}$ appears as an appropriate space where to search for the optimum g function and consequently, for the model's pricing functions $\bar{v}(0,t)$ that best fits the market observed financial instruments.

The minimization problem in terms of the asymptotic forward rate, can be now set as following[5]:

$$\min_{g \in \mathcal{F}} \quad \mathscr{L}_\alpha[g] \tag{2.26}$$

subject to:

$$m_i = \sum_{j=1}^{M} x_{ij}[1 + g(t_j)]e^{-f_\infty t_j}, \tag{2.27}$$

with $i = 1, 2, \ldots, N$.

To lighten the notation, we introduce the following variable:

$$\bar{x}_{ij} = x_{ij} e^{-f_\infty t_j} \tag{2.28}$$

and:

$$\bar{m}_i = m_i - \sum_{j=1}^{M} \bar{x}_{ij}. \tag{2.29}$$

Thus, obtaining a restructured constraint:

$$\sum_{j=1}^{M} \bar{x}_{ij} g(t_j) = \bar{m}_i. \tag{2.30}$$

The key to achieve a satisfying result is to be found in the basic functions $H(t_j, t)$. If chosen correctly the solution can be written in the following terms:

$$g(t) = \sum_{i=1}^{N} \zeta_i \sum_{j=1}^{M} \bar{x}_{ij} H(t_j, t) \tag{2.31}$$

The $g(t)$ function will be used to produce a smooth pricing function that can be easily computed for a generic period of time t, through:

$$v(0,t) = [1 + g(t)]e^{-f_\infty \cdot t}. \tag{2.32}$$

Where f_∞ is an asymptotic parameter that will be a crucial topic of study in the following paragraphs of this chapter. We will also combine to bring about discussion on the main

[5] As reasoned in: **Term structure extrapolation and asymptotic forward rates**

properties of the function $H(t_j, t)$.
The problem has been originally solved by Smith A. and Wilson T. in 2001 and their solution is exposed and still in use in EIOPA's Technical Documentation. EIOPA is adopting Smith and Wilson's results applied to carefully selected market financial instruments to deliver the risk-free interest rate curve since 2010, when the fifth Quantitative Impact Study (QIS5) was enacted.

2.1.1 EIOPA's Calibration Inputs

Before exploring EIOPA's extrapolation model, we require a deeper insight over the market data on which the Smith-Wilson method is to be applied.
The purpose of the estimation is using a market consistent evaluation to determine a risk-free interest rate curve for every relevant currency. The Smith-Wilson method can be applied to any kind of financial instruments to determine the term structure, hence, it is eligible for the construction of risk-free interest rates. By "risk-free" financial instruments we understand those market observable instruments that have a limited or quasi null default risk. When selecting the financial instruments to fit the risk-free curve, unlimited attention must be paid to their credit quality. The most appropriate instruments are the ones that achieve the lowest observed credit risk. To meet this criterion a double condition is set:
1.The investment's return of these financial instruments must be absolutely certain.
2.The instruments are exchanged in markets that are defined by a high frequency of trading, the amount exchanged must be sufficiently high and the conditions must be explicitly defined. In other words, what is actually requested is feasibility at any period of time or in any desired amount.
According to Article 44 of the Delegated Regulation (EU) 2015/35 enacted on 10 October 2014 supplementing the Solvency II directive, these criterions are only met by two financial instruments: Government Bonds and Interest Rate Swap contracts.
In the first three Quantitative Impact Studies, the price of government bonds was considered to be the sole instruments that could represent the highest credit quality. From the QIS4 on, other financial instruments were introduced in the estimation process: swap rates. This transition from government bonds to swaps rates is a direct consequence of the financial crisis. When the financial crisis broke, credit quality related to swaps resulted to be much higher and stable than the credit quality of government bonds. To this date, both the aforementioned instruments are used in the calibration process. EIOPA's favored financial instruments are swap rates, but only when traded in deep, liquid and transparent (DLT) markets. If this condition is not met, government bonds appear as a far better

option in the estimation procedure. An alternative to only using swaps or government bonds, although still not applied to date, has been proposed: the use of a mixture of government bond rates and swap rates for the liquid part of the curve. Such a technique is known as "switching". Employing a combination of both financial instruments may well improve the DLT requirement when selecting proper input data.
An additional and important remark on the difference between using swap rates or government bonds must be made: Empirical evidence has shown a significant decrease in the amount of the technical provision required by the current legislation. Consequently, the choice to prefer swap rates over government bonds eases insurers funding.
For each currency, EIOPA's Technical Documentation specifies the selected financial instrument adopted as an input of the Smith-Wilson method. The provider privileged by the European supervisory authority to collect the required data is Bloomberg.
A further issue is the result of the illiquidity found in the selected instruments for several maturities. Therefore, EIOPA when appropriately choosing the market instrument must also select its deep liquid and transparent maturities. For those maturities not eligible, the Smith-Wilson method proceeds by interpolation up until the Last Liquid Point (LLP). The LLP is nothing less than the last observable market price of the selected financial instrument that EIOPA deemed sufficiently liquid. For each relevant currency, the LLP is defined in the Technical Documentation[6]. Consequently, the LLP is equal to the starting point of the extrapolation sector of the term structure function.

Credit Risk Adjustment

In addition to the observed default free interest rates and in accordance with recital 20 and Article 40 of the Delegated Regulation (UE) 2015/35, dated 10 October 2014, the basic curve calculated by EIOPA is also composed by the Credit Risk Adjustment (CRA). By definition, the CRA is a non-positive parallel shift to the observed market rates and its value depends on the financial instruments used to construct the curve. Therefore, the final interest rates will be smaller or equal to the ones calculated with the sole use of observable market prices in order to eliminate the innate credit risk in swap contracts and government bonds.
According to the aforementioned Delegated Regulation we can determine three main different methods for calculating the adjustment:

1. When considering EEA currency for which the swap rates are used and where the

[6]For all liquid maturities see: Technical documentation of the methodology to derive EIOPAs risk-free interest rate term structures, (30 September 2016), Tables 3, 4, 8 and 9; and the updated document: Risk-free interest rate term structures - Changes of relevant financial instruments, (30 September 2016)

DLT requirement is met, Article 45 states: "The adjustment shall be determined on the basis of the difference between rates capturing the credit risk reflected in the floating rate of interest rate swaps and overnight indexed swap rates (OIS)[7] of the same maturity, where both rates are available from deep, liquid and transparent financial markets. The calculation of the adjustment shall be based on 50 percent of the average of that difference over a time period of one year. The adjustment shall not be lower than 10 basis points and not higher than 35 basis points". As for the EURO (where the floating leg is $EURIBOR_{6months}$), in line with recital 20, we no longer use $OIS_{6months}$ but $OIS_{3months}$[8].
In formula:

$$CRA_{EURO} = \frac{\sum_{i=1}^{N} EURIBOR_{6months,i} - OIS_{3months,i}}{2N}, \tag{2.33}$$

Where N is the total number of observed instruments in the past year on daily data. In case of missing data, it must be proceeded by interpolation. If the missing data is beyond 20%, the market will no longer met the DLT requirement.

2. The second method is to be used when considering EEA-currencies where the selected financial instruments are government bonds. In this case, the CRA shall be equal to CRA_{EURO}.

3. The final method is dependent on the one year yield of financial instrument that meet the DLT requirement. If this yield is greater than the UFR[9], than the CRA is:

$$CRA = CRA_{EURO} \cdot \frac{yield_{mediumterm,currency}}{yield_{mediumterm,EURO}}, \tag{2.34}$$

This scenario is typical of an inflationary system where long term instruments tend to be illiquid. In such a scenario, we need to consider a ratio between a medium term yield (i. e. 3 years) of the given currency and a medium term (i. e. 3 years) yield of the EURO.
In a non-inflationary system, thus the UFR is greater than the yield of the selected financial instrument, we apply a long term ratio, i.e.:

$$CRA = CRA_{EURO} \cdot \frac{yield_{10years,currency}}{yield_{10years,EURO}}, \tag{2.35}$$

Exceptions are practiced for the following currencies:

[7] The OIS is an Interest Swap Rate where the floating leg is indexed to the overnight rate.

[8] $OIS_{6months}$ is the Overnight Index Swap with maturity 6 months and $OIS_{3months}$ is Overnight Index Swap with maturity 3 months.

[9] see 2.1.5

1. For the Norway's Krone we apply the Swedish CRA.
2. For the Liechtenstein's Franc we apply Switzerland's CRA.

And lastly, in the Delegated Regulation, an upper bound of $35bps$ and a lower bound of $10bps$ limit the value of the CRA[10].

Swaps and Credit Quality

The swap rate market is one of the vastest segments of the international security market. Evidence has shown that financial markets consider swap rates with the floating leg linked to the London Interbank Offered Rate (LIBOR)[11] superior in terms of credit quality relative to government bonds. In its most common form, plain-vanilla, a swap contract implies the exchange of a fix interest rate and a variable interest rate related to a notional value, which in itself is never exchanged between the parts of the contract. The indexed leg is most commonly related to the LIBOR[12] with maturity six or three months. Duffie and Huang (1996)[13] studied the impact of credit risk on swap rates and found that the counterparty credit risk affects the interest rates swaps on a rather insignificant scale. As such, spreads between swaps tend to have little correlation with the counterparty's defaultability. This decorrelation between Interest Rate Swaps and credit risk has been furtherly examined. And, the small economical influence of counterparty credit risk on Interest Rate Swaps has been confirmed by even more recent studies[14].
These studies have been proven by the current market behavior and thus leading EIOPA to privilege swap rates over government bonds. Notwithstanding, the market's perception of credit quality is dependent on the current financial and macroeconomic situation and therefore, a reversed situation must not be completely excluded.

2.1.2 The Smith-Wilson Method and Zero Coupon Bonds

The optimization problem proposed by Smith A. and Wilson T, can be reinterpreted in more heuristic terms. The starting point is the hypothesis that the present market value of an asset with a single cash flow in t is equal to the present values calculated through the forward equilibrium rate to which the observed market rates converge. This long

[10]see: Consultation Paper on a Technical document regarding the risk free interest rate term structure

[11]the swap rates used by EIOPA are indexed to the LIBOR, except for the Euro, where the rates are indexed to the Euro Interbank Offered Rate (EURIBOR).

[12]The LIBOR rate is obtained as an average of rates applied by banks to borrow from one another.

[13]see: **Swap Rates and Credit Quality**

[14]Liu J., Longstaff F. A. and Mandell R. E., (2002)

term equilibrium rate is referred to, in EIOPA's technical documentation, as the Ultimate Forward Rate. To this first part of the equation, a Correction is added. Supposing that, in the long run the curve reflecting the forward interest rates will reach asymptotically the UFR, the Correction tends to 0 when $t \to \infty$. Therefore, we can write the following intuitive equation[15]:

$$\text{Present market value} = \text{Present value applying the UFR} + / - \text{Correction}. \tag{2.36}$$

The equilibrium rate reflects the illiquid part of the curve and to this date, EIOPA has decided to treat it like a given constant[16] input. While the Correction is obtained using market information provided by the liquid part of the curve.
The method requires three inputs:

- market price at valuation date of the chosen financial instruments;
- the time vector of the cash flows;
- value of each cash flow.

The Smith-Wilson formulation uses the continuous compounding, thus e^{-UFRt}. As such, in the Euro currency case, by UFR we intend: $ln(1 + 4.2\%)$. Hence, the Smith-Wilson pricing function for a given Zero Coupon Bond with maturity t can be simply expressed as:

$$v(0,t) = e^{-UFR \cdot t} + Correction,\ t \geqslant 0. \tag{2.37}$$

The Correction is reduced to a linear combination with N parameters. Where N is the number of observable assets:

$$\zeta_1 \cdot (w_1) + \zeta_2 \cdot (w_2) + ... + \zeta_N \cdot (w_N),\ t \geqslant 0. \tag{2.38}$$

The generic weight w_j is obtained through a Kernel function $Kernel_j(t)$ that depends on the input's maturity. The value of the Kernel can be considered as a constant since it originates from observable data. In such a case, the Kernel equals a particular function, known as the Wilson function:

$$Kernel_j(t) = W(t,t_j),\ t \geqslant 0. \tag{2.39}$$

Where t_j is the maturity of the j-th asset. The aforementioned Wilson function is formulated as:

$$W(t,t_j) = e^{-UFR \cdot (t+t_j)} \cdot \{\alpha \cdot min(t,t_j) - 0.5 \cdot e^{-\alpha \cdot max(t,t_j)} \cdot [e^{\alpha \cdot min(t,t_j)} - e^{-\alpha \cdot min(t,t_j)}]\}. \tag{2.40}$$

[15] see: Tipos de interés para valorar las provisiones técnicas de seguros

[16] It only changes depending on the currency, see 2.1.5.

Where α is the parameter of mean reversion and it defines the velocity of convergence to the forward equilibrium rate.
The final pricing equation for a single ZCB can be reduced to:

$$v(0,t) = e^{-UFR \cdot t} + \sum_{j=1}^{N} \zeta_j \cdot W(t,t_j),\ t \geqslant 0. \tag{2.41}$$

By construction, we achieve the convergence below:

$$\lim_{t \to \infty} v(0,t) = e^{-UFRt}. \tag{2.42}$$

This comes as a direct consequence of the diminishing information available on the market over time. The more time passes, the less market information tends to be feasible and reliable. Hence, this ensures a convergence to the price provided by employing the sole UFR.
When considering the price of the N different ZCB, the following linear system can be constructed[17]:

$$\begin{aligned}
v(0,t_1) &= e^{-UFR \cdot t_1} + \textstyle\sum_{j=1}^{N} \zeta_j \cdot W(t_1,t_j), \\
v(0,t_2) &= e^{-UFR \cdot t_2} + \textstyle\sum_{j=1}^{N} \zeta_j \cdot W(t_2,t_j), \\
&\vdots \\
v(0,t_j) &= e^{-UFR \cdot t_j} + \textstyle\sum_{j=1}^{j} \zeta_j \cdot W(t_j,t_j), \\
&\vdots \\
v(0,t_N) &= e^{-UFR \cdot t_N} + \textstyle\sum_{j=1}^{N} \zeta_j \cdot W(t_N,t_j).
\end{aligned} \tag{2.43}$$

The solution provided by this system will determine the unknown parameters $\zeta_{1,2,\ldots,N}$.
Using the vector and matrix space, the notation can be eased as below:

$$\boldsymbol{v} = \boldsymbol{\mu} + \boldsymbol{W}\boldsymbol{\zeta}, \tag{2.44}$$

which results in:

$$\boldsymbol{\zeta} = \boldsymbol{W}^{-1}(\boldsymbol{v} - \boldsymbol{\mu}), \tag{2.45}$$

with:

$$\boldsymbol{v} = \begin{bmatrix} v(0,t_1) \\ v(0,t_2) \\ \vdots \\ v(0,t_j) \\ \vdots \\ v(0,t_N) \end{bmatrix}, \qquad \boldsymbol{\zeta} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_j \\ \vdots \\ \zeta_N \end{bmatrix},$$

[17] As reasoned in: **QIS 5 Risk-free interest rates - Extrapolation method**

$$\boldsymbol{\mu} = exp[-UFR \cdot \boldsymbol{t}] = \begin{bmatrix} e^{-UFR \cdot t_1} \\ e^{-UFR \cdot t_2} \\ \vdots \\ e^{-UFR \cdot t_j} \\ \vdots \\ e^{-UFR \cdot t_N} \end{bmatrix}$$

and with the quadratic and symmetric NxN-matrix:

$$\boldsymbol{W} = \begin{bmatrix} W(t_1,t_1), W(t_1,t_2), \ldots, W(t_1,t_N) \\ W(t_2,t_1), W(t_2,t_2), \ldots, W(t_2,t_N) \\ \vdots \\ W(t_j,t_1), \ldots, W(t_j,t_j), \ldots, W(t_j,t_N) \\ \vdots \\ W(t_N,t_1), W(t_N,t_2), \ldots, W(t_N,t_N) \end{bmatrix}.$$

Now, we can apply this method to derive the term structure relying on the information originating from Zero Coupon Bonds.

2.1.3 The Smith-Wilson Method with Assets Generating Multiple Cash Flows

[18] The Smith-Wilson method can be generalized and applied to liquid assets that produce a number of cash flows on a given time vector. The present value of the cash flows can still be calculated using equation 2.36. But, when computing the Correction, a different Kernel function must be defined. The new linear combination has to consider the cash flows.

Given an asset that provides the cash flow $\boldsymbol{x_i}$:

$$\boldsymbol{x_i} : x_{1,1}, x_{1,2}, .., x_{1,M},$$

the Kernel function for the i-th asset can be obtained through:

$$Kernel_i(t) = \sum_{j=1}^{M} x_{i,j} \cdot W(t,t_j),\ t \geqslant 0. \tag{2.46}$$

Therefore, a generic discount factor is given by:

$$v(0,t) = e^{-UFR \cdot t} + \sum_{i=1}^{N} \zeta_i \cdot \sum_{j=1}^{M} x_{i,j} \cdot W(t,t_j). \tag{2.47}$$

[18] As discussed in: **QIS 5 Risk-free interest rates - Extrapolation method**

The present market value of an asset generating the payments $\boldsymbol{x_i} : x_{1,1}, x_{1,2}, .., x_{1,M}$ on the time vector $\boldsymbol{t} : t_1, t_2, .., t_M$ is easily obtained by discounting the cash flow with a known discount factor that depends on time, $v(0, t_j)$, as in the equation below:

$$m_i = \sum_{j=1}^{M} x_{i,j} \cdot v(0, t_j). \tag{2.48}$$

Once set $v(0, t_j)$, we can write the following linear system:

$$\begin{aligned}
m_1 &= \textstyle\sum_{j=1}^{M} x_{1,j} \cdot v(0,t_j) = \sum_{j=1}^{M} x_{1,j} \cdot (e^{-UFR \cdot t_j} + \sum_{l=1}^{N} \zeta_l \cdot \sum_{k=1}^{M} x_{l,k} \cdot W(t_j, t_k)),\\
m_2 &= \textstyle\sum_{j=1}^{M} x_{2,j} \cdot v(0,t_j) = \sum_{j=1}^{M} x_{2,j} \cdot (e^{-UFR \cdot t_j} + \sum_{l=1}^{N} \zeta_l \cdot \sum_{k=1} x_{l,k} \cdot W(t_j, t_k)),\\
&\vdots\\
m_i &= \textstyle\sum_{j=1}^{M} x_{i,j} \cdot v(0,t_j) = \sum_{j=1}^{M} x_{i,j} \cdot (e^{-UFR \cdot t_j} + \sum_{l=1}^{N} \zeta_l \cdot \sum_{k=1}^{M} x_{l,k} \cdot W(t_j, t_k)),\\
&\vdots\\
m_N &= \textstyle\sum_{j=1}^{M} x_{N,j} \cdot v(0,t_j) = \sum_{j=1}^{M} x_{N,j} \cdot (e^{-UFR \cdot t_j} + \sum_{l=1}^{N} \zeta_l \cdot \sum_{k=1}^{M} x_{l,k} \cdot W(t_j, t_k)).
\end{aligned} \tag{2.49}$$

Rearranging the terms in the equations, the following system is obtained:

$$\begin{aligned}
\textstyle\sum_{j=1}^{M} x_{1,j} \cdot v(0,t_j) &= \textstyle\sum_{j=1}^{M} x_{1,j} \cdot^{e-UFR \cdot t_j} + \sum_{l=1}^{N} (\sum_{k=1}^{M} (\sum_{j=1}^{M} x_{1,j} \cdot W(t_j, t_k)) \cdot x_{l,k}) \cdot \zeta_l,\\
\textstyle\sum_{j=1}^{M} x_{2,j} \cdot v(0,t_j) &= \textstyle\sum_{j=1}^{M} x_{2,j} \cdot^{e-UFR \cdot t_j} + \sum_{l=1}^{N} (\sum_{k=1}^{M} (\sum_{j=1}^{M} x_{2,j} \cdot W(t_j, t_k)) \cdot x_{l,k}) \cdot \zeta_l,\\
&\vdots\\
\textstyle\sum_{j=1}^{M} x_{i,j} \cdot v(0,t_j) &= \textstyle\sum_{j=1}^{M} x_{i,j} \cdot^{e-UFR \cdot t_j} + \sum_{l=1}^{N} (\sum_{k=1}^{M} (\sum_{j=1}^{M} x_{i,j} \cdot W(t_j, t_k)) \cdot x_{l,k}) \cdot \zeta_l,\\
&\vdots\\
\textstyle\sum_{j=1}^{M} x_{N,j} \cdot v(0,t_j) &= \textstyle\sum_{j=1}^{M} x_{N,j} \cdot^{e-UFR \cdot t_j} + \sum_{l=1}^{N} (\sum_{k=1}^{M} (\sum_{j=1}^{M} x_{N,j} \cdot W(t_j, t_k)) \cdot x_{l,k}) \cdot \zeta_l.
\end{aligned} \tag{2.50}$$

Using matrixes and vectors:

$$\boldsymbol{m} = \boldsymbol{X}\boldsymbol{v} = \boldsymbol{X}\boldsymbol{\mu} + (\boldsymbol{X}\boldsymbol{W}\boldsymbol{X}^T)\boldsymbol{\zeta}. \tag{2.51}$$

As such, the solution is given by:

$$\boldsymbol{\zeta} = (\boldsymbol{X}\boldsymbol{W}\boldsymbol{X}^T)^{-1}(\boldsymbol{m} - \boldsymbol{X}\boldsymbol{\mu}). \tag{2.52}$$

Were we have the following vectors:

$$\boldsymbol{m} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_i \\ \vdots \\ m_N \end{bmatrix}, \qquad \boldsymbol{v} = \begin{bmatrix} v(0,t_1) \\ v(0,t_2) \\ \vdots \\ v(0,t_j) \\ \vdots \\ v(0,t_M) \end{bmatrix}, \qquad \boldsymbol{\zeta} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_i \\ \vdots \\ \zeta_N \end{bmatrix},$$

$$\boldsymbol{\mu} = exp[-UFR \cdot \boldsymbol{t}] = \begin{bmatrix} e^{-UFR \cdot t_1} \\ e^{-UFR \cdot t_2} \\ \vdots \\ e^{-UFR \cdot t_j} \\ \vdots \\ e^{-UFR \cdot t_M} \end{bmatrix}$$

and with:

$$\boldsymbol{W} = \begin{bmatrix} W(t_1,t_1), W(t_1,t_2), \ldots, W(t_1,t_j) \ldots, W(t_1,t_M) \\ W(t_2,t_1), W(t_2,t_2), \ldots, W(t_2,t_j) \ldots, W(t_2,t_M) \\ \vdots \\ W(t_j,t_1), W(t_j,t_2), \ldots, W(t_j,t_j) \ldots, W(t_j,t_M) \\ \vdots \\ W(t_M,t_1), W(t_M,t_2), \ldots, W(t_M,t_j) \ldots, W(t_M,t_M) \end{bmatrix}, \text{MxM matrix},$$

$$\boldsymbol{X} = \begin{bmatrix} x_{1,1}, x_{1,2}, \ldots, x_{1,j}, \ldots, x_{1,M} \\ x_{2,1}, x_{2,2}, \ldots, x_{2,j}, \ldots, x_{2,M} \\ \vdots \\ x_{i,1}, x_{i,2}, \ldots, x_{i,j}, \ldots, x_{i,M} \\ \vdots \\ x_{N,1}, x_{N,2}, \ldots, x_{N,j}, \ldots, x_{N,M} \end{bmatrix}, \text{NxM matrix.}$$

While, as previously mentioned, $\boldsymbol{W}$ is the quadratic matrix containing the Wilson functions, in this problem we will also require a rectangular matrix containing all cash flows provided by the assets. Each row of $\boldsymbol{X}$ identifies the flux of every single asset and the column identifies the date of payment.
Once computed $\boldsymbol{\zeta}$, we insert the newly found vector in equation 2.51 and therefore, calculate the interest rates needed to determine the term structure[19].

2.1.4 The Wilson Function

As previously claimed, the Wilson function is of crucial importance in determining the "Correction" term in equation 2.36. In this section, we will analyse the Wilson function as implemented by the European supervisory authority in its published documentation[20].

[19]As discussed in: **QIS5**; **Risk-free interest rates-Extrapolation method**

[20]see: Technical documentation of the methodology to derive EIOPAs risk-free interest rate term structures

According to EIOPA's technical specifications, the Wilson function is provided by the equation[21]:

$$W(t,t_j) = e^{-UFR(t+t_j)} \cdot H(t,t_j) = \\ = e^{-UFR(t)} \cdot H(t,t_j) \cdot e^{-UFR(t_j)}, \text{ with } t,t_j \geqslant 0. \tag{2.53}$$

Where:
UFR is is the continuously compounded forward rate, i.e. $UFR = ln(1+4.2\%)$,
t is the time duration to maturity of a generic market price,
t_j is the time duration to maturity of a generic cash flow,
$H(t,t_j)$ is a function known as Heart of the Wilson. In such a case, the Wilson function is nothing less than a scaled version of the Heart of Wilson[22].
Interestingly, the Wilson function is a converging function with an absolute maximum reached before the last liquid point. Its structure can be noticed in the figure below.

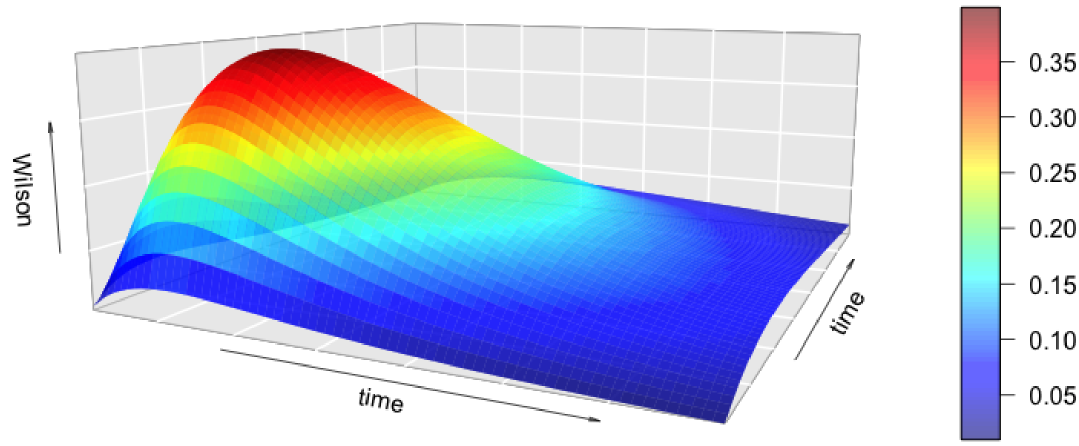

Figure 2.1: Structure of the Wilson function with the LLP set to 20, $UFR = 4.2\%$ and $\alpha = 0.12376$.

[21]Way of reasoning: Technical documentation of the methodology to derive EIOPAs risk-free interest rate term structures

[22]see: **Term structure extrapolation and asymptotic forward rates**

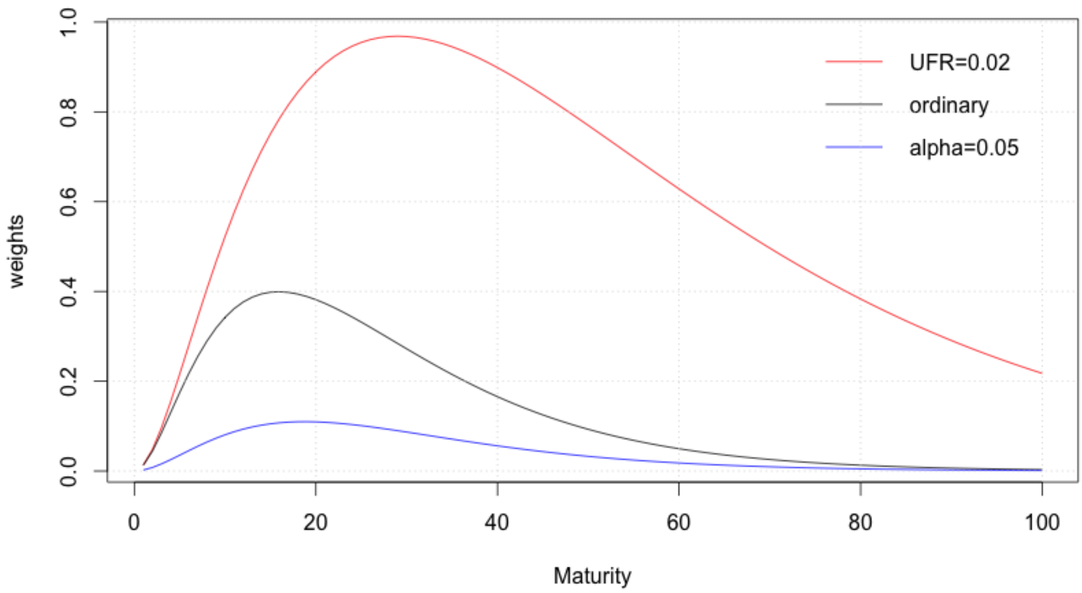

Figure 2.2: Representation of the weights evolution in time. The black curve has a $UFR = 4.2\%$ and an $\alpha = 0.12376$, the red curve has the parameters set at $UFR = 2\%$ and $\alpha = 0.12376$, while the blue curve is built with $UFR = 4.2\%$ and $\alpha = 0.05$.

The second figure above gives a profound insight of the relationship between the Wilson function's evolution, the UFR and the parameter α.
The Heart of Wilson function, $H(t, t_j)$, is deeply related to the interpolating function $g(t)$ in 2.23. The relationship is the result of its structure. The Heart of Wilson is a twice differentiable function that is constructed as a spline function, specifically, as an exponential spline function. This exponential spline function appears particularly useful when modelling the term structure of interest rates, as shown by Andersen (2007), Andersen and Piterbarg (2010) and Pruess (1976).
The Heart of the Wilson function is, indeed, formulated as:

$$H(t, t_j) = \alpha \cdot min(t, t_j) - exp\{-\alpha \cdot max(t, t_j)\} \cdot sinh(\alpha \cdot min(t, t_j)\}. \tag{2.54}$$

Where α is the parameter of the speed of convergence to the asymptotic level.
Expanding $H(t, t_j)$:

$$\begin{aligned} W(t, t_j) = e^{-UFR(t+t_j)} \cdot \Big\{ & \alpha \cdot min(t, t_j) - exp\{-\alpha \cdot max(t, t_j)\} \cdot \\ & \cdot \frac{e^{(\alpha \cdot min(t,t_j)} - e^{(-\alpha \cdot min(t,t_j)}}{2} \Big\}. \end{aligned} \tag{2.55}$$

When including 2.54 in 2.53 we derive:

$$H(t,t_j) = \alpha \cdot min(t,t_j) - exp\{-\alpha \cdot max(t,t_j)\} \cdot \left\{ \frac{e^{(\alpha \cdot min(t,t_j)} - e^{(-\alpha \cdot min(t,t_j)}}{2} \right\}, \quad (2.56)$$

which is exactly the equation 2.40[23].
The Heart of the Wilson function is a symmetric and positive function (given t and t_j's positivity). Additionally, the function is continuous in $t = t_j$ and so are its first two derivatives. Consequently, we obtain the desired smoothness in the liquid part of the model.
We can now show that the Heart of Wilson is only a twice continuously differentiable function, by differentiating respect to t_j:

$$\frac{\partial H(t,t_j)}{\partial t_j} = G(t,t_j) = \begin{cases} \alpha - \alpha \cdot e^{\alpha t} \cdot cosh(\alpha t_j), & t_j \leqslant t \\ \alpha \cdot e^{\alpha t} \cdot sinh(\alpha t_j), & t \leqslant t_j. \end{cases} \quad (2.57)$$

And the second order derivative is given by:

$$\frac{\partial^2 H(t,t_j)}{\partial^2 t_j} = \alpha^2 \cdot H(t,t_j) - \alpha^3 \cdot min(t,t_j). \quad (2.58)$$

The third order derivative results discontinuous in $t = t_j$. As such, it is possible to conclude that, given the continuity of the first two derivatives of the Heart of Wilson function, the resulting curve has a sufficiently smooth behavior in its liquid section.
By analysing the function $G(t,t_j)$, we observe that if $t_j \leq t$, the following equation is obtained:

$$G'(t_j, \boldsymbol{t}) = G(\boldsymbol{t}, t_j) = \alpha \mathbf{1} - \alpha cosh(\alpha t_j) exp[-\alpha \boldsymbol{u}] \quad (2.59)$$

Employing the Wilson function, we con construct a symmetrical matrix, $\boldsymbol{W}$, respect to its principle diagonal, as seen previously. The matrix is positive and it converges to 0 for $t, t_j \to \infty$. This is caused by the definition of $\boldsymbol{W}$ as the weight matrix in the Correction term, 2.36. Thus, reproducing the convergence to the UFR of the forward interest rate.
Through the Wilson function, it is possible to derive the pricing function at any given time t_j. However, this may not result as efficient as only adopting the Heart of Wilson, since the Wilson function depends on both the UFR and the velocity of convergence parameter, α. Whereas, the Heart of Wilson function requires only the parameter α.
We can characterize a pricing function in terms of the Heart of Wilson function by introducing an auxiliary matrix $\boldsymbol{Q}$, which is equal to:

$$\boldsymbol{Q} = \boldsymbol{\mu}_{\triangle} \boldsymbol{X'}. \quad (2.60)$$

[23] The Wilson function is the result of the of the mean reverting and stationary Ornstein-Uhlenbeck stochastic process, see 3.1.

Where $\boldsymbol{\mu}_{\triangle}$ is a diagonal matrix computed through the column vector $\boldsymbol{\mu}$ by applying:

$$\boldsymbol{\mu}_{\triangle}\mathbf{1} = \boldsymbol{\mu}. \tag{2.61}$$

Thus, it is possible to write the following vector:

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_i \\ \vdots \\ \psi_N \end{bmatrix} = \boldsymbol{Q}'\mathbf{1} = \boldsymbol{X}'\boldsymbol{\mu}. \tag{2.62}$$

The vector of market observable prices can be redefined as:

$$\boldsymbol{m} = \boldsymbol{\Psi} + \boldsymbol{Q}'\boldsymbol{H}\boldsymbol{Q}\boldsymbol{\zeta}, \tag{2.63}$$

obtaining the present value function for one generic duration to maturity t:

$$m(t) = e^{-UFRt} + \boldsymbol{W}(0,\boldsymbol{t})\boldsymbol{X}\boldsymbol{\zeta} = e^{-UFRt} + e^{-UFRt}\boldsymbol{H}(0,\boldsymbol{t})\boldsymbol{Q}\boldsymbol{\zeta}. \tag{2.64}$$

When the model is applied to Zero Coupon Bonds, the relation above simplifies to:

$$v(0,t) = e^{-UFRt} + e^{-UFRt}\boldsymbol{H}(0,\boldsymbol{t})\boldsymbol{\zeta} \tag{2.65}$$

Finally, we can determine the equations for the yield to maturity function and the forward intensity function, between periods $\{0,t\}$, in terms of the Heart of Wilson function as below:

$$h(0,t) = UFR - \frac{log(1 + \boldsymbol{H}(t_j,\boldsymbol{t})\boldsymbol{Q}\boldsymbol{\zeta}}{t} \tag{2.66}$$

and

$$\delta(0,t) = UFR - \frac{dlog(1 + \boldsymbol{H}(t_j,\boldsymbol{t})\boldsymbol{Q}\boldsymbol{\zeta})}{dt} = UFR - \frac{G(t_j,\boldsymbol{t})\boldsymbol{Q}\boldsymbol{\zeta}}{1 + \boldsymbol{H}(t_j,\boldsymbol{t})\boldsymbol{Q}\boldsymbol{\zeta}}. \tag{2.67}$$

While at time 0, the equations above can be expressed as in the following equation. Observing that when $t_j = 0$, the Heart of Wilson is reduced to 0:

$$H'(0,\boldsymbol{t}) = G(\boldsymbol{t},0) = 0. \tag{2.68}$$

Therefore, the first order derivative function at time 0 can be easily computed through:

$$G'(0,\boldsymbol{t}) = G(\boldsymbol{t},0) = \alpha\mathbf{1} - \alpha exp[-\alpha\boldsymbol{u}]. \tag{2.69}$$

By using the latter result, the yield to maturity (or yield intensity, as referred to by EIOPA) and the forward intensity function are given by:

$$h(0) = \delta(0) = UFR - \alpha\mathbf{1}'\boldsymbol{Q}\boldsymbol{\zeta} + \alpha e^{-\alpha\boldsymbol{t}'}\boldsymbol{Q}\boldsymbol{\zeta}. \tag{2.70}$$

And to conclude, we have previously claimed that for $t \to \infty$ we obtain:

$$h(\infty) = \delta(\infty), \tag{2.71}$$

which is nonetheless the Ultimate Forward Rate in terms of intensity.
Due to the lack of continuous second order derivative, the forward intensity function is less smooth than the pricing function and the spot rate function in the liquid part of the term structure.

2.1.5 On the Model's Parameters

Two parameters have a significant impact when modelling the term structure as proposed by the Smith A. and Wilson T. The speed of convergence and the long term equilibrium forward rate substantially influence the structure of the curve and the trade-off between smoothness and coherence with liquid market data. We will now outline the features of these parameters relying principally on EIOPA's technical documentation.

UFR

The UFR is an exogenously fixed parameter set to cope with the lack of liquidity of financial instruments in the long run. As such, this parameter is determined under macroeconomic assumptions around the future development of short-term real rates and expected inflation rates. The expected inflation must be in line with the ECB's monetary policies, when considering the EURO currency. Additionally, the Ultimate Forward Rate is a stable parameter, since the motion of present short rates doesn't affect the asymptotic level. Thus, the UFR only varies with a shift in long term expectations. The change in its value should be done in transparent and accountable conditions. And, the procedure requires a clear method that allows scenario calculation. Moreover, quoting the Dutch central bank: "This approach has been chosen because the market for financial products with very long maturities is less extensive, liquid and transparent. Consequently, market rate information about it is insufficient to set the term structure" (De Nederlandsche Bank, 2016). Furthermore, the UFR does not include any sort of term premium that may reflect the risk of holding long term investments[24]. According to several academic studies, four macroeconomic variables have a significant effect on the UFR's value:

1. The previously mentioned expected inflation rate. Currently, there is no reliable statistics reflecting historical inflation rates, past financial structures or monetary

[24]see: **Consultation Paper on the methodology to derive the UFR and its implementation**

policies that could predict an asymptotic long term equilibrium rate. As a result, most technical literature proposes the adoption of the specific inflation level targeted by the central bank, that governs each considered currency. If this is the case, economic policy becomes the most reliable and efficient variable in determining a long term level of inflation given its deep link to macroeconomic dynamics. As a consequence of maintaining economic stability, the economic policy makers need to control the level of the inflation rate. Therefore, inflation becomes a fixed input in the determination of the UFR.

2. Additionally, the second factor that has a significant impact on the asymptotic rate is the real interest rate, when the inflation rate is considered as a separate variable. The precision of the estimated real rate in the long run is once more related to macroeconomic studies and projections. This assumption introduces subjectiveness in the estimation of the UFR and thus, given a tolerated pre-determined corridor of variability, is somehow dependent on the decision-maker's position.

3. In third place, stands the specific request that the term structure has to be an increasing function over time. Hence, the long term interest rates are greater than the last observable interest rate. Several academics have observed that in the very long run the curve can be considered flat, given the little influence present observable rates have over long term rates.

4. The final factor is a technical observation about the innate convexity of the present value to interest rate. Apparently, there is no clear evidence of a linear relationship between interest rates and the prices of bonds.

EIOPA has claimed that the UFR will be analysed annually and when found significantly distant from the one previously fixed, updates will be provided. The parameter is to be published on annual basis each March and subsequently the updated risk-free interest rates are to be released in the following three months[25].
Based on EIOPA's technical specification, the UFR cannot change more than $20bps$ per year. In such a case, the following formula has been set:

$$UFR^l_t = max\big\{UFR^l_{t-1} - 20bps; \quad min\{UFR_t; \quad UFR^l_{t-1} + 20bps\}\big\}, \tag{2.72}$$

with:

UFR^l_t, as the UFR of the current year after limitation;

[25]see: **Consultation Paper on the methodology to derive the UFR and its implementation** (page 9)

UFR^l_{t-1}, as the UFR of the previous year after limitation;

UFR_t, as the UFR of the current year without any kind of limitation.

The UFR without limitation is computed as a sum of the expected real[26] long term yield and the expected inflation rate. This expected real yield in the long run does not differ from one currency to another, on the contrary, the inflation rate does.
Additionally, the expected long term net of inflation interest rate is determined since 2010 through an arithmetic average based on 2009 historical data over 12 countries. However, in 2016 EIOPA changed its approach. From March 2017, the expected real rate will be computed by virtue of the following formula:

$$rr = exp\left\{ \frac{\sum_{i=0}^{n} w_i \cdot ln(1 + rr_i)}{\sum_{i=0}^{n} w_i} - 1 \right\}. \tag{2.73}$$

Thus, using a geometrical mean with:

rr as the expected value of long term real rate;

n as the number of years since 1960;

rr_i as the i-th annual real rate since 1960;

w_i as the i-th weight, defining $w_i = \beta^{n-i}$, with $\beta = 0.99$. [27]

Whereas, the expected inflation rate is the targeted inflation rate by the central bank that governs over the given currency.
This technique was initially proposed in the original technical specifications. On April 2016 EIOPA claimed that the real rate's formula will be modified, declaring that: "Instead of the arithmetic average used under the current approach, a geometrically weighted average with a fixed control parameter of 0.99 should be used."[28]
Regardless of the targeted inflation rate, the following bounds must be respected:

$$ir = \begin{cases} 1\%, & \text{if } it < 1\% \\ 2\%, & \text{if } 1\% < it < 3\% \\ 3\%, & \text{if } 3\% \leq it < 4\% \\ 4\%, & \text{if } 4\% \leq it. \end{cases} \tag{2.74}$$

[26] By real rate we mean a net of inflation rate.

[27] see: **Consultation Paper on the methodology to derive the UFR and its implementation** (page 10)

[28] see: **Consultation Paper on the methodology to derive the UFR and its implementation** (page 22)

Where ir denotes the inflation rate, while it denotes the inflation targeted.
If the central bank governing a specific currency does not have a target, the expected inflation rate is set by default to 2%.
With respect to the previous year, the UFR applied for calculating the term structure for 2016 and for the beginning of 2017 is computed as below:

$$UFR_{t-1}^{l} = \begin{cases} 3.2\%, & \text{for Japanese Yen and Swiss Franc,} \\ 4.2\%, & \begin{array}{l}\text{for EEA currencies and those non-EEA currencies not explicitly} \\ \text{mentioned elsewhere,}\end{array} \\ 5.2\%, & \begin{array}{l}\text{for the Brazilian Real, the Indian Rupee, the Mexican Peso,} \\ \text{the Turkish Lira and the South African Rand.}\end{array} \end{cases} \tag{2.75}$$

These UFRs derive from the fifth Quantitative Impact Study (QIS5) and since no significant impact has emerged throughout the past years, no change has been made to date.
To have a practical computational example: In the specific case of the Eurozone, the ECB has it's fixed inflation target at 2%. The estimated rounded real rate for 2016 and for the beginning of 2017 is 2.2%. By summing these two rates, we obtain 4.2%. The 4.2% level of the long term forward rate is fixed in terms of annual compounding. When requiring a continuously compounded asymptotic rate, we need to apply $ln(1+4.2\%)$, thus obtaining 4.114%.
In the annex to subsection 7.C of EIOPA's technical documentation published in September 2017[29] the methodology and historical data used in the calibration of the expected real rate are illustrated.

Convergence Point and Convergence Period

Once set the LLP and UFR, a period of time must pass up until the interest rate reaches its long term equilibrium. This period of time is a given constant and in the technical documentation is referred to as convergence period. For the EURO the convergence period is explicitly set by the EU's Omnibus II Directive to 40 years. To have a specific example of the computations involved, we can consider the EURO case[30].
Provided the EURO's LLP (20 years maturity), we can easily calculate the convergence point as below:

$$40 + 20 = 60 \text{ years maturity.}$$

[29] see: Technical documentation of the methodology to derive EIOPAs risk-free interest rate term structures
[30] see: Technical documentation of the methodology to derive EIOPAs risk-free interest rate term structures

For any other currency the convergence point, T, is determined through the following stable formula:

$$T = max\{LLP + 40; 60\}. \tag{2.76}$$

Consequently, the convergence period, S, results:

$$S = max\{40; 60 - LLP\}. \tag{2.77}$$

As seen previously, the parameters S and T will be part of the optimization problem that characterizes the speed of convergence, see 2.1.5.

Velocity of Convergence - α

The speed of convergence, denoted by α, is a fixed parameter that is optimized by the European Authority as presented in this paragraph.
Given $U = max(\boldsymbol{t})$, where $\boldsymbol{t}$ is the vector with the time to maturity of the observable financial instruments. We can now calculate the upper end of the risk free forward intensity function at time v employing the formula below:

$$\delta(v) = UFR + \frac{\alpha}{1 - ke^{\alpha v}}, \text{ with } v \geq U. \tag{2.78}$$

Observing that, in such a case, we consider the UFR as an intensity function. As a result, when dealing with the EURO, the UFR will be given by:

$$UFR = ln(1 + 4.2\%). \tag{2.79}$$

Implying that the Heart of Wilson can be simply reduced to:

$$H(\boldsymbol{t}, v) = \alpha\boldsymbol{u} - e^{-\alpha v} sinh[\alpha\boldsymbol{u}]. \tag{2.80}$$

Where k is constant that doesn't depend on S, while it depends on the given UFR and α. Hence, it is commonly referred to as "quasi-constant".
α is calculated requiring that $k \neq 0$. If this condition is not met, the forward rate will not approach the UFR. In EIOPA's technical documentation, we find the following formula used to compute k:

$$k = \frac{1 + \alpha\boldsymbol{t}'\boldsymbol{Q}\boldsymbol{\zeta}}{sinh[\alpha\boldsymbol{t}']\boldsymbol{Q}\boldsymbol{\zeta}}. \tag{2.81}$$

Where $\boldsymbol{Q}$ is an auxiliary matrix given by 2.60 and $\boldsymbol{t}$ is the vector of the observed durations to maturity.
A complete convergence to the UFR is not required but the distance between the UFR and the forward intensity function is restrained by a tolerance parameter, τ, currently set by EIOPA at $1bp$. Therefore, α is a value that must comply with the following condition:

$$|\delta(T) - UFR| \leq 1bp. \tag{2.82}$$

Where $\delta(T)$ denotes the risk-free forward intensity function between the convergence point and a subsequent unitary time interval.
Hence, an optimization problem for determining α can be set by defining: $T = U + S$ as the convergence point (see 2.76), where S is the convergence period (see 2.77), and a function representing the convergence gap in T, $g(\alpha)$, is to be defined as:

$$g(\alpha) = |UFR - \delta(T)| = \frac{\alpha}{|1 - ke^{\alpha T}|}. \tag{2.83}$$

We can now minimize α given the following constraints:

$$\alpha \geqslant a,$$
$$g(\alpha) \leqslant \tau.$$

Where a is a lower bound, set at 0.05.
We now possess all ingredients needed to build a term structure in accordance with the methodology proposed by Smith A. and Wilson T. (2001).

Chapter 3

Properties and Issues of the Smith-Wilson Method

In this chapter, we aim at the in-depth analysis of Smith-Wilson's primary features. We will investigate over the relationship between EIOPA's term structure methodology and two notable models. First, we will concentrate on the link between the Smith-Wilson method and a stochastic process, designed to regulate volatility that goes by the name of Ornstein-Uhlenbeck process. Secondly, an examination of a further relation with a time dependent and mean reverting model for the short rate introduced by Hull and White, will be provided.
The final step will consist of a focus on the critical points occurred since EIOPA's first implementation of the Smith-Wilson method. Some of these problematics underlie the current debate between the European Authority and long-term liabilities managers.

3.1 Smith-Wilson and the Integrated Ornstein-Uhlenbeck Process

[1] This section introduces the Ornstein-Uhlenbeck process and one of its most notable modified versions: the integrated Ornstein-Uhlenbeck process. The main purpose is to highlight the existing connection between the modified version mentioned afore and the Wilson function defined in a Smith-Wilson technique, as suggested by EIOPA (2010). For this statement to be legit, certain assumptions on the Ornstein-Uhlenbeck process must

[1] Way of reasoning: On the relation between the Smith-Wilson method and integrated Ornstein-Uhlenbeck processes

be provided[2].

3.1.1 The Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck process, is a diffusion process originally designed to represent the velocity of particles driven by a Brownian motion. Ornstein-Uhlenbeck processes are regularly used in finance to model the instantaneous volatility, when dealing with stochastic volatility problems. The most extensively studied in the modern technical literature is one particular application of this process to the dynamic of the short rate, commonly referred to as the Vasicek model, (1977)[3]. The Vasicek model is broadly employed to model the volatility underlying commodity prices.

Let $\{r(t),\ t \geq 0\}$, be the time dependent process that is to be modelled. We define this process as an Ornstein-Uhlnebeck process, if it satisfies the stochastic differential equation below :

$$dr(t) = -\lambda r(t)dt + \sigma dZ(t), \text{ with } \lambda, \sigma > 0. \tag{3.1}$$

With:

λ, an arbitrary positive constant;

$dZ(t)$, the standard brownian motion, thus $Z(t) \sim N(0, \sqrt{dt})$;

σ, the volatility of the process.

The equation above is the sum of two specific terms: the drift term and the diffusion term. It is generally assumed that the drift term is the average motion of the process in the following instant of time, while the diffusion term incorporates the width of the motion over time.

Along this line, Vasicek (1977), defined the mean reverting Ornstein-Uhlenbeck process as:

$$dr(t) = k(\theta - r(t))dt + \sigma dZ(t), \text{ with } k, \theta, \sigma > 0. \tag{3.2}$$

Where:

k, is the speed of mean reversion;

and θ, is the mean to which the process tends in the long run.

This version is largely used due to its simplicity since it only needs the estimation of three parameters. And, provides a high performance in terms of market representation. Furthermore, if we let $r(t)$ be the process of the spot rate and the first two moments are sufficient to determine the distribution of the spot rate, empirical evidence has shown that interest rates tend to move towards a long term average rate. Thus, the mean-reversion included in the first term of the equation above. Here, θ will depict the mean of the

[2] see: **QIS5 Risk-Free Interest Rates - Extrapolation Method** Page 13, footnote 16.

[3] see: **An equilibrium characterization of the term structure**

spot rate (the long term rate), while the σ incorporated in the diffusion term will denote the standard deviation of the spot rate over time. The mean reversion drift will act as a decreasing factor when rates are above the long term rate and it will cause a positive drift, when rates are below the long term rate.
Additionally, we can give the process an alternative mathematical structure by using the integral representation of the process as following:

$$r(t) = r(0)e^{-kt} + \frac{k\theta}{k}(1 - e^{-kt}) + \sigma \int_0^t e^{-k(t-s)} dZ(s). \tag{3.3}$$

With $r(0)$ denoting a stochastic variable independent of the brownian motion $dZ(s)$. In particular, when considering the process of the spot rate, $r(0)$ is regarded as the rate observable at the evaluation date.
Computing the expected value, the variance and the covariance, we obtain:

$$\mathbb{E}[r(t)] = \mathbb{E}[r(0)]e^{-kt} + \frac{k\theta}{k}(1 - e^{-kt}); \tag{3.4}$$

$$Var(r(t)) = Var(r(0))e^{-2kt} + \frac{\sigma^2}{2k}(1 - e^{-2kt}); \tag{3.5}$$

$$Cov(r(t), r(s)) = Var(r(0))e^{-k(t+s)} + \frac{\sigma^2}{2k} e^{-k(t+s)}(e^{2k(t \wedge s)} - 1). \tag{3.6}$$

A disadvantage of the process lies in the impossibility of preventing the process to assume negative values. This comes as a direct consequence of the structure incorporated in the diffusion term in equation 3.2. As such, we can calculate the probability of negative values as:

$$\mathbb{P}\{r(s) < 0\} = \int_{-\infty}^0 \mathbb{N}(x, \mathbb{E}[r(s)], Var[r(s)])dx. \tag{3.7}$$

3.1.2 The Integrated Ornstein-Uhlenbeck Process

An Integrated Ornstein-Uhlenbeck, depicted by $r^*(t)$, is a stochastic process that is defined as an integer of a strictly stationary process on the real line, $r(t)$. This is represented in the following equation:

$$r^*(t) = \int_0^t r(s)ds. \tag{3.8}$$

By specifying $r(t)$ as in 3.3, it can be demonstrated that the expected value, the variance and covariance of $r^*(t)$, can be computed through:

$$\begin{aligned} \mathbb{E}[r^*(t)] &= \mathbb{E}\left[\int_0^t r(s)ds\right] = \\ &= \frac{1}{k}\left(\mathbb{E}[r(0)] - \frac{k\theta}{k}\right)(1 - e^{-kt}) + \frac{k\theta}{k}t; \end{aligned} \tag{3.9}$$

$$\begin{aligned} Var[r^*(t)] &= Var\left[\int_0^t r(s)ds\right] = \\ &= \frac{1}{k^2}\left(Var[r(0)] - \frac{\sigma^2}{k}\right)(1-e^{-kt}) + \frac{\sigma^2}{k^3}(kt - e^{-kt}sinh[kt]); \end{aligned} \tag{3.10}$$

$$\begin{aligned} Cov[r^*(s), r^*(t)] &= Cov\left[\int_0^s r(y)dy, \int_0^t r(z)dz\right] = \\ &= \frac{1}{k^2}\left(Var[r(0)] - \frac{\sigma^2}{k}\right)(1-e^{-k(s\vee t)})(1-e^{-k(s\wedge t)}) + \\ &+ \frac{\sigma^2}{k^3}(kt - e^{-k(s\vee t)}sinh[k(s\wedge t)]). \end{aligned} \tag{3.11}$$

This definition of the covariance of an integrated Ornstein-Uhlenbeck process will be discussed and linked to the Wilson function in the next section.

3.1.3 The Relation Between Smith-Wilson and the Integrated Ornstein-Uhlenbeck Process

[4] The relationship between the Smith-Wilson method and the Integrated Ornstein-Uhlenbeck process lies in the Wilson function. For our purpose, we consider the Wilson function related to Zero Coupon Bonds, as in 2.40. If this is the case, it can be demonstrated that the Wilson function is equal to a scaling of the covariance as computed in the integrated Ornstein-Uhlenbeck process.
Let:

$$\frac{\sigma^2 e^{-\alpha(s+t)}}{\alpha^3}$$

be the scaling factor applied to the covariance.
Hence, we can write the following equation:

$$\frac{\alpha^3 W(s,t)}{\sigma^2 e^{-\alpha(s+t)}} = Cov[r^*(s), r^*(t)]. \tag{3.12}$$

We can now set a pricing problem under risk neutral evaluation by letting $r(s)$ be the short term rate underlying a Vasicek stochastic process. As previously demonstrated, the Vasicek model is equivalent to a mean reverting Ornstein-Uhlenbeck process. By choosing to model the evolution of the short term rate through an Ornstein-Uhlenbeck process, we assume a polynomial exponential structure of the discount factor with added noise.

[4] As discussed in: On the relation between the Smith-Wilson method and integrated Ornstein-Uhlenbeck processes

Therefore, the equation for the stochastic discount factor legit from period t to 0, can be simply reduced to:

$$D(t) = exp\Big\{ - \int_0^t r(s)ds \Big\}. \tag{3.13}$$

The price is obtained by applying the risk-neutral expected value to the stochastic discount factor as following:

$$P(t) = \mathbb{E}^{\mathbb{Q}}[D(t)]. \tag{3.14}$$

Introducing a new variable, $r^\circ(t)$, such that $\mathbb{E}[r^\circ(t)] = 0$, we can compute:

$$r^\circ(t) = -\mathbb{E}[r^*(t)] + \int_0^t r(s)ds, \tag{3.15}$$

with the expected value $\mathbb{E}[r^*(t)]$ given by equation 3.9.
By inserting 3.15 in equation 3.13 and proceeding with a Taylor series expansion around 0 short term rates, we obtain the following approximation of the stochastic discount factor:

$$D(t) \simeq e^{-\mathbb{E}[r^*(t)]}(1 - r^\circ(t)) = D^\circ(t). \tag{3.16}$$

Where $D^\circ(t)$ is the approximated stochastic discount factor.
By reparametrising:

$$\alpha = k,$$

$$\alpha^3 = \sigma^2,$$

$$\alpha^2 = Var[r(0)],$$

$$\alpha UFR = k\theta,$$

$$UFR = \mathbb{E}[r(0)].$$

We can now redefine the approximated stochastic discount factor as:

$$D^\circ(t) = e^{-UFRt}(1 - r^\circ(t)). \tag{3.17}$$

When computing the covariance of the approximated stochastic discount factor, we get:

$$\begin{aligned} Cov[D^\circ(s), D^\circ(t)] &= Cov[e^{-UFRs}r^\circ(s), e^{-UFRt}r^\circ(t)] = \\ &= e^{-UFR(s+t)}Cov[r^*(s), r^*(t)]. \end{aligned} \tag{3.18}$$

This final equation can be assimilated to the Wilson function as in 2.40. If this is the case, $Cov[r^*(s), r^*(t)]$ is to be interpreted as the Heart of the Wilson function.
The pricing function 2.41 can be written as the expected value of an interest rate model based on an integrated Ornstein-Uhlenbeck process, provided that the values of the process are adequately estimated. This result is a mere consequence of the gaussianity incorporated in the integrated Ornstein-Uhlenbeck process. Thus, the position of the integrated

Ornstein-Uhlenbeck process at times $t_1, t_2, \ldots, t_j$ is distributed as a multivariate Normal. It is possible to proceed with further analysis by letting the mean of a multivariate Normal, $\boldsymbol{X} = \{X_1, \ldots, X_j\}$, be represented through the following vector:

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_A \\ \mu_B \end{bmatrix}. \tag{3.19}$$

As such, the covariance matrix is given by:

$$\boldsymbol{\Omega} = \begin{bmatrix} \Omega_{AA}\Omega_{AB} \\ \Omega_{BA}\Omega_{BB} \end{bmatrix}. \tag{3.20}$$

Consequently, we obtain:

$$\mathbb{E}[X_A | X_B = x_B] = \mu_A + \Omega_{AB}\Omega_{BB}^{-1}(x_B - \mu_B). \tag{3.21}$$

Additionally, the stochastic discount vector can be written as a multivariate Normal:

$$\boldsymbol{D}^{\circ} \sim \mathbb{N}(\boldsymbol{\mu}, \boldsymbol{\Omega}),$$

with:

$$\boldsymbol{D}^{\circ} = \begin{bmatrix} D^{\circ}(t) \\ D^{\circ}(t_1) \\ D^{\circ}(t_2) \\ \vdots \\ D^{\circ}(t_i) \\ \vdots \\ D^{\circ}(t_N) \end{bmatrix},$$

$$\mu_A = e^{-UFRt},$$

$$\boldsymbol{\mu_B} = \begin{bmatrix} e^{-UFRt_1} \\ e^{-UFRt_2} \\ \vdots \\ e^{-UFRu_i} \\ \vdots \\ e^{-UFRt_N} \end{bmatrix},$$

$$\boldsymbol{\Omega_{AB}} = \{e^{-UFR(t+t_j)} Cov[r^*(t), r^*(t_j)]\}_{j=1}^{N}$$

and

$$\boldsymbol{\Omega_{BB}} = \{e^{-UFR(t_i+t_j)} Cov[r^*(t_i), r^*(t_j)]\}_{j=1}^{N},$$

where $t_1, t_2, \ldots, t_N$ are the maturities of the N Zero Coupon Bonds, while t is the evaluation date. In this case, the pricing equation, 2.41, can be now written as:

$$\begin{aligned} P(t) &= \mathbb{E}[D^{\circ}(t)|(D^{\circ}(u_1), D^{\circ}(u_2), \ldots, D^{\circ}(u_N))' = \boldsymbol{m}'] = \\ &= \mu_A + \boldsymbol{\Omega_{AB}}\boldsymbol{\Omega_{BB}}^{-1}(\boldsymbol{m} - \boldsymbol{\mu_B}). \end{aligned} \tag{3.22}$$

This leads to the conclusion that the pricing function, as specified in a Smith-Wilson model, can be interpreted as a conditional expected value of a given non-stationary Gaussian stochastic process.

Alternatively, a re-formulated version of the pricing equation can be given by[5]

$$P(t) = \beta_0 + \boldsymbol{\beta}'\boldsymbol{m}. \tag{3.23}$$

Thus:

$$\beta_0 = \mu_A - \boldsymbol{\Omega_{AB}}\boldsymbol{\Omega_{BB}}^{-1}\boldsymbol{\mu_B} \tag{3.24}$$

and

$$\boldsymbol{\beta}' = \boldsymbol{\Omega_{AB}}\boldsymbol{\Omega_{BB}}^{-1}. \tag{3.25}$$

Moreover, β_0 and $\boldsymbol{\beta}'$ are functions of both t and $\boldsymbol{m}$ only when α is the result of the optimization problem set in 2.83. Otherwise, β_0 and $\boldsymbol{\beta}'$ are only functions of the maturity. The evolution of the β parameters will be of interest in the section dealing with the issues linked to the Smith-Wilson method.

And finally, the vector of parameters $\boldsymbol{\zeta}$, can be calculated as:

$$\boldsymbol{\zeta} = \boldsymbol{\Omega_{BB}}^{-1}(\boldsymbol{m} - \boldsymbol{\mu_B}). \tag{3.26}$$

In the equation above the vector $\boldsymbol{m}$ is the vector of the observed prices, while the prices of unobserved assets, under risk neutral assumptions, are computed by applying the risk neutral expected value to the stochastic discount factors conditional on the observed assets prices.

With this newly discovered result, we have found an alternative procedure to compute the parameters in a Smith-Wilson methodology.

3.2 Smith Wilson and the Hull and White Model

In 2016 De Kort J. and M.H. Vellekoop found an existing relationship between the exponential splines used to interpolate the market part of the term structure in a Smith-Wilson model and the Hull and White discount factor. We will firstly proceed with the characterization of the Hull and White model as originally synthesized by the two authors. Subsequently, we will move to the aforementioned relationship between the two models.

[5] Way of reasoning: **Issues with the Smith-Wilson Method**

3.2.1 The Hull and White Model

The Hull and White [6] model (1990) is a relatively simple arbitrage-free model used to describe the evolution of stochastic interest rate over time by means of continuous-time processes or jump-diffusion processes. It is typically employed to price interest rate derivatives as it ensures that the market value of a generic bond is exact. Furthermore, the Hull and White Model can be interpreted as a general extension of the Ho and Lee model (1986)[7] by adding mean-reversion to the dynamic of the short term interest rates. Thus, in the long run the short rate will turn to its long term equilibrium. The long term equilibrium is only affected by major macroeconomic changes. Alternatively, the model can be interpreted as a modified version of the Vasicek model, where the long term equilibrium rate and the speed of convergence to the equilibrium become variables depending on time.

In a Hull and White model, the dynamic of the short rate can be synthetized by the following equation:

$$dr(t) = k(t)(\theta(t) - r(t))dt + \sigma(t)dZ(t) = (\theta^*(t) - k(t)\rho(t))dt + \sigma(t)dZ(t), \tag{3.27}$$

supposing that:

$Z(t)$ is the standard Brownian motion, with $Z(t) \sim \mathcal{N}(0, \sqrt{dt})$;

$k(t)$ is the time dependent speed of mean reversion and is related to the slope of the curve;

$\theta(t)$ and $\theta^*(t) = k(t)\theta(t)$ represent the time dependent mean to which the process tends in the long run.

When $k(t)$ and $\theta(t)$ (or $\theta^*(t)$) are constant we can easily track the Vasicek model.

It is now possible to apply the model to risk-neutral pricing problems. A zero coupon bond in t with a generic maturity T, under risk-neutrality assumptions, can be easily priced through:

$$v(t,T) = \mathbb{E}\Big[exp\Big(-\int_t^T r(u)du\Big)\Big|\mathcal{F}(t)\Big], \tag{3.28}$$

thus, we need to compute an expected value dependent on the filtration $\mathcal{F}(t)$, which summarizes the information available in t.

In a Hull and White model, the pricing function is reduced to[8]:

$$v(t,T) = A(t,T)e^{-B(t,T)r(t)}, \tag{3.29}$$

where it is proven that:

$$B(t,T) = \frac{1 - exp\{-k(t)(T-t)\}}{k(t)} \tag{3.30}$$

[6]see **Pricing Interest-Rate-Derivatives Securities**

[7]see: **Term Structure Movements and Pricing of Interest Rate Claims**

[8]As reasoned in: **Pricing interest-rate derivative securities**

and:

$$A(t,T) = exp\left\{ \frac{[B(t,T) - T + t][k(t)\pi(t) - \frac{\sigma(t)^2}{2}]}{k^2(t)} - \frac{\sigma^2.(t)B^2(t,T)}{4k(t)} \right\}. \tag{3.31}$$

$A(t,T)$ depends on the parameter $\pi(t)$ that is:

$$\pi(t) = k(t) + \theta^*(t) - \lambda(t)\sigma(t), \tag{3.32}$$

where $\lambda(t)$ is the market price of the interest rate risk at period t.

3.2.2 On the Relationship between the Hull and White Model and the Smith-Wilson Method

We will now proceed to illustrate how a scaled version of the Heart of Wilson function can be used to determine the interest rate in a money market account and most importantly, the discount factor defined through a Hull and White model. To this end, we will require the previously defined integrated Ornstein-Uhlenbeck process.
The first step is to introduce the following processes:
A standard brownian motion: $Z(t)$;
The Ornstein-Uhlenbeck Process $X(t)$:

$$X(t) = \int_0^t e^{-\alpha(t-u)} dW(u). \tag{3.33}$$

Where $W(u)$ is a standard brownian motion independent of $Z(t)$ and α is a positive fixed constant. Provided the above definitions, the Heart of Wilson function con be linked to the covariance of the processes:

$$H(t,s) = \alpha \cdot cov\big(Z(t) + X(t); Z(s) - X(s)\big). \tag{3.34}$$

To demonstrate the relationship, we need to determine the scaling of the Heart of Wilson function. This is only possible by means of an integrated Ornstein-Uhlenbeck process with the following structure:

$$\Upsilon(t) = \int_0^t X(s) ds, \tag{3.35}$$

where the process $X(s)$ is defined by 3.33.
Consequently, the conditional expected value will be computed through:

$$\mathbb{E}\big[\Upsilon(t)|\Upsilon(t_1), \Upsilon(t_2), \ldots, \Upsilon(t_N\big] = \sum_{j=1}^{N} \epsilon_j \Upsilon_{t_j}. \tag{3.36}$$

In this equation, the vector $\boldsymbol{\epsilon}$ can be defined in terms of an invertible matrix $\boldsymbol{G}$ related to the Wilson function:

$$\boldsymbol{\epsilon} = \bar{\boldsymbol{G}}^{-1}\boldsymbol{\beta}(t), \tag{3.37}$$

with the vector $\boldsymbol{\beta}_j(t)$ given by:

$$\boldsymbol{\beta}_j(t) = G(t, t_j) \tag{3.38}$$

and

$$\bar{G}_{ij} = G(t_i, t_j), \tag{3.39}$$

providing the scaling:

$$G(t_i, t_j) = H(t_i, t_j) - (1 - e^{-\alpha t_i})(1 - e^{-\alpha t_j}). \tag{3.40}$$

The claim that $G(t_i, t_j)$[9] can be used as the best linear predictor of the yield, $\frac{1}{t}log(B_t)$, produced by a money market account in a Hull and White model it is now easily provable. Given the generic stochastic process $\chi(t)$, the best linear predictor, conditional to $\chi(t_1), \chi(t_2), \ldots, \chi(t_N)$, is:

$$g[\chi(t)] = \sum_{j=1}^{N} \vartheta_j \chi(t_j) \tag{3.41}$$

and supposing that the weights ϑ_j are the solution of:

$$\min_{\vartheta_1, \vartheta_2, \ldots, \vartheta_N} \quad \mathbb{E}\left[\left(\chi(t) - \sum_{j=1}^{N} \vartheta_j \chi(t_j)\right)^2\right]. \tag{3.42}$$

If we let $log(B_t)$ be the mean reverting integrated Ornstein-Uhlenbeck process, the conditional expected value is directly obtained through:

$$\begin{aligned} &\mathbb{E}\big[log(B_t) \big| log(B_{t_1}), log(B_{t_2}), \ldots, log(B_{t_N})\big] = \\ &= \mathbb{E}\big[log(B_t)\big] + \sum_{j=1}^{N}\left(\sum_{i=1}^{N} \boldsymbol{G}_{ij}^{-1} G(t, t_j)\right)(log(B_{t_j}) - \mathbb{E}[log(B_{t_j})] \end{aligned} \tag{3.43}$$

that is the best linear predictor.
It is also possible to prove that the best linear predictor of the stocastic discount factor in a Hull and White model, $D(t)$, can also be represented in terms of the interpolating functions of the Smith-Wilson method. To this end, we require to define:

$$\begin{aligned} log\hat{G}(t_i, t_j) =& \mathbb{E}[log(D(t_i)) + \mathbb{E}[log(D(t_j)) + \\ &+ \frac{1}{2}k^{-3}(t)(G(t_i, t_i) + G(t_j, t_j) + 2G(t_i, t_j)), \quad \textit{for all} \quad i, j = 1, 2, \ldots, N \end{aligned} \tag{3.44}$$

[9]Reasoning as in: **Term structure extrapolation and asymptotic forward rates**

where $k(t)$ is the time dependent mean reversion parameter in a Hull and White model. When the matrix $\hat{\boldsymbol{G}}$ is invertible the best linear predictor for the Hull and White discount factor $D(t)$ conditional to $D(t_1), D(t_2), \ldots, D(t_N)$, is:

$$g[D(t)] = \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \hat{G}_{ij}^{-1} \hat{G}(t, t_j) \right) D(t_i). \tag{3.45}$$

Further mathematical analysis can also lead to represent the Heart of Wilson function in terms of a covariance between two process.[10].
We will rely upon the results achieved in this section, when discussing the smoothness of the Smith-Wilson term structure of interest rates.

3.3 Issues with the Smith-Wilson Method

The Smith-Wilson method is far from being flawless. Some of it's issues can potentially have a negative impact when evaluating market consistent liabilities and consequently, produce an incoherent valuation of the Solvency Capital Requirement. This may refrain insurance companies from taking proper actions when dealing with external market shocks and deliver increasing long-term costs on policyholders.
The purpose of this section is to determine the main problematics arose from EIOPA's privileged extrapolation method. Recent scientific literature and research on the subject will be discussed[11] along with EIOPA's Technical Documentation.

3.3.1 Hedging

A serious critique to the Smith-Wilson method regards the perfect hedging assumption. When dealing with evaluations concerning financial instruments, we assume that a unit of liability to be paid at time t can be perfectly hedged by a free credit risk Zero Coupon Bonds with maturity t. Therefore. when acquiring a credit risk free Zero Coupon Bond it is certain that the debt will be extinguished no matter how interest rates move. Under this assumption and in a complete market, when the financial instrument has more than a single cash flow, each cash flow can be replicated by a Zero Coupon Bond with maturity matching each payment date.
In practice, the complete market hypothesis is not met. Hence, it is almost impossible to observe default free Zero Coupon Bonds for each tenor, especially when modeling long run maturities.

[10]For proof see: **Term structure extrapolation and asymptotic forward rates** Proposition 4
[11]We will mainly focus on Lageras A. and Lindholm M., (2016), findings

The Smith-Wilson method is an interpolation method for the DLT financial instruments with maturities $t_1, t_2, \ldots, t_N$, if we consider only the first part of the curve. In terms of the perfect hedging assumption, we can therefore, replicate all liabilities with maturities $t_1, t_2, \ldots, t_N$ with Zero Coupon Bonds of same maturity.
Equation 3.23 is the key to produce a replication strategy: have β_0 units in cash and buy β_i units of default free Zero Coupon Bond, each with maturities t_i, where $i = 1, 2, \ldots, N$. Regardless the behavior of the Zero Coupon Bond value, the final price of the portfolio will remain stable at $P(t)$.
Nevertheless, liabilities with maturities outside the observed set of financial instruments are priced using only the model. As such, empirical evidence has shown that for all Smith-Wilson curves the replication strategy has a strange oscillating pattern when the method deals with maturities larger than the observed maturities t_N. In this case, the strategy requires a negative amount of Zero Coupon Bonds with tenor t_{N-1}, a positive amount of Zero Coupon Bonds with tenor t_{N-2}, again a negative amount of Zero Coupon Bonds with tenor t_{N-3}, and so on. In mathematical terms, for $t > t_N$, this translates to:

$$sign(\beta_i) = (-1)^{N-i}, \text{ for } i = 1, 2, \ldots, N. \tag{3.46}$$

Nonetheless, the sum of the absolute value of each amount will be greater than the present value of the liability that is to be hedged[12]
A further issue related to the hedge strategy comes when considering a time interval of Δt, say a month. When Δt passes, the pricing function $v(0, t + \Delta t)$ is the result of Zero Coupon Bonds with maturities $t_i + \Delta t$ for $i = 1, 2, \ldots, N$, while $v(0, t_i)$ was calculated using Zero Coupon Bonds with maturities t_i for $i = 1, 2, \ldots, N$. A practical hedging strategy is almost impossible to conceive when dealing with negative amounts of financial instruments.
Moreover, by virtue of 3.46, all the possessed instruments must be changed to holdings with opposite sign in the time span of one year, if all t_i are all one year apart. However, in some cases t_i are 2, 3 or even 5 years apart.

3.3.2 Discount Factors

The discount factor function, $v(0, t)$, which is the outcome of the Smith-Wilson model, may well present the following undesired features:

1. Despite the fact that, $v(0, t)$ may appear to be a decreasing function, for certain values of t, on the liquid part of the term structure, it could easily become a locally

[12] see **Issues with the Smith-Wilson Method** Example 1

increasing function for several input points, since there is no innate constraint on the discount function that forces a decreasing behavior. Thus, introducing the possibility of reaching negative values of the forward interest rate. This issue is common in term structure models, particularly when fitting the curve on two liquid market inputs that result in an insignificantly distant value of $v(0, t)$.

2. The second problem with the discount factor function comes when extrapolating the term structure. The function $v(0, t)$ may become negative when the final forward yield tend to be higher than the sum of UFR and alpha and negative values of the discount factor don't allow real values of interest rates. There is nothing specified by EIOPA on how to manage the model when such a situation occurs. One may simply check manually the input data. Consequently, when considering inputs computed through simulations or Economic Scenario Generators, checks must be made before applying the Smith-Wilson method.

A possible solution is an increase in the value of α, which leads to an increase of the speed of mean-reversion, thus producing a stiffer term structure.
We can demonstrate that for large values of α the pricing function returns to positive values by observing that: when the speed of convergence reaches a high enough level and when $t \geq s$, the Wilson function can easily be approximated to:

$$W(s,t) \simeq \alpha \cdot min\{s,t\} \tag{3.47}$$

This will imply that $v(0, t)$ con be approximated through e^{-UFRt}. Hence, if t is the convergence point, therefore $t > t_i$, an appropriate increase in the speed of convergence will eventually provide positive values of the pricing function.

3.3.3 Optimization of the Velocity of Convergence

The speed of convergence has been for long a subject of debate between market operators. There is a major concern about the speed of convergence as a potential source of volatility for insurance companies.
A first topic of furor is the the "40 + 20" convergence period. The drawback of such a large convergence point is related to the lack of sensitivity to financial shocks. To introduce more stability in the term structure curve, both insurance companies and the European Parliament are proposing a reduced value of "10 + 20".
The second concern regarding the speed of convergence is related to the mathematical approach used by the European Regulator. In the aforementioned optimization problem

set in EIOPA's technical documentation, there is a major risk: singularities may arise when searching for the optimal α. This will be discussed in the following paragraph.
Using the notation of 2.1.5, we will have to cope with a singularity when:

$$1 - ke^{\alpha T} = 0. \tag{3.48}$$

Therefore:

$$1 + (\alpha \boldsymbol{t'} - e^{\alpha T} sinh[\alpha \boldsymbol{t'}])\boldsymbol{Q\zeta} = 0. \tag{3.49}$$

In this case, the discount factor for the convergence point can be written as:

$$v(0,T) = e^{-UFR \cdot T}[1 + (\alpha \boldsymbol{t'} - e^{\alpha T} sinh[\alpha \boldsymbol{t'}])\boldsymbol{Q\zeta}]. \tag{3.50}$$

If the above $v(0,T)$ is equal to 0, this will result in a singularity for $g(\alpha)$.
It can furthermore be observed that $\delta(T) - UFR \leq 0$ is only true when $v(0,T) \leq 0$. Consequently, we obtain $\delta(T) - UFR > 0$, only if $v(0,T) > 0$. In such a case, the problem may produce singularities, for some $\alpha \geq a$, in α's optimization domain depending on the evolution of the input rates.
Additionally, De Kort G and Vellekoop M.H. (2016) notice that when the Smith-Wilson method is applied to zero coupon financial instruments, for some values of the parameter α, on which the velocity of convergence to the Ultimate Forward Rate depends, the resulting prices could become negative. Unlike in the afore discussed problematic, in this case, the issue is related to the value given to α.
The final point of criticism is related to the solution given to the above issue: an increased velocity of convergence may avoid negative discount factors. As a results, it may well produce a singularity that has to be avoided by the selected optimization algorithm[13].

3.3.4 Lack of Smoothness

We have previously examined, the existing relationship between the exponential splines used by the Smith-Wilson method and the best linear predictor in a Hull and White model. Through this relationship, we can prove that the smoothness achieved using Smith-Wilson's interpolation functions may not result sufficient (as discussed by De Kort G and Vellekoop M.H., 2016).
Considering a Hull and White model with an evolutionary process that follows a standard Brownian motion with mean reversion to the long term time dependent equilibrium $\theta(t)$ at a speed measured by the positive time dependent parameter $k(t)$ and with volatility summarized by σ_t, we can simply employ $\theta(t)$ to calibrate the term structure given by $v(0,t)$. To this end, we will require the forward intensity function, $\delta(0,t)$ and its first order

[13]see **Issues with the Smith-Wilson Method** Page 6

derivative.[14].
Supposing that:

$$\theta(t) = \frac{1}{k(t)} \cdot \frac{\partial\delta(0,t)}{\partial t} + \delta(0,t) + \frac{\sigma_t^2[1 - e^{-2k(t)\cdot t}]}{2k^2(t)}, \tag{3.51}$$

when the term structured is modelled through the Smith-Wilson methodology, $\theta(t)$ will depend upon a mixture of second order derivatives of the Heart of Wilson function, provided that:

$$\frac{\partial\delta(0,t)}{\partial t} = -\frac{\partial^2 log[v(0,t)]}{\partial^2 t}. \tag{3.52}$$

As aforementioned, these interpolating functions are continuous but still their differentiability is limited to the second order. Hence, we incur an undesirable low level of smoothness that can only be overcome by increasing differentiability.
In the following chapter, we will discuss a methodology created to specifically cope with this problematic. The result will induce a different specification of the interpolating function.

3.3.5 Exogenously Given UFR

Since it's introduction in 2010, the constant long term forward rate, UFR, has raised concerns among academics and insurance companies alike. The most severe weakness of the Smith-Wilson model is, indeed, the exogeneity and constancy of the long term forward rate that contradicts the very first principle on which Solvency II is based: market consistency. There is no financial instrument available on the present financial market providing an interest rate in any way related to the UFR. As a result, its high level produces a significantly unrealistic increase in the extrapolated part of the term structure of interest rates. In such a case, the interest rates, provided by the regulator beyond the last liquid point, suffers from a lack of connection to available market instruments and of severe market distortion. The UFR has been described as both an "accounting trick" and as specifically designed to "camouflage risk". Consequently, it substantially contrasts with EIOPA's initial intentions of employing a model whose outcome would be a direct image of the financial market behavior. This will inevitably produce a shortage of sensitivity to swap rates of the capital required to result solvent in accordance with the Solvency II legislation. As such, the UFR introduces a significant degree of uncertainty and thus, volatility in risk managing companies who offer long term guarantees. Furthermore, the subjectiveness of the UFR leaves enough space for its potential revision due to political preasure rather than through an objective depiction of the evolution of financial instruments available on the market. Arbitrary changes of the asymptotic forward rate produce

[14] As discussed in: **Term structure extrapolation and asymptotic forward rates**

regulatory risks that no insurance company has the ability to hedge. Hence, leaving considerable room for exposure to unexpected variations of the long term forward rate. The main consequence will be failing to provide long term liabilities insurers and pension funds with proper instruments to avoid market crisis and produce inevitable higher costs on policyholders.

To achieve a structure that better reproduces the behavior of the market, a lower asymptotic rate could be fixed. This will translate into a lower discount curve that may lead to higher long term liabilities. As a matter of fact, EIOPA has proposed a reduced UFR for the Euro area, 3.7%. This has prompted opposition from the German long term liability market who "argue the rate reflects assumptions about long-term inflation and growth in the EU and that EIOPA is wrong to compare it against long-term spot rates" and "cutting the UFR would force insurers to buy more long-dated assets, forcing down rates at the long end and setting in motion a pro-cyclical feedback loop of lowering rates and lowering UFR resets" (Mannix, 2016). Whereas, Insurance Europe favored the current 4.2% until a complete review of Solvency II is delivered.

On the other side of the debate, stand Danish and Dutch insurance companies as well as policy makers, who clearly support a lower UFR. They warn against an artificially high discount factor "with potentially undesired effects on different generations" (Investment and Pensions Europe, 2016) that may support future costs of current evaluation errors. Moreover, as Chrisou E., (2016), claims "lower discount curve driven by the UFR cut may result in higher cash generation for the industry".

However, the mere reduction of the level of the UFR may not emerge as a sufficient manoeuvre. In 2013, the DNB (De Nederlandische Bank) has proposed a moving average of the 20 year forward rates during the previous 10 years to be used as an asymptotic rate for Dutch pension funds. In July 2015, the UFR produced by the DNB was 3.3%. However, market operators have required further clarification on this result, since the current observable 20 years forward rate was 1.8% (Preesman, 2015).

A far more reasonable option has been put forward by De Kort J. and Vellekoop M.H. (2016). Their proposal involves a moving UFR and relies on setting the UFR as a second target to be optimized using the solution given by Smith A. and Wilson T. Through this procedure, the asymptotic forward rate will reflect information given by market observed data. This alternative version of the Smith-Wilson method requires to abandon the assumption of a fixed UFR by restructuring the optimization problem such that the new problem would lead to an open UFR parameter directly produced by the resolutive algorithm. By letting the UFR behave as a moving variable defined by a model that derives its outcome using market data, it will definitely generate a larger and desirable degree of market consistency in the extrapolation part of the term structure.

Although, this approach may lead the UFR methodology in the right direction it will

necessarily have to integrate with the other crucial principle claimed by the current legislation: the steadiness of the interest rate curve. Hence, the market's fluctuation must be limited when designing a term structure of interest rates in order to avoid introducing unnecessary volatility in the balance sheet. Thus, ensuring stability in the long-term liability market.
In the next chapter we will characterize this modified version of EIOPA's applied method using De Kort J. and Vellekoop M.H. initial design and examining its results. Moreover, a second problem, also implemented by Kort J. and Vellekoop M.H. will be introduced. This additional model is a more complex version of the Smith-Wilson method which, not only relies on an open asymptotic rate, but has the supplementary advantage of producing a smoother term structure in its interpolation section.

Chapter 4

Alternative Methods and Alternative Versions of the Smith-Wilson Method

This chapter presents a study of several possible solutions that may reduce the points of critique arose from the discussions provided in the previous chapter. We will research for the best possible model to rely upon when seeking to construct a term structure that still falls under Solvency II's requirements. At first, we will illustrate a marginally modified Smith-Wilson method that will produce an optimized open asymptotic forward rate. Then, we will move towards a slightly more complex version of the same method. To finally conclude the chapter, with the most notable term structure methodology: The Nelson, Siegel and Svensson model.

4.1 The Smith Wilson Model with Moving Ultimate Forward Rate

The fixed Ultimate Forward Rate has been a subject of debate in the long term liability management environment. This led EIOPA to publish at the beginning of 2016, a public consultation paper on the current inter - and extrapolation method.
The initial proposals have submitted a lower level of the UFR. But still, maintaining the asymptotic rate at a constant level.
A more satisfying suggestion would consider the long term equilibrium rate as the result of a newly defined optimization problem where the UFR is not given a priori. By permitting the long term rate to act as an open parameter, the restructured algorithm chooses a value

for the UFR that will lead to the highest possible level of smoothness in the interpolation section of the interest rates curve. Redefining the problem in these terms, will prevent the non-consistency with market information of the long term equilibrium rate and will increase the level of smoothness around the last liquid point related to each selected financial instruments.

4.1.1 The Optimization Problem

[1] We have previously constructed the function $\mathscr{L}_\alpha$ that allowed the definition of a smoothness criterion for the curve between liquid market data. There is no need to modify this function. But, what is to be altered is the space over which the two solution are to be minimized. The new algorithm will require an additional minimzation of $\mathscr{L}_\alpha$ for the free parameter f_∞. The f_∞ parameter represents the Ultimate Forward Rate in terms of intensity yield. If this is the case, the UFR can be easily expressed in terms of f_∞, as below:

$$UFR = e^{f_\infty} - 1. \tag{4.1}$$

The smoothness optimization problem when searching for an additional minimization of f_∞, and for a given α, can be redefined as:

$$\min_{f_\infty} \quad \min_{g \in \mathscr{H}(f_\infty)} \quad \mathscr{L}_\alpha[g] \tag{4.2}$$

In such a problem, we seek for the optimum g function on a space that depends on a free parameter optimized in the previous step. The space $\mathscr{H}(f_\infty)$ has the following structure:

$$\begin{aligned} \mathscr{H}(f_\infty) \quad = \quad & \Big\{ g \in \mathcal{C}^2(\mathbb{R}^+) \quad | \quad g \in \mathcal{F}_0, \\ & \sum_{j=1}^{M} \bar{x}_{ij} g(t_j) = \bar{m}_i \quad \textit{for all} \quad i = 1, 2, \ldots, N \Big\} \end{aligned} \tag{4.3}$$

Where $\bar{x}_{ij}$ is a function of the cash flow generated by each i-th instrument at time j, in line with 2.28, and $\bar{m}_i$ is related to the i-th prices of the market price, as in equation 2.29. To solve the optimization algorithm, 4.2, we need to provide a matrix containing Heart of Wilson functions applied to different periods of time:

$$\boldsymbol{H}_{ij} = \begin{bmatrix} H(t_1,t_1), H(t_1,t_2), \ldots, H(t_1,t_i) \ldots, H(t_1,t_M) \\ H(t_2,t_1), H(t_2,t_2), \ldots, H(t_2,t_i) \ldots, H(t_2,t_M) \\ \vdots \\ H(t_i,t_1), H(t_i,t_2), \ldots, H(t_i,t_j) \ldots, H(t_i,t_M) \\ \vdots \\ H(t_M,t_1), H(t_N,t_2), \ldots, H(t_M,t_i) \ldots, H(t_M,t_M) \end{bmatrix}, \quad \text{MxM matrix}, \tag{4.4}$$

[1]Way of reasoning: **Term structure extrapolation and asymptotic forward rates**

where each element, $H(t_i, t_j)$, is calculated through 2.54. Additionally, to solve the final equation we will require to produce two diagonal matrixes:

$$\boldsymbol{D}_{ij}^{UFR} = \begin{bmatrix} e^{-UFR*t_1} & 0 & 0 & \dots & 0 \\ 0 & e^{-UFR*t_2} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & e^{-UFR*t_{j-1}} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & e^{-UFR*t_M} \end{bmatrix}, \text{MxM matrix} \tag{4.5}$$

and

$$\boldsymbol{U}_{ij} = \begin{bmatrix} t_1 & 0 & 0 & \dots & 0 \\ 0 & t_2 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & t_{j-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & t_M \end{bmatrix}, \text{MxM matrix.} \tag{4.6}$$

The matrix $\boldsymbol{D}_{ij}^{UFR}$ will be of crucial importance, since only this matrix contains the free parameter UFR over which the minimization algorithm must vary.
Moreover, this $\boldsymbol{D}_{ij}^{UFR}$ is related to the matrix that can be obtained through the redefinition of each cash flow produced by the N available financial instruments. The redefined cash flow matrix is, therefore:

$$\bar{\boldsymbol{X}}_{ij} = \begin{bmatrix} x_{11}e^{-UFR*t_1} & x_{12}e^{-UFR*t_2} & \dots & x_{1m}e^{-UFR*t_m} & \dots & x_{1M}e^{-UFR*t_M} \\ x_{21}e^{-UFR*t_1} & x_{22}e^{-UFR*t_2} & \dots & x_{2m}e^{-UFR*t_m} & \dots & x_{2M}e^{-UFR*t_M} \\ & & & \vdots & & \\ x_{i1}e^{-UFR*t_1} & x_{i2}e^{-UFR*t_2} & \dots & x_{im}e^{-UFR*t_m} & \dots & x_{iM}e^{-UFR*t_M} \\ & & & \vdots & & \\ x_{N1}e^{-UFR*t_1} & x_{N2}e^{-UFR*t_2} & \dots & x_{Nm}e^{-UFR*t_m} & \dots & x_{NM}e^{-UFR*t_M} \end{bmatrix}. \tag{4.7}$$

This NxM matrix is the simple result of the product:

$$\bar{\boldsymbol{X}}_{ij} = \boldsymbol{X}_{ij}\boldsymbol{D}_{ij}^{UFR}. \tag{4.8}$$

This matrix enables the determination of a new parmeter η, that is to be calculated as:

$$\boldsymbol{\eta} = \bar{\boldsymbol{X}}'(\bar{\boldsymbol{X}}\boldsymbol{H}\bar{\boldsymbol{X}}')^{-1}\bar{\boldsymbol{m}} = \bar{\boldsymbol{X}}'\boldsymbol{\zeta}. \tag{4.9}$$

Where $\bar{\boldsymbol{m}}$ is the column vector which contains the N redefined prices as in equation 2.29:

$$\bar{\boldsymbol{m}} = \begin{bmatrix} m_1 - \sum_{j=1}^{M} \bar{x}_{1j} \\ m_2 - \sum_{j=1}^{M} \bar{x}_{2j} \\ \vdots \\ m_{i-1} - \sum_{j=1}^{M} \bar{x}_{i-1,j} \\ \vdots \\ m_N - \sum_{j=1}^{M} \bar{x}_{Nj} \end{bmatrix}. \tag{4.10}$$

For a square invertible matrix $\bar{\boldsymbol{X}}$, a typical case would be the cash flow matrix generated by Zero Coupon Bonds, the $\boldsymbol{\eta}$ simplifies to:

$$\boldsymbol{\eta} = \boldsymbol{H}^{-1}\bar{\boldsymbol{X}}^{-1}\bar{\boldsymbol{m}}. \tag{4.11}$$

Hence, given the previous definition, $\boldsymbol{\eta}$ will have the structure of a column vector with M elements:

$$\boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{j-1} \\ \vdots \\ \eta_M \end{bmatrix}. \tag{4.12}$$

When approaching the original problem where the UFR is set as a fixed parameter, the interpolating curve g for a generic period of time t can be reinterpreted through the M ηs as below:

$$g(t) = \sum_{j=1}^{M} \eta_j H(t, t_j) \tag{4.13}$$

and thus, the pricing function for a given period of time t, can be written in terms of η:

$$v(0,t) = \left(1 + \sum_{j=1}^{M} \eta_j H(t, t_j)\right) e^{-UFRt}. \tag{4.14}$$

Finally, before moving to the solution of the problem, we need to define a column vector of ones, $\boldsymbol{e}$, in the space $\mathbb{R}^M$:

$$\boldsymbol{e} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{4.15}$$

Observing that $\bar{\boldsymbol{m}}$ can be easily expressed in terms of $\boldsymbol{e}$ and $\boldsymbol{D}_{ij}^{UFR}$, through:

$$\bar{\boldsymbol{m}} = \boldsymbol{m} - \boldsymbol{X}\boldsymbol{D}^{UFR}\boldsymbol{e}, \tag{4.16}$$

we can proceed with the solution.

4.1.2 The Solution

[2] De Kort J. and Vellekoop M.H. (2016) proved the existence of a unique and real solution to problem 4.2. If this is the case, the intensity UFR is interpreted as the asymptotic free parameter f_∞.
Given the definition of the smoothness regulator $\mathscr{L}_\alpha$ as shown in equation 2.21, for any $g \in \mathcal{F}_0 \supseteq \mathscr{H}(f_\infty)$, $\mathscr{L}_\alpha$ can be mathematically reformulated as:

$$\mathscr{L}_\alpha[g] = \int_0^\infty [g'(u)(-g'''(u) + \alpha^2 g'(u))]du \tag{4.17}$$

Reinterpreting this result through the η_js:

$$\begin{aligned}\mathscr{L}_\alpha[g] &= 0.5\Big(\alpha^3 \sum_{j=1}^{M} \eta_j \int_0^{t_j} g'(u)du\Big) = \\ &= 0.5\Big(\alpha^3 \sum_{j=1}^{M} \eta_j \sum_{i=1}^{M} \eta_i \boldsymbol{H}_{ji}\Big)\end{aligned} \tag{4.18}$$

When expressing all elements of the previous equation in terms of vectors and matrices, this will produce:

$$\begin{aligned}\mathscr{L}_\alpha[g] &= 0.5\Big(\alpha^3 \bar{\boldsymbol{m}}'(\bar{\boldsymbol{X}}\boldsymbol{H}\bar{\boldsymbol{X}}')^{-1}(\bar{\boldsymbol{X}}\boldsymbol{H}\bar{\boldsymbol{X}}')(\bar{\boldsymbol{X}}\boldsymbol{H}\bar{\boldsymbol{X}}')^{-1}\bar{\boldsymbol{m}}\Big) \\ &= 0.5\Big(\alpha^3 \bar{\boldsymbol{m}}'(\bar{\boldsymbol{X}}\boldsymbol{H}\bar{\boldsymbol{X}}')^{-1}\bar{\boldsymbol{m}}\Big)\end{aligned} \tag{4.19}$$

To simplify the notation, matrix $\boldsymbol{\Lambda}$ is introduced:

$$\boldsymbol{\Lambda} = (\bar{\boldsymbol{X}}\boldsymbol{H}\bar{\boldsymbol{X}}')^{-1} \tag{4.20}$$

The next step is deriving 4.19 respect to f_∞, and as such, obtaining:

$$\frac{\partial \mathscr{L}_\alpha[g]}{\partial f_\infty} = -2\bar{\boldsymbol{m}}'\boldsymbol{\Lambda}^{-1}\boldsymbol{X}\boldsymbol{U}(\boldsymbol{e} + \boldsymbol{H}\boldsymbol{X}'\boldsymbol{\Lambda}^{-1}\bar{\boldsymbol{m}}) \tag{4.21}$$

The final step will be substituting in the previous formula the expressions 4.8, 4.16 and 4.20 and equaling to 0:

$$\begin{aligned}&(\boldsymbol{m} - \boldsymbol{X}\boldsymbol{D}^{UFR}\boldsymbol{e})'(\boldsymbol{X}\boldsymbol{D}^{UFR}\boldsymbol{H}\boldsymbol{D}^{UFR}\boldsymbol{X}')^{-1}\boldsymbol{X}\boldsymbol{D}^{UFR}\boldsymbol{U} \cdot (\boldsymbol{e}+ \\ &+ \boldsymbol{H}\boldsymbol{D}^{UFR}\boldsymbol{X}'(\boldsymbol{X}\boldsymbol{D}^{UFR}\boldsymbol{H}\boldsymbol{D}^{UFR}\boldsymbol{X}')^{-1}(\boldsymbol{m} - \boldsymbol{X}\boldsymbol{D}^{UFR}\boldsymbol{e})) = 0\end{aligned} \tag{4.22}$$

[2] As discussed in: **Term structure extrapolation and asymptotic forward rates**

The optimum asymptotic forward rate f_∞ will be the root of this formula. According to the previous definitions, when multiplying all the matrices and vectors, the equation above will reduce itself into a scalar. And as such, easing the computations. Therefore, calculating the minimizer f_∞, will result a simple procedure of numerical optimization.
A further reduction of the algorithm can be implemented when the cash flow matrix $\boldsymbol{X}$ is invertible. By introducing the vector $\boldsymbol{\phi}$ that is to be calculated through:

$$\boldsymbol{\phi} = \boldsymbol{X}^{-1}\boldsymbol{m}. \tag{4.23}$$

This will lighten the computation since the root can be easily found solving:

$$(\boldsymbol{D}^{UFR}\boldsymbol{\phi} - \boldsymbol{e})'\boldsymbol{H}^{-1}\boldsymbol{U}(\boldsymbol{D}^{UFR}\boldsymbol{\phi}) = 0. \tag{4.24}$$

When renouncing to the matrix and vector notation, we can write this last equation as:

$$\sum_{j=1}^{M}\sum_{i=1}^{M}(t_j\phi_j e^{UFRt_j})\boldsymbol{H}_{ij}(\phi_j e^{UFRt_j} - 1) = 0. \tag{4.25}$$

Consequently, when the cash flow matrix is invertible the numerical algorithm is reduced to an even simpler form. Considering that only $\boldsymbol{D}^{UFR}$ depends on the parameter that is to be rooted, there is no significant evidence of numerical complications in the optimization process.
With this simple solution found by De Kort J. and Vellekoop M.H. (2016), we have overcome what was one of the most critical points of the Smith-Wilson function. We have introduced market consistency in the term structure model when setting the ultimate equilibrium rate without abandoning a certain degree of smoothness and still providing interpolation between all market data.

4.2 A Modified Model

To overcome problems related to negative forward rates, negative prices of the Zero Coupon Bond and primarly to increase the model's smoothness in the market part of the curve, De Kort J and Vellekoop M.H. (2016) have proposed a different optimization approach to the problem solved by Smith A. and Wilson T.
By challenging the criterion of smoothness imposed by the Smith-Wilson method on the first and second derivatives of the pricing function, De Kort J and Vellekoop M.H. prefer to focus on the smoothness, thus, on the differentiability, of the yield to maturity function and of the instantaneous forward rate function. Provided that:

$$v(0,t) = e^{-\int_0^t \delta(o,u)du} = e^{-h(0,t)t}. \tag{4.26}$$

This implies that focusing on the interest rate curve rather than on the pricig function curve will avoid negative values of the Zero Coupon Bond pricing function without decreasing differentiability in the present value.
This criterion will result in a modified structure of the Heart of Wilson function. As shown previously, the Heart of Wilson function was originally designed as an exponential spline function which should regulate the interpolation part of the interest rate curve.

4.2.1 The Optimization problem

[3] To obtain the desired degree of smoothness in the forward intensity and yield to maturity term structure, it is required to re-design the minimization and converging problem. We can interpret the new problem either as the minimization of the interpolation regulator $g(t)$ for the yield to maturity curve or we can minimize $g(t)$ to obtain the smoothest forward intensity rate curve. This, mathematically translates to:

$$\min_{g \in \mathscr{H}^{\delta}} \mathscr{L}_{\bar{\alpha}}[g] \tag{4.27}$$

and

$$\min_{g \in \mathscr{H}^{h}} \mathscr{L}_{\alpha}[g]. \tag{4.28}$$

The spaces where the function $g(t)$ is to be searched for, are defined as:

$$\begin{aligned} \mathscr{H}^{\delta} \quad = \quad & \Bigg\{ g \in \mathcal{C}^2(\mathbb{R}^+) \quad | \quad g \in \mathcal{F}_a, \\ & \sum_{j=1}^{M} x_{ij} e^{-\int_0^{t_j} g(u)du} = m_i \quad \textit{for all} \quad i = 1, 2, \ldots, N \Bigg\} \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} \mathscr{H}^{h} \quad = \quad & \Bigg\{ g \in \mathcal{C}^2(\mathbb{R}^+) \quad | \quad g \in \mathcal{F}_a, \\ & \sum_{j=1}^{M} x_{ij} e^{-g(t_j)t_j} = m_i \quad \textit{for all} \quad i = 1, 2, \ldots, N \Bigg\}. \end{aligned} \tag{4.30}$$

An innate assumption is necessary for the development of a resolutive algorithm: the initial value of the yield to maturity, $h(0)$ must be given, provided that $g(0) = a$.
Furthermore, when minimizing $\mathscr{L}_{\bar{\alpha}}$, we will need a newly defined velocity of convergence regulator given by $\bar{\alpha}$. Its value will be determined by the same algorithm implemented to obtain α.

[3] Way of reasoning: **Term structure extrapolation and asymptotic forward rates**

4.2.2 A New Class of Interpolating Functions

[4] The earlier defined problem leads to an explicit result that depends on the Heart of Wilson function and a modified version of these interpolating functions, $\bar{H}(t,t_j)$:

$$\bar{H}(t,t_j) = \begin{cases} 1 - e^{-\bar{\alpha}t}\Big(\dfrac{cosh(\bar{\alpha}t_j)-1}{0.5\bar{\alpha}^2 t_j^2}\Big) + \\ \quad + \dfrac{cosh(\bar{\alpha}(t_j-t)) - 1 - 0.5\bar{\alpha}^2(t_j-t)^2}{0.5\bar{\alpha}^2 t_j^2}, & t \leqslant t_j, \\ \\ 1 - e^{-\bar{\alpha}t}\Big(\frac{cosh(\bar{\alpha}t_j)-1}{0.5\bar{\alpha}^2 t_j^2}\Big), & t > t_j. \end{cases} \tag{4.31}$$

With the first order derivative given by:

$$\frac{\partial \bar{H}(t,t_j)}{\partial t} = \bar{G}(t,t_j) =) = \begin{cases} \bar{\alpha}e^{-\bar{\alpha}t}\Big(\dfrac{cosh(\bar{\alpha}t_j)-1}{0.5\bar{\alpha}^2 t_j^2}\Big) + \\ \quad + \dfrac{-\bar{\alpha}sinh(\bar{\alpha}(t_j-t)) + \bar{\alpha}^2(t_j-t)}{0.5\bar{\alpha}^2 t_j^2}, & t \leqslant t_j, \\ \\ \bar{\alpha}e^{-\bar{\alpha}t}\Big(\frac{cosh(\bar{\alpha}t_j)-1}{0.5\bar{\alpha}^2 t_j^2}\Big), & t > t_j. \end{cases} \tag{4.32}$$

This newly defined version of the exponential spline function can be interpreted as an affine combination of integrals of Heart of Wilson functions, thus producing a higher rate of smoothness in the market section of the term structure.
An additional feature, also related to smoothness, of this class of functions lies in its differentiability: $\bar{H}(t,t_j)$ is in fact three times differentiable for all $t_j > 0$. Further properties of the function can be derived through differentiation:

$$\partial_1^2 \bar{H}(0,t_j) = \lim_{t\to\infty} \partial_1^2 \bar{H}(t,t_j) = 0, \tag{4.33}$$

$$\lim_{t\to\infty} \bar{H}(t,t_j) = 1, \tag{4.34}$$

$$\bar{H}(0,t_j) = 0. \tag{4.35}$$

This leads to the conclusion that the function $\bar{H}(t,t_j)$ becomes linear for very small and positive values of t_j.
Whereas, we obtain the following convergence for $\bar{H}(t,t_j)$:

$$\lim_{t\to\infty} \bar{H}(t,t_j) = 1, \quad \textit{for all} \quad t > t_j. \tag{4.36}$$

Finally, it can be demonstrated through the third derivative that:

$$\partial_1^3 \bar{H}(t,t_j) + \bar{\alpha}^2 \partial_1 \bar{H}(t,t_j) = \frac{2\bar{\alpha}^2}{t_j^2}(t_j - t), \quad \textit{if} \quad t < t_j. \tag{4.37}$$

[4] As reasoned in: **Term structure extrapolation and asymptotic forward rates**

The graph below shows a comparison, in terms of smoothness, between the Heart of Wilson and the new class of interpolating function. The orange section is dedicated to the Heart of Wilson's structure. These curves are computed employing a scaled version of the Heart of Wilson. The scaling is provided by:

$$\frac{-H(t,t_j)}{\alpha t_j}.$$

On the left side, the shapes are obtained with $\alpha = 0.5$ while on the right side, the curves originate from a value of α set to 0.1. As for the green figures on the upper side of the graph, they illustrate the new class of splines computed, once again, through the previously mentioned values of the velocity of convergence parameters.

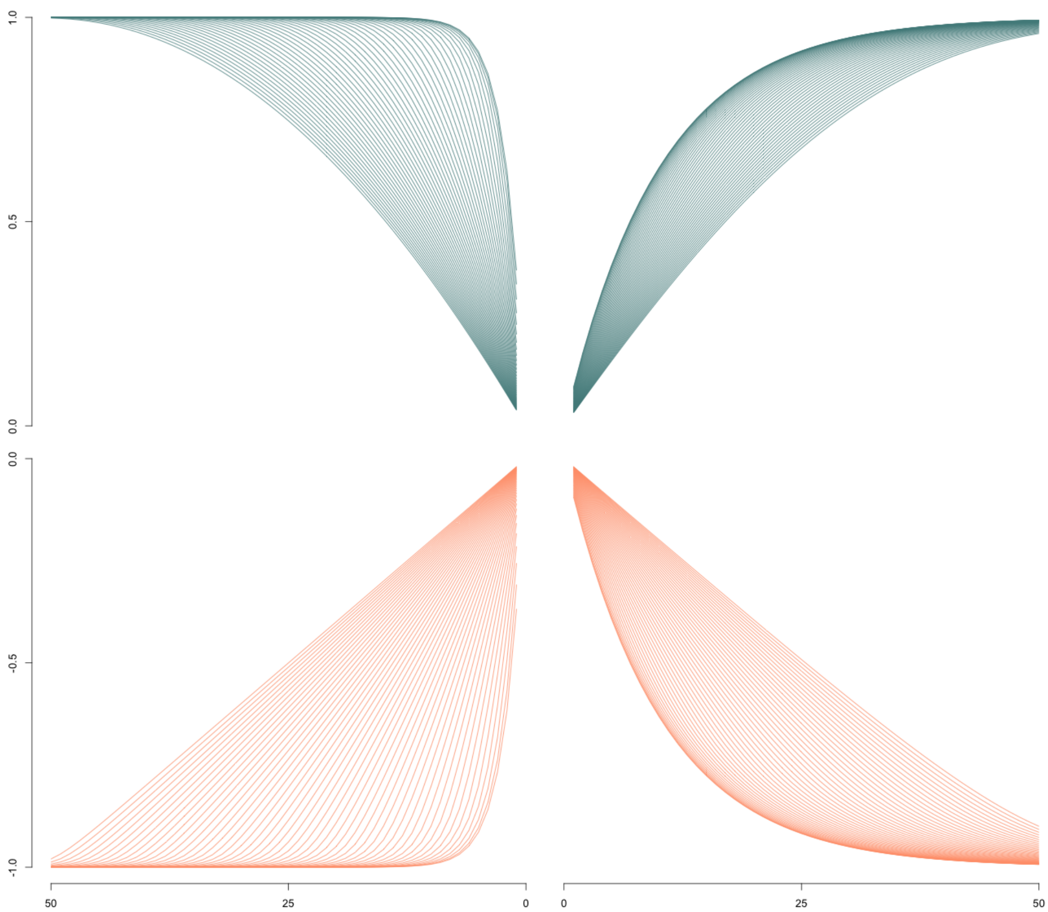

Figure 4.1: Orange lines correspond to the Heart of Wilson and olive lines to the new set of interpolating functions. On the left, the figure illustrates curves computed with $\alpha = 0.5$, while on the right $\alpha = 0.1$ is used.

Given the definition of these freshly construed spline functions, we can now move forward and analyse the result of the optimization problem.

4.2.3 The Solution

De Kort J. and Vellekoop M.H. proved that the minimizers $g^{\delta}(T)$ and $g^{r}(t)$ of the two aforementioned optimization problems, if they exist, must have the form of:

$$g^{\delta}(t) = g(0) + \sum_{i=1}^{N} \zeta_i^{\delta} \sum_{j=1}^{M} \phi_j^{\delta} x_{ij} t_j^2 \bar{H}(t, t_j) \tag{4.38}$$

and

$$g^{h}(t) = g(0) + \sum_{i=1}^{N} \zeta_i^{h} \sum_{j=1}^{M} \phi_j^{h} x_{ij} t_j H(t, t_j). \tag{4.39}$$

To proceed with the computations required, we need to define the structure of the two ϕ parameters and that of the two ζs. While for the function $\bar{H}(t, t_j)$, the considerations above are still legit.
In terms of the instantaneous forward rate function, ϕ_j^h and ζ_i^h are computed by rooting the following equations:

$$m_i = \sum_{j=1}^{M} x_{ij} \phi_j^{\delta}, \quad \textit{for all} \quad i = 1, 2, \ldots, N, \tag{4.40}$$

while

$$\begin{aligned} -\frac{log(\phi_k^{\delta})}{t_k} =& g(0) + \sum_{i=1}^{N} \zeta_i^{\delta} \sum_{j=1}^{M} \phi_j^{\delta} x_{ij} t_j^2 \frac{1}{t_k} \cdot \\ & \cdot \int_0^{t_k} \bar{H}(u, t_j) du, \quad \text{for all} \quad k = 1, 2..., M. \end{aligned} \tag{4.41}$$

Where the k-th integral $\int_0^{t_k} \bar{H}(u, t_j) du$ can be easily determined through:

$$\begin{aligned} \int_0^{t_k} \bar{H}(u, t_j) du = t_k + \frac{e^{-\bar{\alpha} t_k}}{\bar{\alpha}} \left[\frac{cosh(\bar{\alpha} t_j)}{0.5 \bar{\alpha}^2 t_j^2} \right] - \frac{1}{\bar{\alpha}} \left[\frac{cosh(\bar{\alpha} t_j)}{0.5 \bar{\alpha}^2 t_j^2} \right] + \\ + \mathbb{1}_{\{t_k \leq t_j\}} \left\{ \frac{\frac{-sinh[\bar{\alpha}(t_j - t_k)]}{\bar{\alpha}} - t_k + \frac{1}{6} \bar{\alpha}^2 (t_j - t_k)^3}{0.5 \bar{\alpha}^2 t_j^2} - \frac{\frac{-sinh[\bar{\alpha}(t_j)]}{\bar{\alpha}} + \frac{1}{6} \bar{\alpha}^2 t_j^3}{0.5 \bar{\alpha}^2 t_j^2} \right\}. \end{aligned} \tag{4.42}$$

When reasoning in terms of the yield to maturity curve, the parameters ζ_i^r and ϕ_j^r are to be rooted from:

$$m_i = \sum_{j=1}^{M} x_{ij} \phi_j^{h}, \quad \textit{for all} \quad i = 1, 2, \ldots, N \tag{4.43}$$

and

$$\begin{aligned} -\frac{log(\phi_k^{h})}{t_k} =& g(0) + \sum_{i=1}^{N} \zeta_i^{h} \sum_{j=1}^{M} \phi_j^{h} x_{ij} t_j H(u, t_j) \\ & \textit{for all} \quad k = 1, 2..., M. \end{aligned} \tag{4.44}$$

Substituting the interpolating function $g^{\delta}(t)$ or $g^{r}(t)$ in the pricing function 2.27, it is now possible to create the term structure, if the Ultimate Forward Rate is a pre-defined parameter.

4.2.4 The Ultimate Forward Rate

When the asymptotic forward rate is not given a free parameter it can be easily computed for both interpolating functions $g^{\delta}(t)$ and $g^{h}(t)$.
When dealing with the yield to maturity optimization function, we notice that:

$$\lim_{t\to\infty} H(t,t_j) = \alpha t_j, \quad \textit{for all} \quad t_j > 0. \tag{4.45}$$

This leads to the conclusion that the ultimate forward rate can be easily calculated as:

$$UFR^{h} = \lim_{t\to\infty} g^{h}(t) = g(0) + \sum_{i=1}^{N} \zeta_i^{h} \sum_{j=1}^{M} \phi_j^{h} x_{ij} \alpha t_j^2. \tag{4.46}$$

On the other hand, when reasoning in terms of the forward intensity rate, the function we obtain provides the following definition of the asymptotic forward rate:

$$UFR^{\delta} = \lim_{t\to\infty} g^{\delta}(t) = g(0) + \sum_{i=1}^{N} \zeta_i^{r} \sum_{j=1}^{M} \phi_j^{r} x_{ij} t_j^2. \tag{4.47}$$

This result is a consequence of 4.36.
The notation used in the two equations of the Ultimate Forward Rates can be eased through their reinterpretation as a linear combination of yields to maturity with different tenors. To proceed further, we require the definition of the yield to maturity between periods $\{0, t_s\}$ as below:

$$h_s = h(0,t_s) = \frac{-log[v(0,t_s)]}{t_s}, \tag{4.48}$$

for all $t_s > 0$.
While at time 0, we get the following relationship:

$$h_0 = h(0,0) = \delta(0), \tag{4.49}$$

Using the above definitions, the optimized Ultimate Forward Rate for both methodologies can be easily computed as a linear combination of n spot rates. As a matter of fact, it can be proven that the Ultimate Forward Rate can also be computed through:

$$UFR^{\delta} = g(0) + \sum_{j=1}^{N} \sum_{s=1}^{N} [\omega^{\delta}]_{js}^{-1} (h_s - g(0)), \tag{4.50}$$

in terms of the forward intensity function, and:

$$UFR^h = g(0) + \sum_{j=1}^{N} \sum_{s=1}^{N} [\omega^h]^{-1}_{js} (h_s - g(0)), \tag{4.51}$$

in terms of the yield to maturity function.
In the equations above, the functions ω^δ_{js} and ω^h_{js} are given by:

$$\omega^\delta_{sj} = \frac{1}{t_s} \int_0^{t_s} \bar{H}(u, t_j) du \tag{4.52}$$

and

$$\omega^h_{sj} = \frac{1}{\alpha t_j} H(t_s, t_j). \tag{4.53}$$

When rearranging the terms, the notation simplifies to:

$$UFR = \sum_{s=0}^{N} w_s h_s, \tag{4.54}$$

where $\{w_s, s = 1, 2, \ldots, N\}$ are the weights of the linear combination.
Each s-th weight, w_s, is the result of:

$$w_s = \sum_{j=0}^{N} [\omega]^{-1}_{js}, \quad \textit{for all} \quad s > 0. \tag{4.55}$$

While for w_0 we have:

$$w_0 = 1 - \sum_{s=0}^{N} \omega_s, \tag{4.56}$$

where the matrix $\boldsymbol{\omega}$ can be easily built either using elements defined by ω^δ_{js}:

$$\boldsymbol{\omega} = \boldsymbol{\omega}^\delta = \begin{bmatrix} \frac{1}{t_1} \int_0^{t_1} \bar{H}(u, t_1) du, \frac{1}{t_1} \int_0^{t_1} \bar{H}(u, t_2) du, \ldots, \frac{1}{t_1} \int_0^{t_1} \bar{H}(u, t_N) du \\ \frac{1}{t_2} \int_0^{t_2} \bar{H}(u, t_1) du, \frac{1}{t_2} \int_0^{t_2} \bar{H}(u, t_2) du, \ldots, \frac{1}{t_2} \int_0^{t_2} \bar{H}(u, t_N) du \\ \vdots \\ \frac{1}{t_s} \int_0^{t_s} \bar{H}(u, t_1) du, \ldots, \frac{1}{t_s} \int_0^{t_s} \bar{H}(u, t_j) du, \ldots, \frac{1}{t_s} \int_0^{t_s} \bar{H}(u, t_N) du \\ \vdots \\ \frac{1}{t_N} \int_0^{t_N} \bar{H}(u, t_1) du, \frac{1}{t_N} \int_0^{t_N} \bar{H}(u, t_2) du, \ldots, \frac{1}{t_N} \int_0^{t_N} \bar{H}(u, t_N) du \end{bmatrix}, \tag{4.57}$$

or by using the resulting components from the application of ω^h_{sj}:

$$\boldsymbol{\omega} = \boldsymbol{\omega}^h = \begin{bmatrix} \frac{1}{\alpha t_1} H(t_1, t_1), \frac{1}{\alpha t_2} H(t_1, t_2), \ldots, \frac{1}{\alpha t_N} H(t_1, t_N) \\ \frac{1}{\alpha t_1} H(t_2, t_1), \frac{1}{\alpha t_2} H(t_2, t_2), \ldots, \frac{1}{\alpha t_N} H(t_2, t_N) \\ \vdots \\ \frac{1}{\alpha t_1} H(t_s, t_1), \ldots, \frac{1}{\alpha t_j} H(t_s, t_j), \ldots, \frac{1}{\alpha t_N} H(t_s, t_N) \\ \vdots \\ \frac{1}{\alpha t_1} H(t_N, t_1), \frac{1}{\alpha t_2} H(t_N, t_2), \ldots, \frac{1}{\alpha t_N} H(t_N, t_N) \end{bmatrix}. \tag{4.58}$$

When the cash flow matrix $\boldsymbol{X}$ is invertible, we can compute the yield to maturity function much more easily, by employing:

$$h_s = -\frac{log\phi_s}{t_s}, \tag{4.59}$$

where ϕ_s is the s-th element in the $\boldsymbol{\phi}$ vector:

$$\boldsymbol{\phi} = \boldsymbol{X}^{-1}\boldsymbol{m} \tag{4.60}$$

By reason of this result, it is possible to calculate the UFR with both definition as a simple linear combination of available market data and the weights w_s. In such a case, the weights can be easily determined through the available maturities t_i, for $i = 1, 2, \dots, N$.[5]

4.2.5 The Short Rate

As previously mentioned, the yield to maturity at time 0, h_0 can be provided as an input variable. When this is not the case, the yield to maturity at time 0 can be determined through the optimization of a particular function dependent on h_0. Hence, we should search on the space of h_0 for that specific value that produces the smoothest curve in the short term.
If the cash flow matrix is invertible, thus $M = N$, and the yield to maturity at time 0, which produces the smoothest curve is unknown, we can easily calculate it through:

$$h_0 = \frac{\sum_{j=1}^{n} \frac{1}{t_j} \sum_{s=1}^{n} \boldsymbol{\omega}_{js}^{-1} \frac{r_{t_j} + r_{t_s}}{2}}{\sum_{j=1}^{n} \frac{1}{t_j} \sum_{s=1}^{n} \boldsymbol{\omega}_{js}^{-1}}, \tag{4.61}$$

[6] where the matrix $\boldsymbol{\omega}$ is provided either by 4.57 or 4.58.
Once determined this final element through the equation above or simply by setting it as an a priori constant, we have all the ingredients required to construct the term structure in accordance with a model that relies upon the yield to maturity function or the forward intensity function to produce the desired smoothness. Such a structure, in the particular case of the forward intensity function, also depends on a newly designed smoother class of interpolating splines.

4.3 The Nelson, Siegel and Svensson Model

Nelson C.R and Siegel A. F. (1987)[7] introduced a parsimonious and simple model to shape the term structure of interest rate by using three primary components: slope, curvature

[5] For proof of the results see: **Term structure extrapolation and asymptotic forward rates**
[6] This result has been proven in: **Term structure extrapolation and asymptotic forward rates**
[7] see: **Parsimonious Modeling of Yield Curves**

and level. This model allows mispricing of market data to achieve smoother curves in return. Moreover, with a simple modification of the method consisting in an additional term introduced by Svensson (1994)[8], the Nelson-Siegel method has become the most credited zero coupon yield curve methodology. As a matter of fact, it is largely used by central banks around the globe to describe the behavior of yields produced by Zero Coupon Bonds.
We will move through this section, by initially giving an insight into the original Nelson and Siegel method and proceeding afterwards to characterize the alternative studied by Svensson.
In the final section of this chapter, we will provide a comparison between the models generally preferred by central banks and EIOPA's methodology.

4.3.1 The Nelson-Siegel Model

[9] The model introduced by Nelson C.R and Siegel A. F. is a parsimonious methodology. With only four parameters it allows to build a term structure that appropriately describes the behavior over time of zero coupon interest rates, providing a sufficiently high level of smoothness and a good fit to market data. This is the key reason for its popularity among central banks worldwide.
The Nelson-Siegel Model has been created to fit the instantaneous forward rate function, between two different period of time $\{t, T\}$ through a specific functional form. The result was achieved using Hermite polynomials and applying a series expansion in the polynomials space. Normally, this type of solution is linked to differential equations' resolutive algorithms.
With the formula below we can easily express the final result obtained by Nelson and Siegel:

$$\delta(t,T) = \beta_0 + \beta_1 e^{-\frac{\tau}{\tau_1}} + \beta_2 \frac{\tau}{\tau_1} e^{-\frac{\tau}{\tau_1}}. \tag{4.62}$$

Where τ is given by:

$$\tau = T - t. \tag{4.63}$$

Thus, denoting the time to maturity on which the instantaneous forward rate is applied. The implied yield to maturity curve can be easily shaped through the formula below:

$$h(t,T) = \beta_0 + \beta_1 \left(\frac{1 - e^{-\frac{\tau}{\tau_1}}}{\frac{\tau}{\tau_1}} \right) + \beta_2 \left(\frac{1 - e^{-\frac{\tau}{\tau_1}}}{\frac{\tau}{\tau_1}} - e^{-\frac{\tau}{\tau_1}} \right), \tag{4.64}$$

[8]see: **Estimating and Interpreting Forward Interest Rates: Sweden 1992-1994**
[9]Way of reasoning: **Parsimonious Modeling of Yield Curves**

while the implied pricing function curve follows:

$$v(t,T) = exp\Bigg\{ -\beta_0\tau - (\beta_1+\beta_2)\left(\frac{1-e^{-\frac{\tau}{\tau_1}}}{\frac{1}{\tau_1}}\right) + \beta_2\tau e^{-\frac{\tau}{\tau_1}}\Bigg\}. \tag{4.65}$$

Hence, the yield curve can be estimated through a simple parsimonious model composed by three exponential components and with $\eta = \{\beta_0, \beta_1, \beta_2, \tau_1\}$ as the parameter space to be estimated. The methodology upon which we rely to accurately estimate these parameter consists in: implementing an algorithm that minimizes the squared difference between Nelson-Siegel's interest rates, $i_{ns}(t, t+\tau_j, \eta)$, and the observed zero coupon market interest rates, $i(t, t+\tau_j)$:

$$\min_{\eta} = \sum_{j=1}^{N}[i(t, t+\tau_j) - i_{ns}(t, t+\tau_j, \eta)]^2. \tag{4.66}$$

If this is the case, the resulting curve is not the product of an interpolation problem since the market points are loosely incorporated in the term structure.
It is now possible to characterize the parameters that need to be estimated. β_0 has the most intuitive economic interpretation. Observing that:

$$\lim_{\tau\to\infty} \delta(t,T) = \beta_0 \tag{4.67}$$

β_0 it is regularly referred to as the long term level of the forward intensity function.
While, given:

$$\lim_{\tau\to 0} \delta(t,T) = \beta_0 + \beta_1 \tag{4.68}$$

$\beta_0 + \beta 1$ it is interpreted as the short rate.
Furthermore, the parameter τ is the time decaying factor that governs the monotonic and exponential decay assumption made in a Nelson and Siegel formulation.
A dynamic extension of this model has been proposed by Diebold F.X. and Lee C, (2006). They suggest to incorporate a time-varying parameter space $\eta(t) = \{\beta_0(t), \beta_1(t), \beta_2(t), \tau_1(t)\}$. In such a case, it can be proven that $\beta_0(t), \beta_1(t)$ and $\beta_2(t)$ determine respectively the level, slope and curvature of the term structure and are regularly modelled through three different autoregressive models. Many existing studies have shown how this model performs better than others in terms of bond-fitting and prediction-making, e.g. Guo B., Han Q. and Zao B. (2014), Ullah W., Matsuda Y. and Tsukuda Y (2014) and Steeley J.M. (2014).

4.3.2 Svensson's Modification

Svensson L.E. (1994) introduces an alternative version of the model characterized by Nelson and Siegel. With such an alternative model, which imposes a further term to the

formulation, Svensson achieves a better fit for long maturities. Additionally, Svensson's proposal increases the curve's flexibility, and thus generating a far more realistic term structure. This re-modeling is not driven by the series expansion originally implemented by Nelson and Siegel, but by empirical evidence that led to the adoption of a forth term. This additional term is nothing less then a mere reproduction of the last term in a regular Nelson and Siegel model.
Using Svensson's approach, the formulation of the extrapolating curve, legit between times $\{t, T\}$, is to be computed through a redefined imstantaneous forward rate function[10]:

$$\delta(t,T) = \beta_0 + \beta_1 e^{-\frac{\tau}{\tau_1}} + \beta_2 \frac{\tau}{\tau_1} e^{-\frac{\tau}{\tau_1}} + \beta_3 \frac{\tau}{\tau_2} e^{-\frac{\tau}{\tau_2}}. \tag{4.69}$$

By restructuring the Nelson and Siegel model as afore, we added two more parameters, β_3 and τ_2, to the parameter space. The latter being a separate decay regulator. While, β_3 is an additional parameter that rules over the movement of the term structure. Their estimated value is still to be determined through problem 4.66.
The implied yield to maturity curve is thus, given by:

$$\begin{aligned} h(t,T) = \beta_0 + \beta_1 \left(\frac{1 - e^{-\frac{\tau}{\tau_1}}}{\frac{\tau}{\tau_1}} \right) + \beta_2 \left(\frac{1 - e^{-\frac{\tau}{\tau_1}}}{\frac{\tau}{\tau_1}} - e^{-\frac{\tau}{\tau_1}} \right) + \\ + \beta_3 \left(\frac{1 - e^{-\frac{\tau}{\tau_2}}}{\frac{\tau}{\tau_2}} - e^{-\frac{\tau}{\tau_2}} \right). \end{aligned} \tag{4.70}$$

And finally, the present value function is easily obtained through:

$$\begin{aligned} v(t,T) = exp \Bigg\{ - \beta_0 \tau - (\beta_1 + \beta_2) \left(\frac{1 - e^{-\frac{\tau}{\tau_1}}}{\frac{1}{\tau_1}} \right) + \\ + \beta_3 \left(\frac{1 - e^{-\frac{\tau}{\tau_2}}}{\frac{1}{\tau_2}} \right) + \beta_2 \tau e^{-\frac{\tau}{\tau_1}} + \beta_3 \tau e^{-\frac{\tau}{\tau_2}} \Bigg\}. \end{aligned} \tag{4.71}$$

This version of the Nelson-Siegel Model is probably the most notable extrapolation methodology among financial institutions worldwide. Its popularity relies on its parsimonious structure, its flexibility, on a good in-sample fitting of observable market rates and on out-of-sample forecasting of future interest rates. But, the properties of the Nelson, Siegel and Svensson appear less desirable when wishing to include all market points in the yield term structure. This problematic will be discussed in the next section.
In the figure below we have an example of a Nelson, Siegel and Svensson extrapolating curve. The model is applied to data over United States treasury bonds with valuation date 12-16-2016.[11]

[10] Way of reasoning: **Estimating and Interpreting Forward Interest Rates: Sweden 1992-1994**
[11] source: The Wall Street Journal

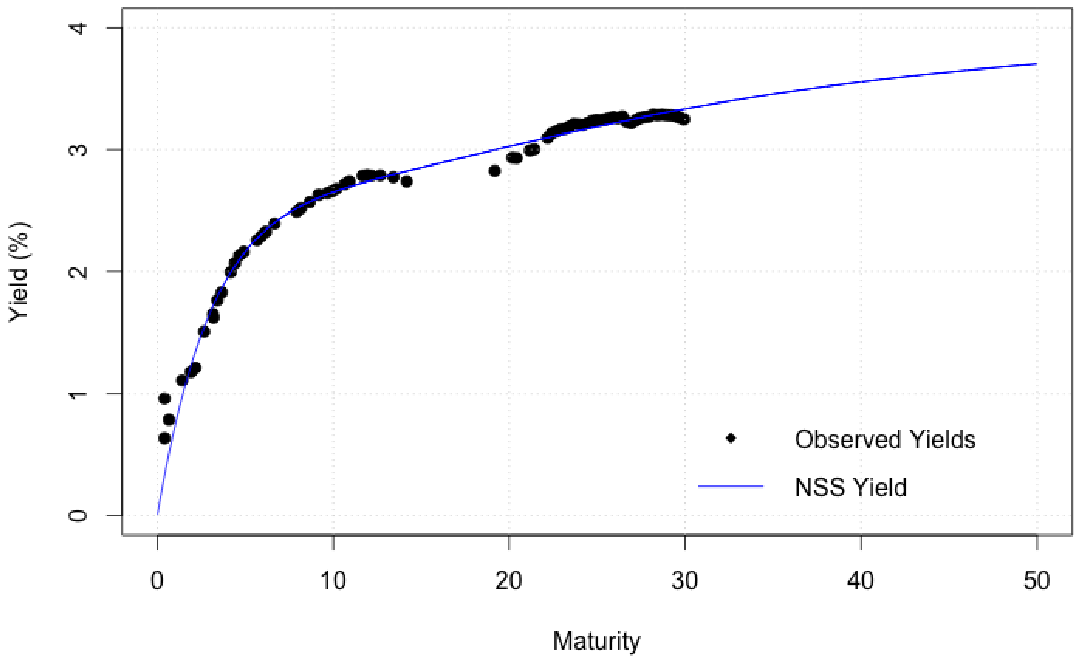

Figure 4.2: Extrapolation of the yield curve using data over United States Treasury Bonds available on 12-16-2016. The black dots show the market data as observed, while the blue line corresponds to the Nelson, Siegel and Svensson extrapolation curve.

We can easily notice, how this curve fails to include all market data in it's structure but achieves a high degree of smoothness beyond the last observed liquid maturity and principally before reaching the final market maturity.
With a further empirical example, we illustrate the Nelson, Siegel and Svensson term structure as obtained by the European Central Bank to extrapolate the Euro area yield curve. In this case, Nelson, Siegel and Svensson term structure has a peculiarity that can be observed in Figure 2.22. This curve has been computed using the ECB's estimated parameters on AAA bonds. To produce the required parameters, the ECB chose to minimize the squared difference between zero coupon prices instead of interest rates. The resulting parameters are updated on a daily basis. In table below, we illustrate the parameters dated 12-16-2016[12]:

[12]source: European Central Bank.

Parameter	β_0	β_1	β_2	β_3	τ_1	τ_2
Value	1.601312	-2.454312	11.930172	-16.170803	1.290330	1.525010

Table 4.1: NSS parameters estimated by the ECB.

While Figure 5.14 depicts the curve obtained by the ECB:

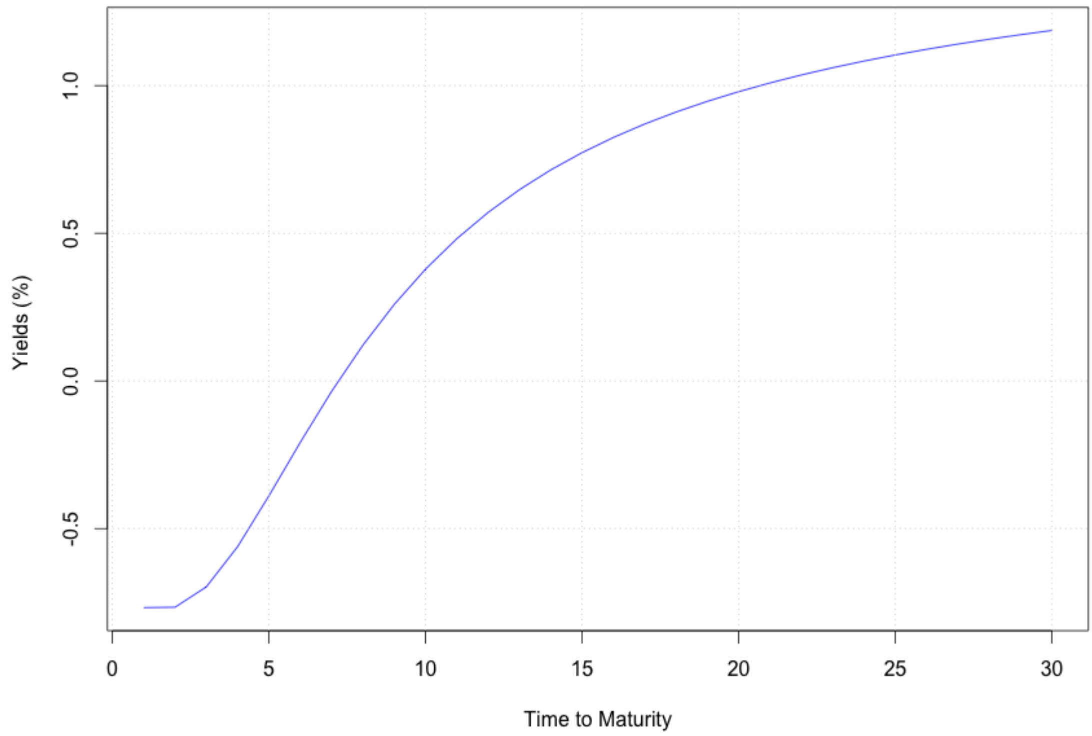

Figure 4.3: The yield curve as computed by the ECB on 12-20-2016.

The above structure is relevant since we can notice negative yield values for short term maturities revealing a low interest rate tendency in the Euro area. This behavior doesn't come as surprise. But, is rather the result of equilibriums achieved within the current government bond market where negative yields have occurred in the past years.

4.3.3 The Nelson, Siegel and Svensson Model and the Smith-Wilson Method

The Nelson, Siegel and Svensson model is a generally acclaimed model that produces a rather smooth curve through the information provided by the financial market. However, the fit to market data imposed in a Nelson, Siegel and Svensson is far from being perfect. As a matter of fact, it is the result of a minimization process that will inevitably produce deviations from market data. This feature is appropriate for extrapolation proposes when needing to produce a macroeconomic analysis across the globe and to extract long term expectations. Nevertheless, it fails to produce a pricing function that relies completely on market consistency. For this set of problems, a spline approach with smoothness constraints is far more convenient. In recent years, many spline based approaches have been proposed.[13]. The most notable interpolating functions are cubic splines.

While the Nelson, Siegel and Svensson model loosely incorporates market points in its structure, the Smiths-Wilson method perfectly matches market point. This is utterly in line with the current legislation. Market consistency of the term structure of interest rates is crucial in the computation process of the market present value of a long term liability manager's portfolio.

The Smith-Wilson method is in fact a mixture of two approaches generally used to build the term structure. As such, the method adopts thoroughly exponential splines. And thus, it applies an interpolating approach for the market part of the curve, with little regard for smoothness. Similarly to the Nelson, Siegel and Svensson model, but with a fixed asymptotic rate, the final part of the term structure is created through on an extrapolation approach. In a Smith-Wilson model this will inevitably require a larger amount of parameters: one for each financial instrument incorporated in the curve with an additional parameter when considering the extended open-UFR version of the Smith-Wilson method. Hence, EIOPA's privileged methodology results less parsimonious in terms of estimated parameters, but a far better interpreter of the market's behavior[14].

The final observation is related to the functional form achieved by both methods. The Nelson, Siegel and Svensson method approaches a mathematical formulation of the instantaneous forward rate function. Consequently, the pricing function and yield function is computed in a subsequent stage. On the contrary, in a Smith-Wilson approach, the outcome of the model is the pricing function. While, we obtain the forward intensity function through further computations.

To conclude, neither of the models mentioned above produce both a smooth and a perfectly in-sample fitted curve. The criterion used to select the best method must be thoroughly based on a context analysis to which the term structure is to be applied.

[13] e.g. Anderson N. and Sleath J., (2001) and Koopman S. J., Mallee M. I., Van der Wel P. M. (2007)

[14] Another notable example of this type of mixed approach is applied by Barrie&Hibbert. In this case, splines are employed for the market part of the structure, while Nelson and Siegel for the extrapolation level of the curve.

Chapter 5

Application and Results

In this chapter, the Smith Wilson method and all alternatives afore presented, will be discussed in empirical terms. We will pursue a deep comprehension of the method and will attempt to produce all required comparisons between the different term structures models to achieve a definitive response to all possible problematics. Before being applied to real market data, the Smith Wilson algorithm has been tested through the cash flow of given financial instruments specified in EOPA's technical documentation. [1]. The results achieved have been satisfying allowing its further application to market financial instruments.
In the present chapter, the Smith-Wilson methodology and the alternatives mentioned will be primarily implemented using one particular set of data. Our main dataset will consist of swap rates relative to the EURO currency, observed on 17-12-2016. The table below contains the selected Euroswaps.

Maturity	1	2	3	4	5	6	7	8	9	10	12	15	20
Rates (%)	-0.19%	-0.15%	-0.08%	0.01%	0.13%	0.26%	0.39%	0.52%	0.64%	0.75%	0.93%	1.12%	1.27%

Table 5.1: Observable rates

[2] This table allows a geometrical representation that is illustrated in Figure 5.1,

[1]see: Technical documentation of the methodology to derive EIOPAs risk-free interest rate term structures
[2]source: SEB group

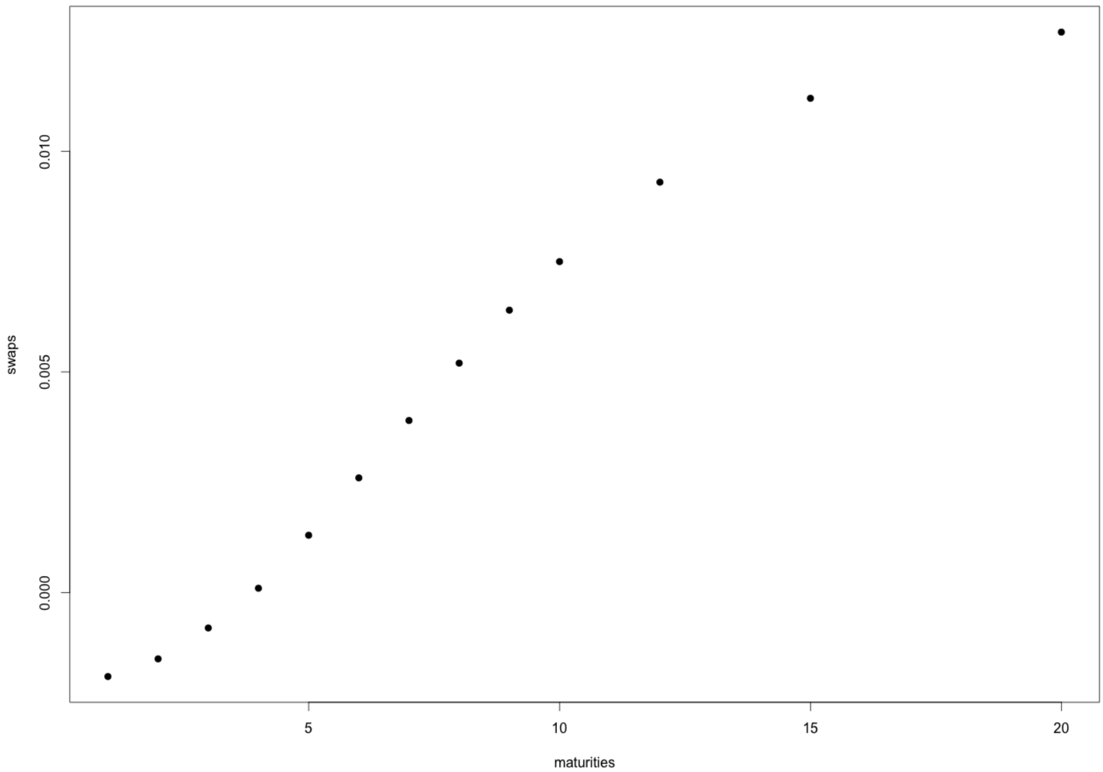

Figure 5.1: The black dots illustrate the Euroswaps available on 17-12-2016 up to a 20 year maturity.

As previously mentioned, the rates are selected in line with the liquidity criterion imposed by the current legislation. This set of data will be used throughout the hole application chapter.
The first section will focus on the plain empirical study of the Smith-Wilson method. We will then proceed to represent the term structure under different values of the UFR before moving to the results obtained from the free-UFR algorithm and from the yield curve smoothening alternative. The concluding section will include the results achieved through the Nelson, Siegel and Svensson model and a final comparison between all methods employed throughout the hole chapter.

5.1 The Smith-Wilson Method as applied by EIOPA

[3] To characterize the features of EIOPA's privileged extrapolation technique we will now illustrate a practical example of the Smith Wilson method using market swap rates shown

[3] The R code used to produce the Smith-Wilson curves is included in Appendix A.

in Table 5.1.
In Figure 5.2, three types of term structure are illustrated. The red curve is the structure of the forward intensity rate. Meanwhile, the blue curve and the green curve correspond respectively to the structure of the annual interest rate and the yield to maturity function. In this case, the asymptotic interest rate is set at 4.2%. For our purpose, we will require its redefinition in terms of intensity: $ln(1 + 0.042)$. Moreover, the last interpolation point equals 20 years since we operate with the Euro currency. Provided all previous assumptions, we obtain an optimized velocity of convergence parameter set at $\alpha = 0.128325$.

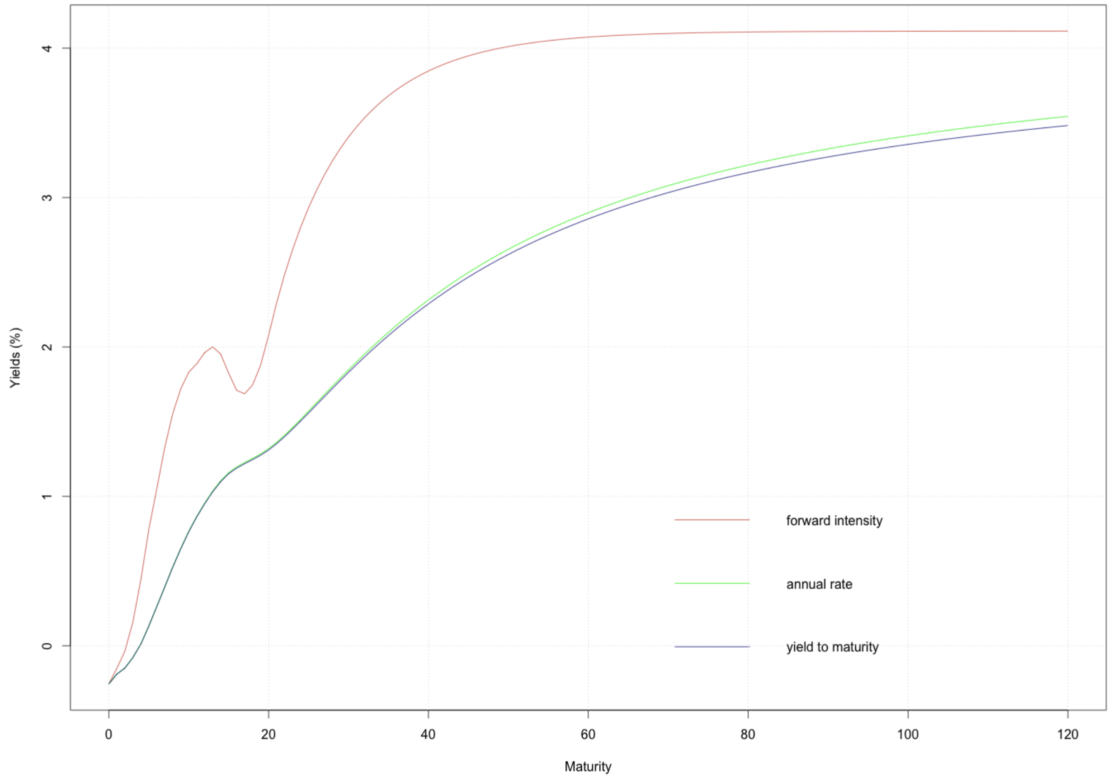

Figure 5.2: Extrapolation of the term structure from market data as observed on 17-12-2016. The red line shows the forward intensity rate, while the green and blue line correspond respectively to the the annual rate and the yield to maturity, with an optimized $\alpha = 0.128325$.

We have earlier claimed the necessity of reaching all market points imposed by the current legislation. Provided that all points denoting the observed swap rates are perfectly included in the curve through interpolation, a "pointy" curvature in the market section of the shape is inevitably generated. Subsequently, the term structure becomes far smoother in the convergence sector. At the end of the convergence period, the Ultimate Forward Rate is not reached by the interest rate curve. Even at the end of the depicted 120 years

term structure, the interest rate curve still doesn't approach the UFR value. The reason lies in the definition of the asymptotic rate in terms of the forward rate. While the outcome provided by EIOPA is the term structure of interest rates. With time the interest rate will converge to the Ultimate Forward Rate but regularly, it will happen close to economically irrelevant maturities.
To have a deeper understanding of the method and its convergent evolution, the forward intensity rate curve is represented in Figure 5.3.

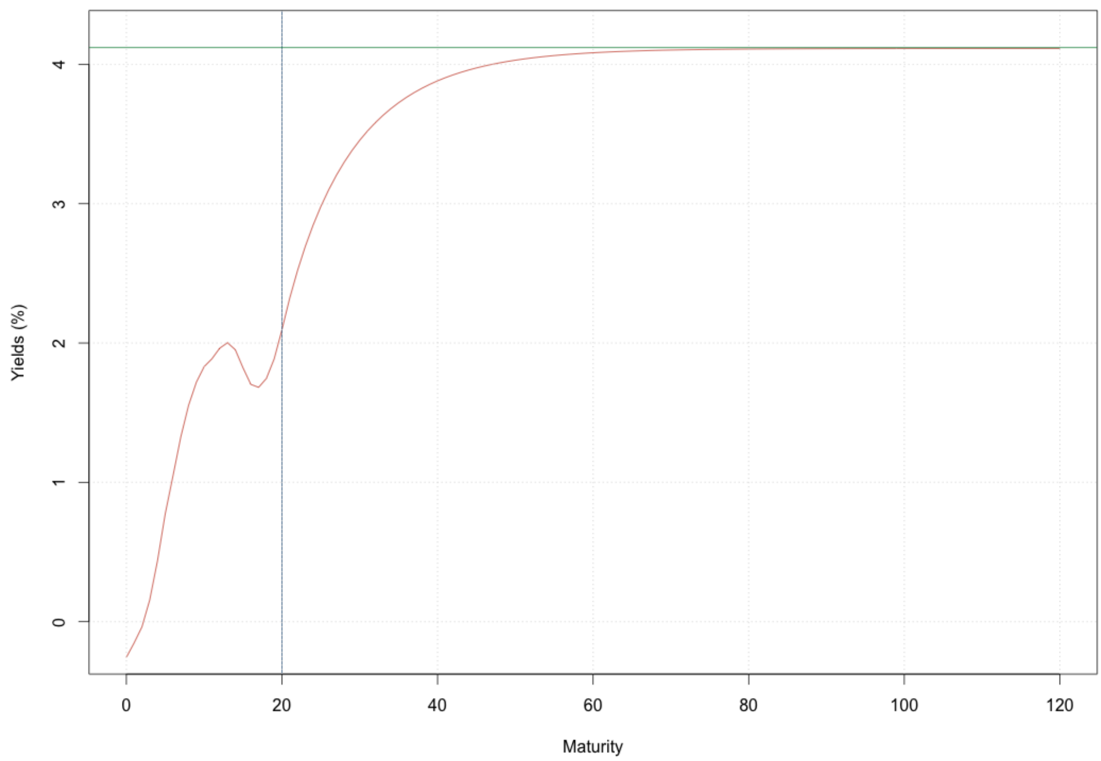

Figure 5.3: Convergence of the forward intensity rate to the fixed UFR.

In the above graph we can notice the convergence to the fixed asymptotic forward rate in terms of ultimate forward intensity, $ln(1 + 0.042)$. Beyond the last liquid point, the behavior of the forward intensity function is governed by the parameter α, by the fixed convergence period and consequently by the likewise fixed convergence point. The number of vertices in the interpolative section of the forward intensity curve depend on the liquid observable maturities, which for the EURO currency are limited to 13. To give a different example, if we consider the data mentioned in the Technical Specifications we have 20 available swap rates. This will inevitably produce a higher number of vertices in the curve.

5.2 Different Levels of the UFR Parameter

The results of the public consultation launched by EIOPA over the value of the UFR have been published on April 2016. In this document, EIOPA approaches different levels of the asymptotic rate on which to apply the Smith-Wilson method.

Following this line, we will now produce the term structure for a range of UFRs. The starting value of the UFR is 3.2%, while the final value will be 5.2%. Additionally, the analysis will be based on real market data over the Euro currency. In the Figure below we illustrate the results obtained with the different selected levels of UFR.

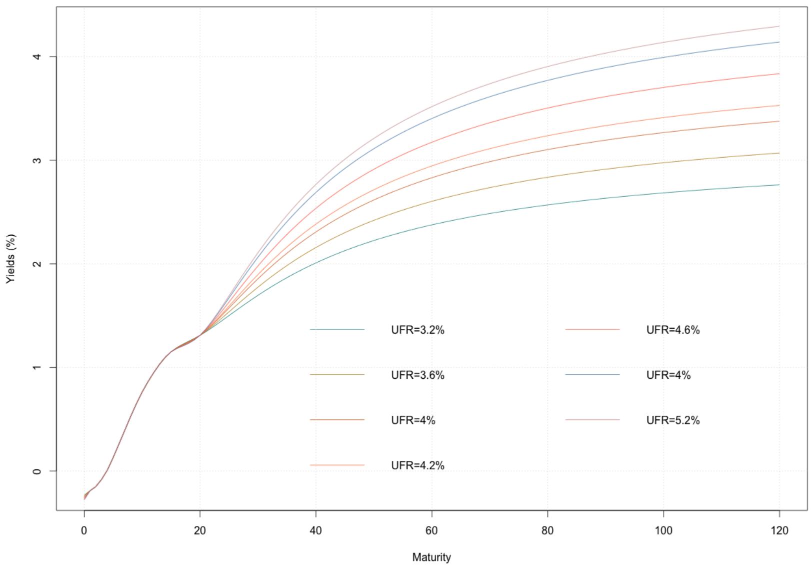

Figure 5.4: Representation of the Smith-Wilson yield curve's evolution under different levels of UFR.

The value of the UFR has an impact primarily on the section beyond the last liquid point, producing no relevant change in the interpolation section. This effect increases with time. The top curve is the result of the highest level of UFR. Whereas, the lowest curve is set at the lowest UFR considered.

Moreover, an increase in the level of UFR causes an increase in the optimum velocity of convergence. This behavior can be noticed in Table 5.2 where we achieve different values of αs for different fixed values of the asymptotic rate.

UFR	3.2%	3.6%	4%	4.2%	4.6%	5%	5.2%
α	0.117186	0.12434	0.12656	0.128325	0.131413	0.134039	0.135214

Table 5.2: Values of α under different UFRs

Further effects driven by the UFR can also be easily analysed in terms of the unity pricing function as in Figure 5.5.

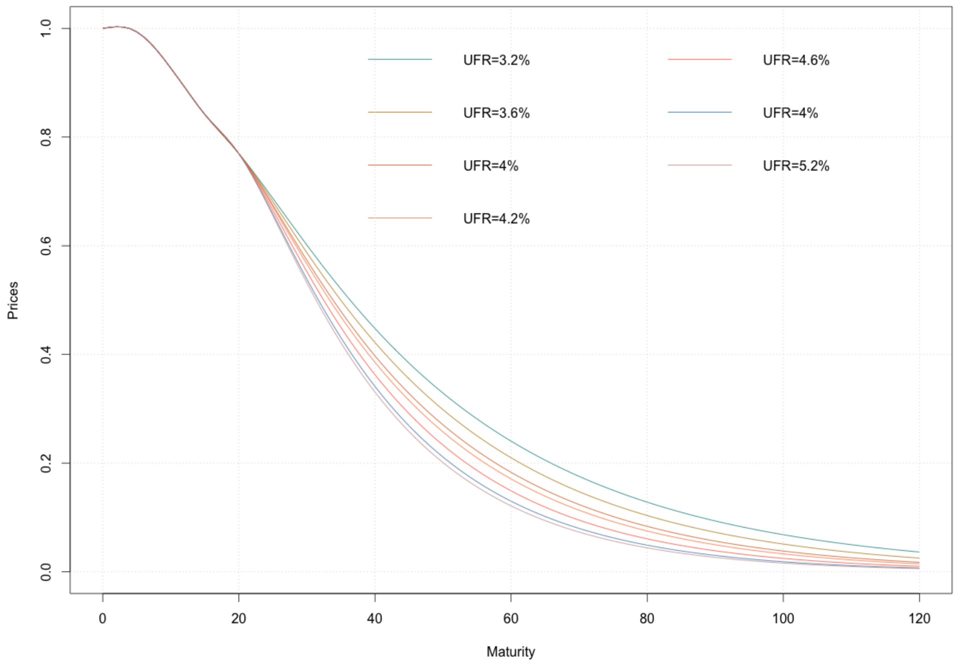

Figure 5.5: Representation of the Smith-Wilson pricing function under different levels of UFR.

The time value of money has a decreasing development. This decreasing process is faster for higher levels of the asymptotic rate. Although, the difference between the curves tends to be less accentuated in the long run. This is the result of the curve's natural convergence to zero.
In April 2016 and as confirmed in September of the same year, EIOPA as a response to consultations ran in the previous mouths, announced a probable change in the real rates computation methodology that could be applied from March 2017. This will inevitably produce different values of the asymptotic rates for each currency as reported in the consultation paper[4]. In terms of our empirical example, the Euro will have a 3.7% UFR re-estimated through this new method.
We can show an immediate comparison between the present curve and the future structure in Figure 5.6. The revised UFR will necessarily cause a decrease in the extrapolation sector of the term structure.

[4]see: **Consultation Paper on the methodology to derive the UFR and its implementation** page 53

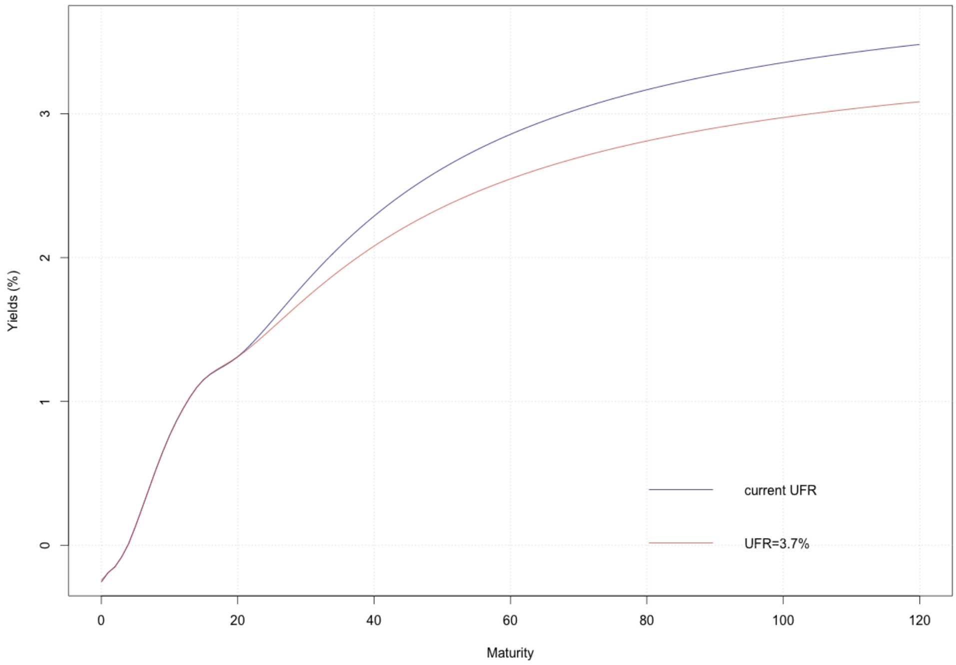

Figure 5.6: Comparison between the present curve, blue line, and the curve at level $UFR =$ 3.7%, red line. The values of the velocity of convergence are respectively 0.128325 and 0.123552.

5.3 Alternative Versions of the Smith-Wilson Method

In this section, we will investigate over the empirical characterization of the alternative methods proposed by De Kort and Vellerkoop. Additionally, we will approach a comparison between the results achieved with the current term structure and the alternative methods, by also employing a different set of data.

5.3.1 Open UFR using Current Market Data

Using the free asymptotic forward rate resolutive algorithm[5] we incur in a major problematic. Since the optimum alpha depends on the value of the UFR, while the optimized UFR depends on the value of an a priori alpha, it is impossible to optimize both values at the same time. To achieve an acceptable result, we proceeded heuristically. First, we

[5]The R code used for this purpose is listed in Appendix B

defined different values of α a priori. Then, we computed the optimizing UFR and the optimum α. The results achieved are illustrate in Table 5.3

α a priori	0.05	0.085	0.1	0.15	0.2	0.25
UFR intensity	0.0144	0.0140	0.0138	0.0136	0.0134	0.0133
α optimized	0.0820	0.0851	0.0861	0.0882	0.0894	0.0902

Table 5.3: Values of UFRs under different a priori αs.

Values below 0.05 are restricted by the optimizing algorithm, while values beyond 0.2 seem unrealistic given the present level of a 0.1 α defined by EIOPA. However, we have included a value of 0.25, to have a better margin of comparison.
Furtherly, in Figure 5.7 we illustrate the behavior of the term structure curve under different values of an a priori velocity of convergence avoiding the α optimization procedure.

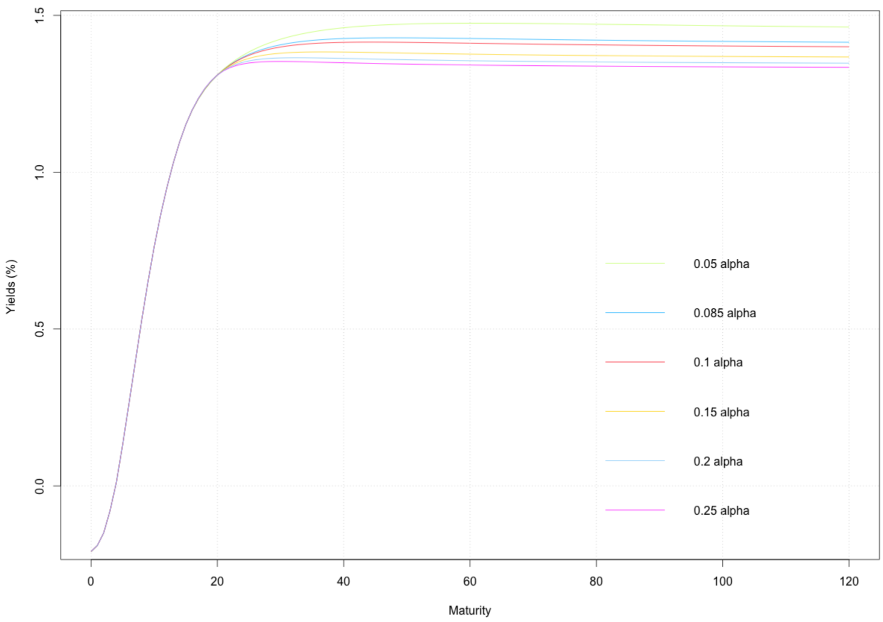

Figure 5.7: Representation of the term structure curve under different values of an a priori velocity of convergence.

From the above table, we can easily trace the most coherent values for the UFR, in terms of intensity, and for the velocity of convergence. As such, we choose to elaborate our analysis on a UFR equal to 1.4%[6] and on a value of α of 0.085. In Figure 5.8, we illustrate the resulting interest rate curve.

[6]The Ultimate Forward Rate is computed through $e^{0.014} - 1$.

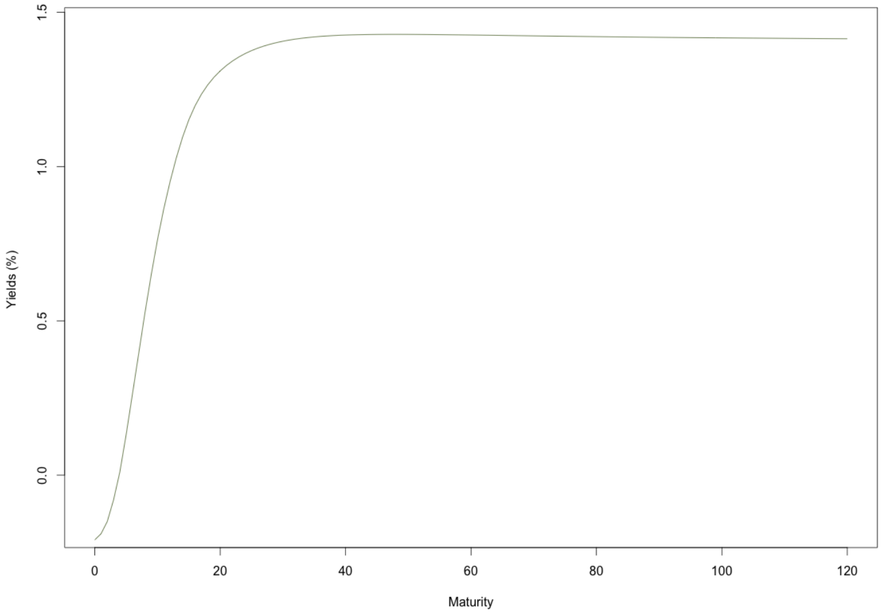

Figure 5.8: Representation of the annual interest rate curve resulting from the free-UFR algorithm.

To better comprehend the consequences of such a modelling, the curves for data available on the market on 17-12-2016 in terms of forward intensity rate, yield to maturity and annual interest are displayed in Figure 5.9.

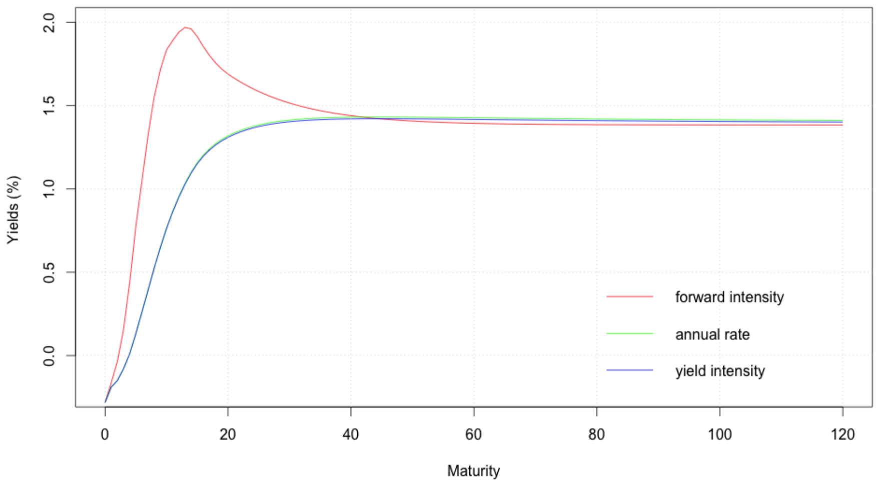

Figure 5.9: Illustration of the free-UFR extrapolated curves. The red curve corresponds to the forward intensity rate curve, while the green and blue curves correspond respectively to the annual yield and to the yield to maturity.

A first important remark is the drop of the forward intensity function below the other two curves in the extrapolation part of the structure. Furthermore, the level of the asymptotic rate is by far under the UFR's current value. Thus, resulting in lower extrapolated rates and inevitably causing a sudden decline in the yield curve when considering its non-market section.
This behavior is immediately noticeable in the comparison between the UFR optimizing curve and the present term structure, see Figure 5.10.

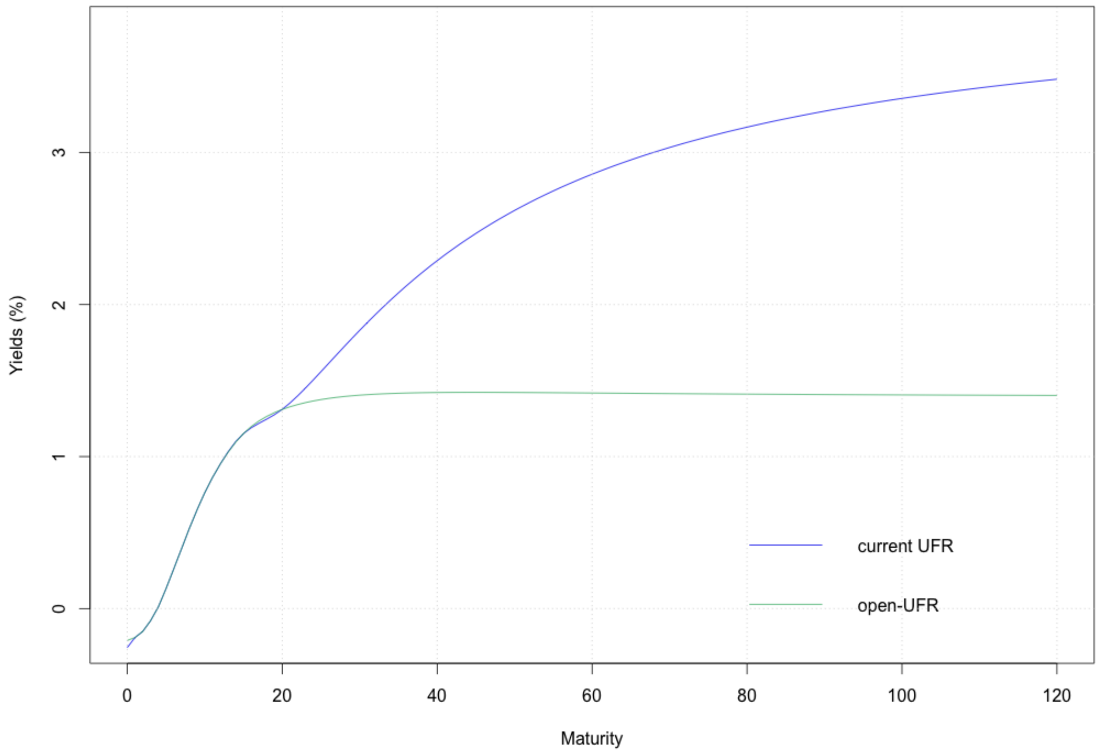

Figure 5.10: Comparison between the present curve and the UFR optimizing curve.

The evolution of the two curves begin to differ before reaching the last available market point. A first sufficiently large gap is already produced beyond maturity 15. In this section of the figure, we notice higher values generated by the optimizing-UFR method. This can be observed in Figure 5.11. Still, after surpassing the last market maturity we notice a change in the behavior of the curves. While the curve with an UFR set at 4.2% increases with time, the free-UFR curves moves in the exact opposite direction. Thus, causing a wide and increasing gap between the curves.

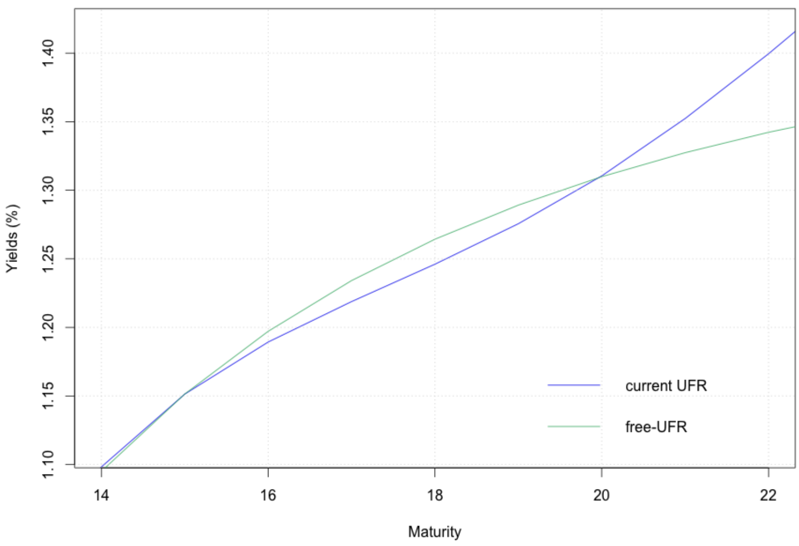

Figure 5.11: More details on the comparison between the present curve and the UFR optimizing curve.

Unsurprisingly, there appears to be no significant difference in the existing gap between the two curves in the extrapolation section, when imposing a constant speed of convergence, e.g. $\alpha = 0.1$.

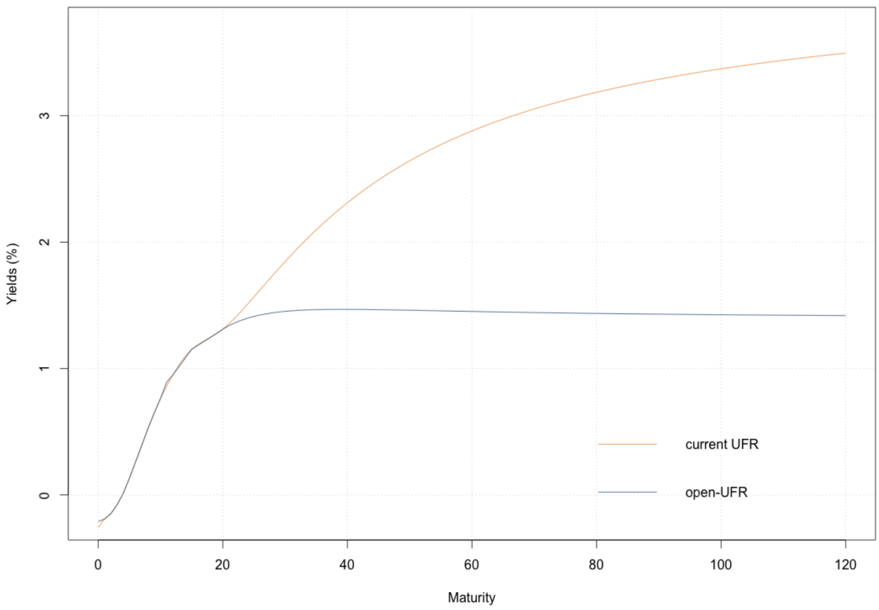

Figure 5.12: Comparison between the present term structure and the UFR optimizing curve, both calculated with $\alpha = 0.1$.

The Figure above shows the evolution of the term structure with EIOPA's fixed UFR for the Euro currency and the curve that converges to the minimum UFR, both determined under $\alpha = 0.1$. Nevertheless, we still obtain a quite similar distance between the the two shapes. This exact pattern also occurs before reaching the established convergence point. Hence, leading to the conclusion that the speed of convergence has little influence on leveling the convergence between two curves. Instead, the sole parameter governing the convergence rests the UFR. Since the aforementioned gap between the two curves seems quite peculiar we can proceed with further analysis using a different set of data.

5.3.2 Open UFR using only Positive Market Data

This section is dedicated specifically to the results achieved through different market data. To select proper input rates, Figure 5.13 can be used to examine the evolution of the Euro swap rates in the past years. The graph shows the values of the swap rates with maturity 1,5,10 and 20 years between December 1998 and the December 2016.

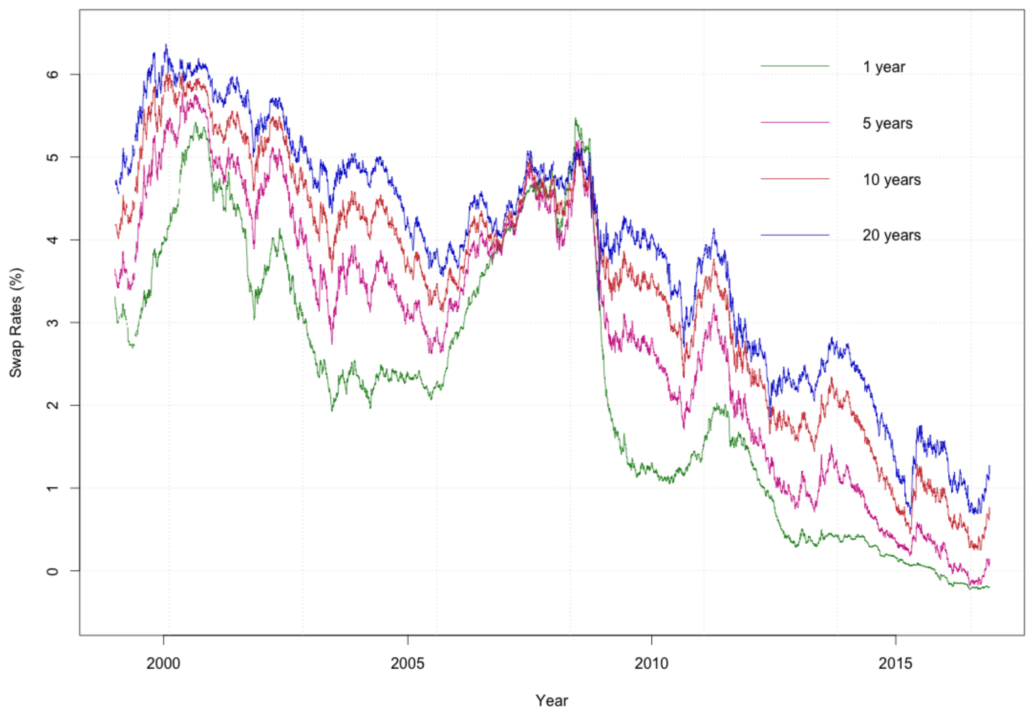

Figure 5.13: Illustration of the Euro swap rates' evolution from year 1998 to 2016 for several maturities. The lower green curve represents the 1 year maturity swap rate. The magenta, orange and blue lines correspond respectively to swap rates with 5, 10 and 20 years maturity.

In the past few years, the rates have dropped consistently, producing the negative values we currently observe on the market. The drop didn't affect all rates with the same intensity. But, we observe a more severe decrease for the short term swaps.
An interesting analysis could be conducted on non-negative swap rates. For this reason, we chose to operate with Euro swap rates recorded on 20-12-2013[7]. The rates are displayed in Table 5.4.

Maturity	1	2	3	4	5	6	7	8	9	10	12	15	20
Rates (%)	0.4215%	0.5345%	0.7295%	0.975%	1.221%	1.4405%	1.6415%	1.818%	1.9765%	2.117%	2.328%	2.5465%	2.675%

Table 5.4: Swap rates recorded on 12-20-2013.

By employing the same heuristical approach shown in the previous section, the minimizer intensity UFR is approximately equal to 0.028. With such a value of the asymptotic rate, the most coherent speed of convergence results $\alpha = 0.079$. Using this particular set of parameters and data, we can easily generate the structure of the annual interest rates.

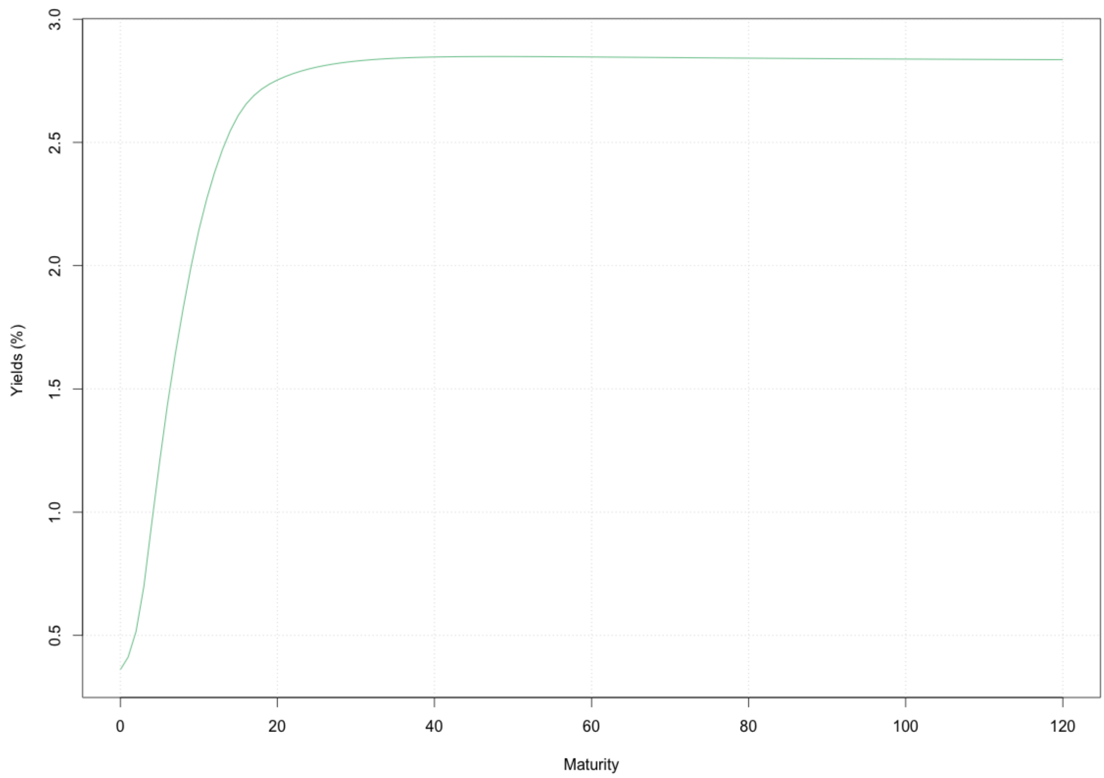

Figure 5.14: Representation of the free-UFR extrapolated interest rate structure for 2013 data.

[7]source: Bloomberg

For further analysis, we also depict the forward intensity function, the yield to maturity curve and the annual interest rate curve in a single graph, see Figure 5.15.

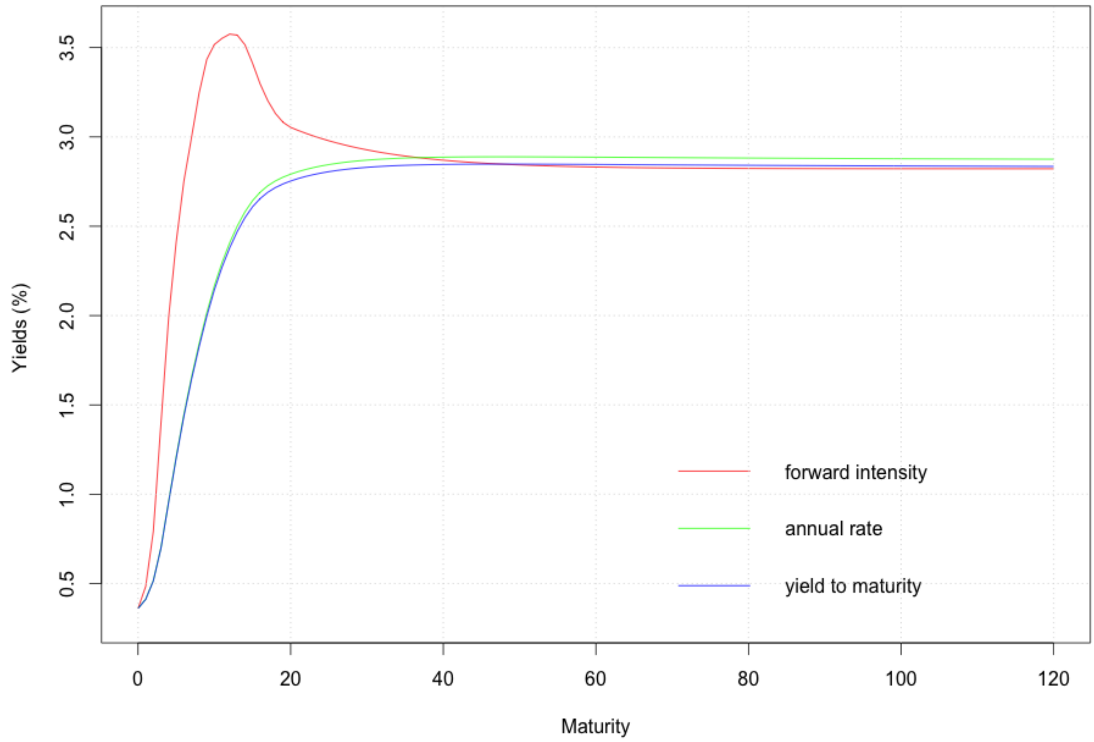

Figure 5.15: Illustration of the free-UFR extrapolated curves. The red curve corresponds to the forward intensity rate curve, while the green curve and blue curve correspond respectively to the annual interest rate and the yield to maturity function.

The evolution of the structure is admittedly similar to the behavior of the curves produced with more recent data. As a result, the forward intensity function is descending below the yield to maturity in the extrapolation part after an initial domination of the other two curves. But clearly, there is no longer evidence of negative rates.
We will now further proceed with the comparison between the yield term structure under an optimum UFR and the yield curve under the current UFR level. In such a case, we need to generate a regular Smith-Wilson curve based on the 2013 data, imposing a $UFR = 4.2\%$. Consequently, the velocity of convergence results equal to 0.109926.
The Figure below is the empirical representation of the two obtained structures.

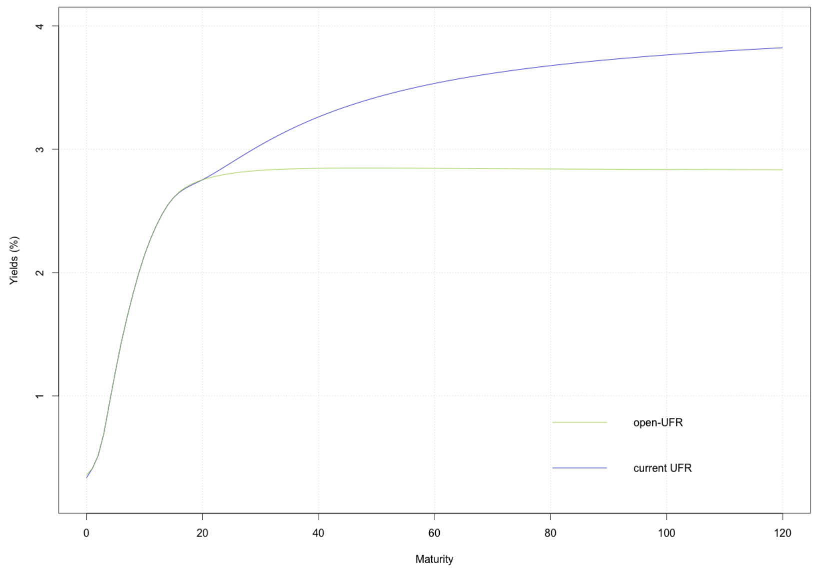

Figure 5.16: Comparison between the term structure with the current Euro UFR and the UFR optimizing curve.

With this new dataset, we obtain a figure that presents no significant difference between the two structures for the market sector of the curves. Whereas, there continues to be an increasing wide gap between the two lines for maturities beyond the last market point. Nevertheless, the distance appears far shorter than the that in the negative market case. To have further insight on the term structures, in the next figure we will delineate the two above mentioned structures for several periods of time.

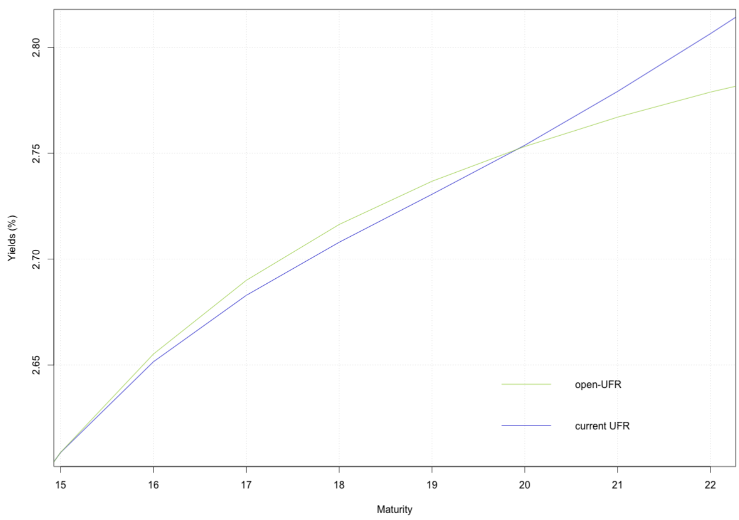

Figure 5.17: More details on the comparison between the term structure with the current Euro UFR and the UFR optimizing curve.

As before, the rates are practically identical before reaching maturity 15 years, when the free-UFR tends to slightly dominate the second curve for a short period of time. But when reaching a 20 years maturity, we have a reversed situation with a gap between the curves that widens with time.
Once more, there appears to be no significant difference between the evolution of the curves, when imposing a speed of convergence of 0.1 for both structures, see Figure 5.18.

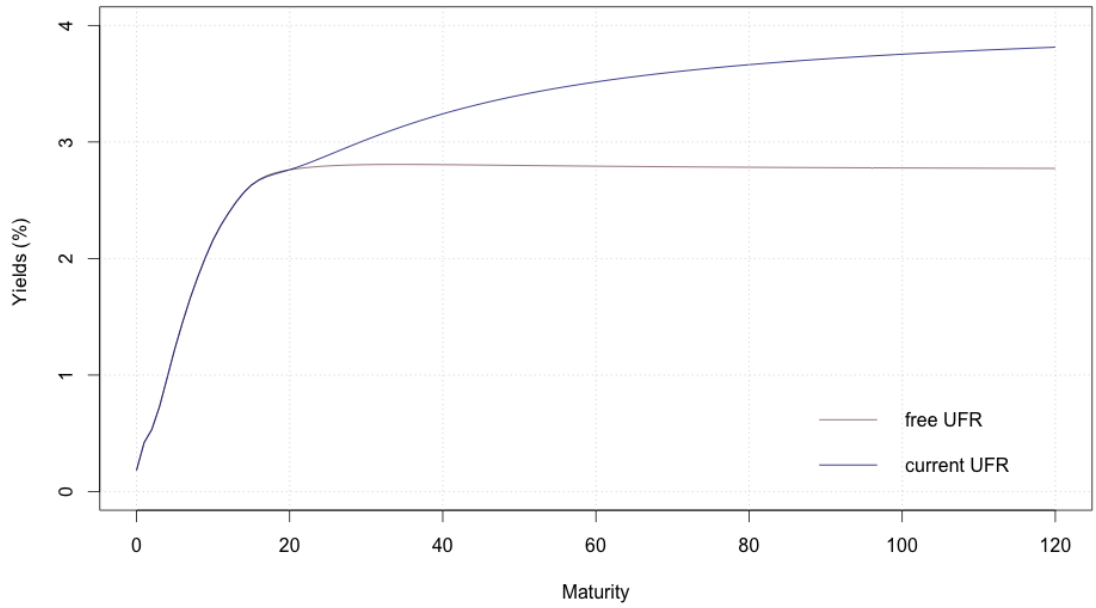

Figure 5.18: Comparison between the term structure with the current Euro UFR and the UFR optimizing curve with $\alpha = 0.1$.

With this final figure, the empirical analysis of the optimizing-UFR problem has been concluded. We will now move to the alternative method discussed by De Kort J. and Vellekoop M. H.

5.3.3 The Yield Smoothening Alternative Method

In this part of the empirical study, we will present the curves generated through one of the smoothening methods proposed in the previous chapter. The structures produced are based on the model that smoothens the yield curve.[8]. In the Figure 5.19, we illustrate the annual interest rate. Yet again, we incur in the catch-22 related to the definition of the optimize UFR and the velocity of convergence. An heuristical approach to the resolutive

[8]The R code in inserted in Appendix B.

algorithm produces an approximate value of the asymptotic rate, in terms of intensity, is equal to 1.7%, resulting in an $\alpha = 0.08$.

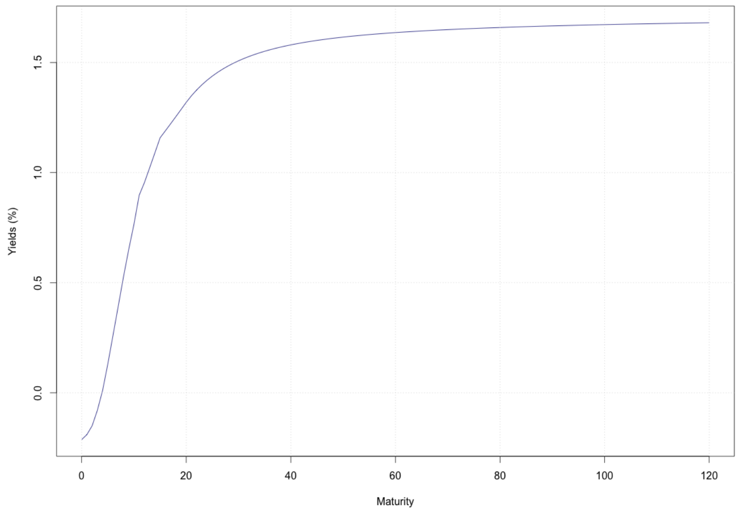

Figure 5.19: Interpolation and Extrapolation of the term structure of interest rates using the smoothest yield to maturity curve approach.

This approach can be furtherly characterized through its comparison with the current application of the Smith-Wilson Method.

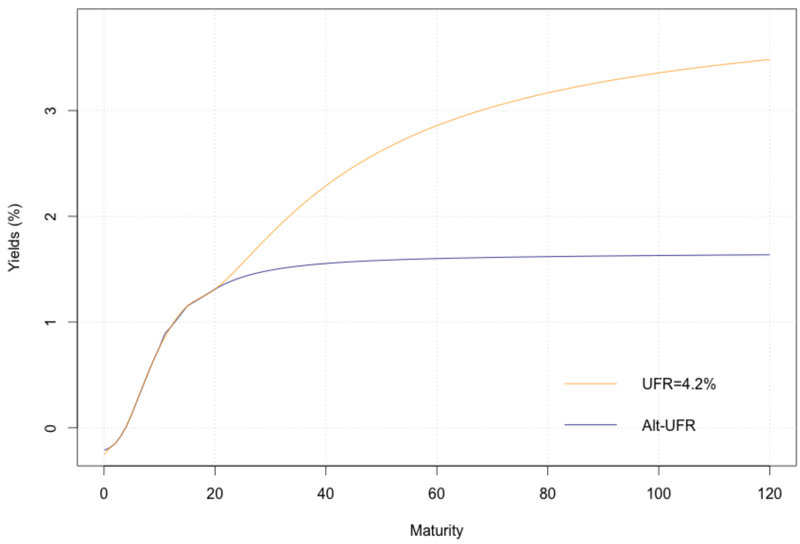

Figure 5.20: Interpolation and Extrapolation using EIOPA's current Smith-Wilson method and the yield to maturity smoothening method. The yellow line depicts the current method, while the violet curve corresponds to the alternative method curve.

As expected and formally proved in the previous chapter, the resulting yield curve is far smoother in the interpolation section of the term structure. However, we can observe several vertices corresponding to market data. But still, this method completely avoids the humped shape displayed in the Smith-Wilson term structure.
As in the open-UFR technique, the increasing large gap between the two curves persists. Leading to the conclusion that the current level of the long term equilibrium rate is just beyond credibility. Once more, and for exactly the same reason as afore, we propose the comparison of these two methods in terms of a different set of data. Thus, avoiding negative values of input rates. The curves obtained through the data illustrated in Table 5.4 are represented in Figure 5.21.

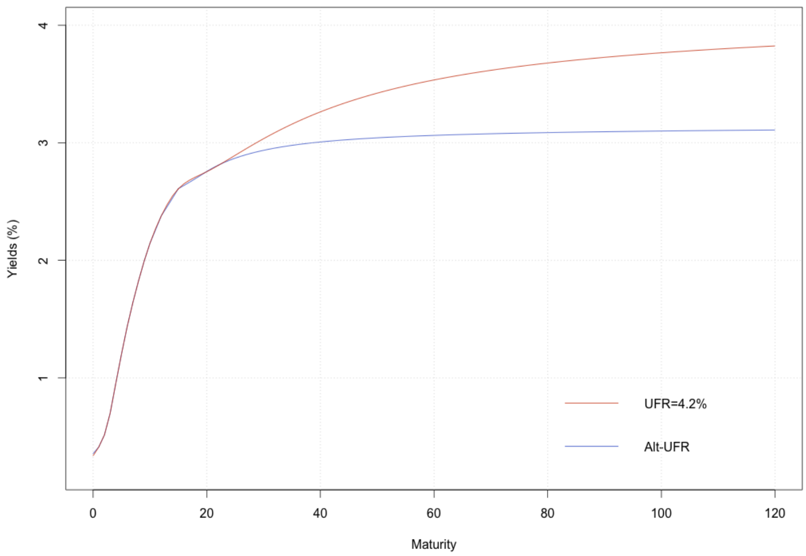

Figure 5.21: Interpolation and Extrapolation for data available on 17-12-2013, using EIOPA's current Smith-Wilson implementation and the yield lo maturity smoothening method. The brown line represents the current method and the violet line is the structure created through the alternative method.

With this particular dataset, the yield to maturity smoothening approach produces a value of the asymptotic rate equal approximately to 3.15% and an optimum speed of convergence of 0.075. And once again, we obtain a smaller distance between rates computed through the regular Smith-Wilson approach and one of the proposed alternatives. This leads to a first consideration over the value chosen for the asymptotic rate: A 4.2% rate might have been adequate in a high interest rates market period. However, it utterly fails to capture the current market situation.

5.4 Nelson, Siegel and Svensson

In this section we will investigate the empirical results of the Nelson Siegel and Svensson model mentioned earlier. The inputs are the Euro area swap rates recorded on 17-12-2016. The initial step consists in determining the parameters of the model. To this end, we cannot use raw market data but we need to extract interest rates from the swap rates employing a widespread technique referred to as "bootstraping". The procedure implies the computation of the zero coupon discount factors for each unity of time between 0 and the last observed maturity. The first discount factor is given by:

$$v(0,1) = \frac{1}{1+j_1}, \tag{5.1}$$

and the second:

$$v(0,2) = \frac{1-v(t,t+1)j_2}{1+j_2}. \tag{5.2}$$

Consequently, the general formula is reduced to:

$$v(0,k) = \frac{1-j_k\sum_{s=1}^{k-1} v(0,s)}{1+j_k}. \tag{5.3}$$

The induced interest rate for a one year maturity is, therefore, computed through:

$$i(0,1) = j_1, \tag{5.4}$$

while the second interest rate is:

$$i(0,1) = \left[\frac{1+j_2}{1-v(0,1)j_2}\right]^{1/2} - 1, \tag{5.5}$$

and the remaining values are given by:

$$i(0,k) = \left[\frac{1+j_k}{1-j_k\sum_{s=1}^{k-1} v(0,s)}\right]^{1/k} - 1. \tag{5.6}$$

When using the bootstrap methodology, we require values of swap rates for all tenors between 0 and the last observed rate. The dataset we posses consists of $j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8, j_9, j_{10}, j_{12}, j_{15}, j_{20}$, where the generic element j_k denotes the Euro swap rate with maturity k.

We can easily determine the missing swap rates employing linear interpolation. To have a simple example, the swap rate with maturity 16 years is:

$$j_{16} = \frac{20-16}{20-15}j_{15} + \frac{16-15}{20-15}j_{20} \tag{5.7}$$

Once obtained the needed interest rates, we can move to the optimization of Nelson, Siegel and Svensson's parameters[9]. The values obtained are illustrated in the table below:

[9] The R code used to produce the Smith-Wilson curves is included in Appendix C.

Parameter	β_0	β_1	β_2	β_3	τ_1	τ_2
Value	1.928647	-2.138804	12.378568	-15.987591	1.550565	1.746287

Table 5.5: Estimated NSS parameters.

The parameters don't differ significantly from the ones achieved by the ECB, see Table 4.1, and thus, resulting in a similar term structure.

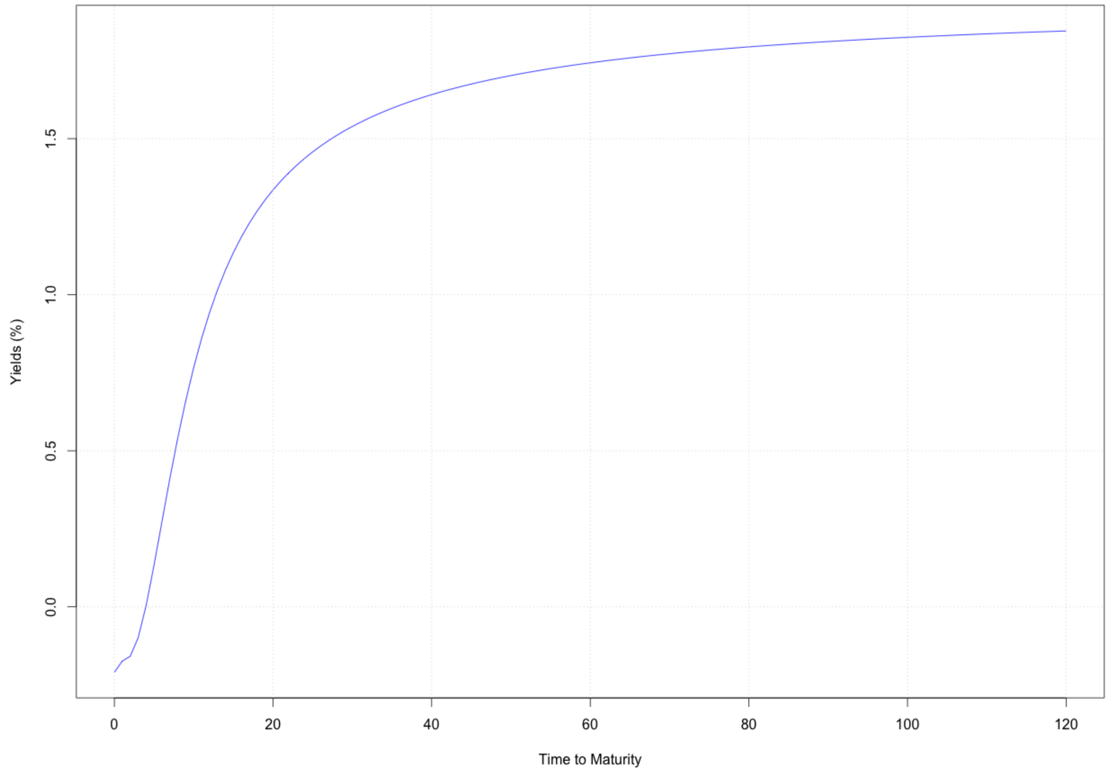

Figure 5.22: Reperesentation of the Nelson, Siegel and Svensson term structure of interest rates up to a 120 years maturity.

The extreme values for the instantaneous forward rate are (in %):

$$\delta(0) = -0.2101571 \tag{5.8}$$

and

$$\delta(\infty) = 1.928647. \tag{5.9}$$

Additionally, in Figure 5.23 we illustrate the term structures of the forward intensity, the yield to maturity and the annual interest rate for all maturities up to 120 years.

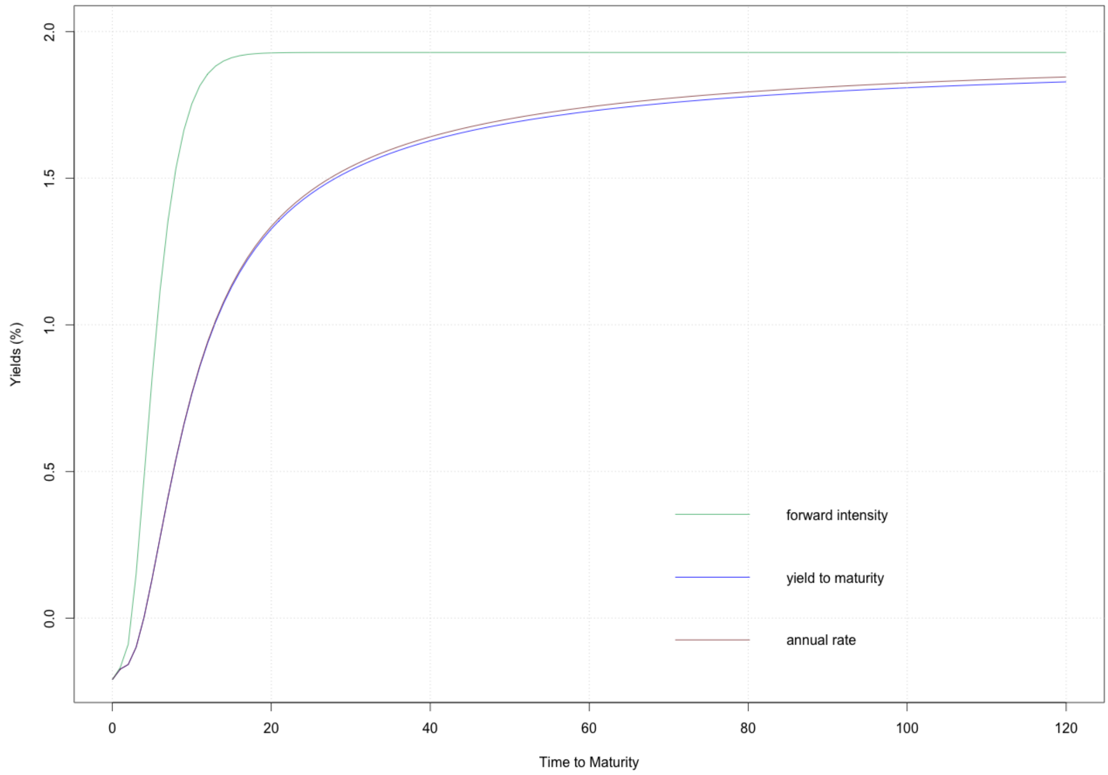

Figure 5.23: Extrapolation of the term structure according to the Nelson, Siegel and Svensson model. The green curve corresponds to the forward intensity structure while the blue and brown line denote respectively the yield to maturity and the annual interest rate curve.

We can easily notice the smoothness of the curves, particularly in the market area of the term structure, where there is no constraint on reaching all market points. As a matter of fact, what this methodology achieves is a linear representation of interest rates avoiding Smith-Wilson's angular behavior. Additionally, the three curves in the long rung are closer than the Smith-Wilson's curves with same tenors.

5.4.1 Empirical Comparison between the Nelson, Siegel and Svensson Model and the Smith-Wilson Method

Using 2016 Data

A comparison between the term structure of interest rates achieved through all the above mentioned methodologies will be presented in this final section of the empirical study. We will start by approaching the term structures using the data available on 17-20-2016. As

for the Nelson, Siegel and Svensson method, we prefer a comparison in terms of the curve produced by the ECB, to have a better understanding of risk-free term structures.
In Figure 5.24 all curves produced by the aforementioned methods are illustrated.

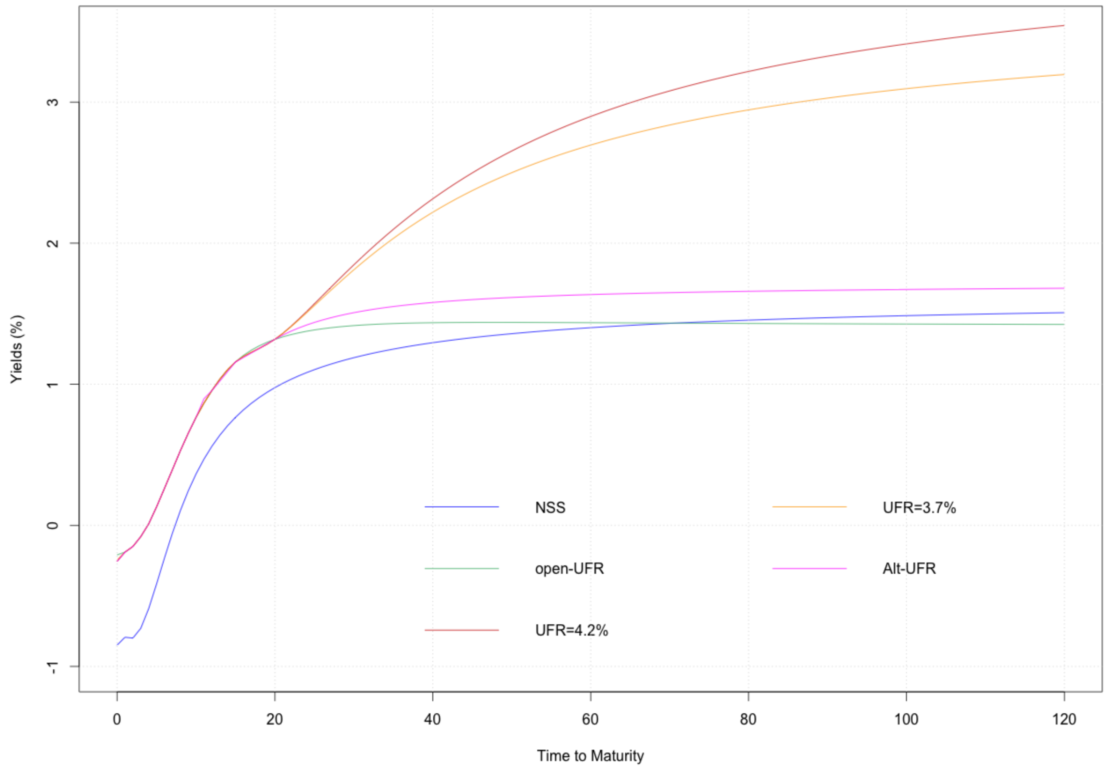

Figure 5.24: Extrapolation of the term structure according to all analyzed models. The low green, magenta and blue curves correspond respectively to the open-UFR method, the yield smoothening approach and the Nelson, Siegel and Svensson structure. The red and orange lines depict the regular Smith-Wilson method produced with UFR levels of 4.2% and 3.7%.

The first feature we notice in the graph is the gap between the two regular Smith-Wilson curve's calculated with a UFR of 4.2% and 3.7% on the upper side of the figure and the lower curves designed according to the Nelson, Siegel and Svensson model, the yield to maturity smoothening method and free-UFR algorithm. The values between two principle sectors begin to separate before reaching the last observed maturity but initially, without causing a significant distance between rates in the interpolation part of the structure.
Considerable forks evolve beyond the last liquid maturity. The largest gap divides the graph in the two principal sectors. However, gaps evolve between all curves and widen with time. But, while the the four higher lines have an increasing evolution, the open-UFR curve moves towards the opposite side of the graph. Moreover, we can easily observe how

the Nelson, Siegel and Svensson model, as estimated by the ECB, occupies the lowest position in the graph.
For the benefit of the present analysis, Table 5.6 shows all asymptotic rates, in terms of intensity, originating from the curves in Figure 5.24.

Method	Current SW	2017 SW	Open-UFR	Smoothenss Yield	NSS
Asymptotic Rate	4.1%	3.6%	1.4%	1.7%	1.6%

Table 5.6: Asymptotic Rates

The proximity between the asymptotic rate generated by the Nelson, Siegel and Svensson model and by the yield to maturity smoothener approach confirms what already observed in Figure 5.24: The yield to maturity smoothener model almost emulates the results obtained by the Nelson, Siegel and Svensson term structure.
Additionally, in the following figure we can compare the extrapolated short rates provided by all models.

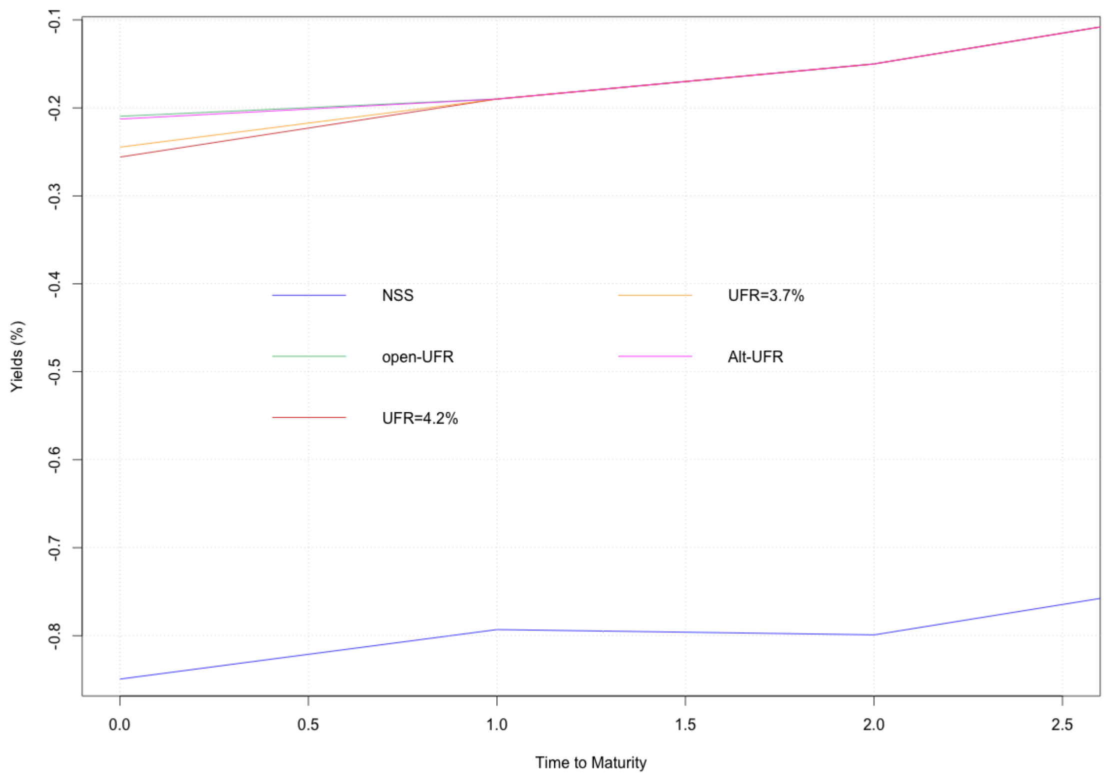

Figure 5.25: Details related to the extrapolation of the term structure according to all analyzed models from 0 to 2.5 years maturities.

Further and essential details relative to this particular set of term structures are shown in Figure 5.26, where we illustrate the interest rates with maturities from 14 years up until 26 years.

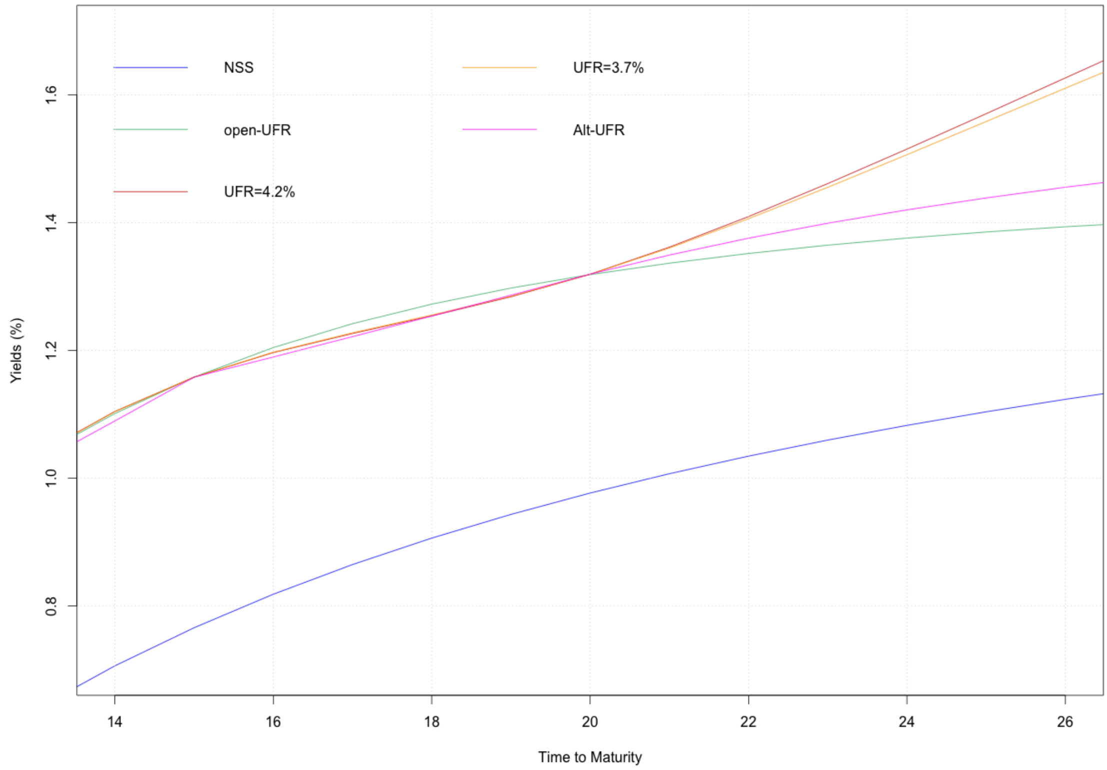

Figure 5.26: Details related to the extrapolation of the term structure according to all analyzed models on maturities between 14 years and 26 years.

For short maturities, the position of the curves cannot be clearly defined. By comparison, Nelson, Siegel and Svensson appears to be the lowest curve for all maturities up until around the 70-th. From this point on, it dominates the open UFR curve for all long term maturities. This comes as a result of the estimated parameters produced by the ECB. The dataset on which the estimation has been generated provides much lower rates.
Additionally, in the specific section illustrated in Figure 5.26, we also notice what appears to be the highest curve in Figure 5.24, becoming the lowest Smith-Wilson term structure, before rapidly increasing from maturity 20 on. A final observation is to be made on the value of the curves at time 20. All curves, but the Nelson, Siegel and Svensson curve, reach the same point. This is a direct consequence of the interpolation constraints that don't apply to Nelson, Siegel and Svensson's case.

Using 2013 Data

An interesting conclusive analysis can be made for data that lack on negative values. For this purpose, we will once more require the Euro area swap rates recorded on 20-12-2013 and illustrated in Table 5.4.

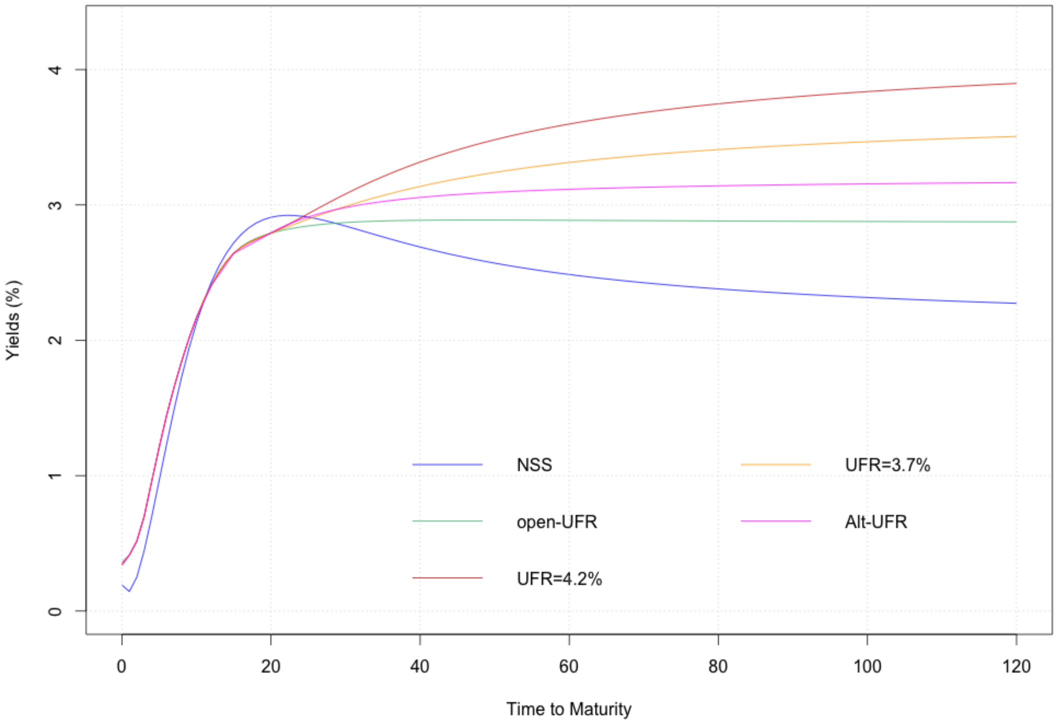

Figure 5.27: Interpolation and extrapolation of the term structure provided by all analyzed models. The low green, magenta and blue curves correspond respectively to the open-UFR method, the yield to maturity smoothening approach and the Nelson, Siegel and Svensson structure. The red and orange lines depict the regular Smith-Wilson method produced with levels of UFR set at 4.2% and 3.7%.

In this specific case, we can no longer perfectly distinguish two sectors in the graph. The highest curve is still obtained imposing a 4.7% UFR to the regular Smith-Wilson method. Although, we obtain lower distances between the curves. All Smith-Wilson methods achieve closer rates in the long term compared with the long term rates produced by the previous dataset, as shown in Table 5.7.

Method	Current SW	2017 SW	Open-UFR	Smoothess Yield	NSS
Asymptotic Rate	4.1%	3.6%	2.8%	3.15%	2.0%

Table 5.7: Asymptotic Rates with 2013 Data.

Albeit, the curve with generatd by the open UFR algorithm is closer to the Nelson, Siegel and Svensson asymptotic rate, we can no longer denote a single structure that better acts as a Nelson, Siegel and Svensson curve. In such a case, the Nelson, Siegel and Svensson curve has a rather peculiar diminishing evolution in the very long run. Although, one can easily notice it's very high level in the medium term.
A closer look to the results is given in the two final graphs of this chapter. In Figure 5.28 we find the resuts achieved for short maturities.

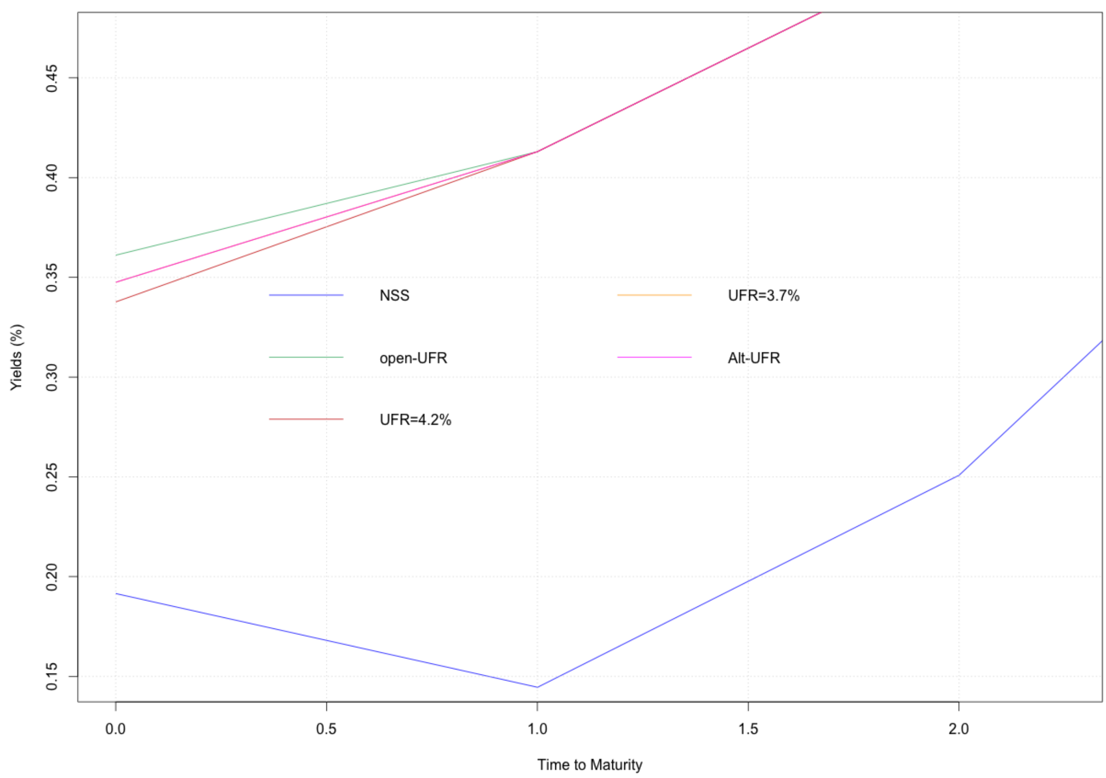

Figure 5.28: Details of the extrapolation of the term structures produced by all analyzed models on term short maturities between 0 years and 2 years.

In the above graph, we can once again easily notice the wide difference produced between the ECB model and Smith-Wilson methodologies.

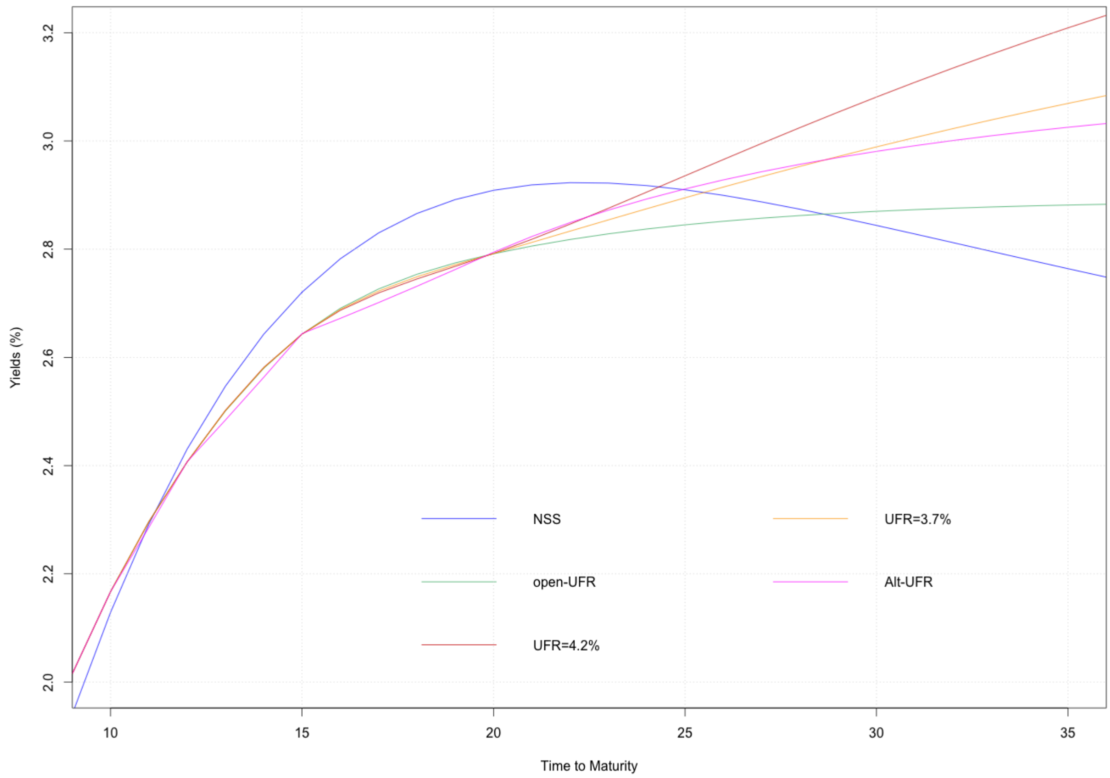

Figure 5.29: Details of the extrapolation of the term structures produced by all analyzed models on medium term maturities between 10 years and 35 years.

The curves' position in the lower graph has little resemblance with what observed previously in Figure 5.26 for short term maturities. Although, the ECB strcuture produces a particularly high hump between maturity 12 and 25, there is no clear dominance of any curve up until a 25 year maturity, when the curves reposition as in Figure 5.26. But still, without generating the same gap between the structures. Consequently, this supports the assumption that for positive market rates the outcome of all mentioned models tend to produce more homogeneous interest rates. Whereas, when the input data includes negative rates, a serious divergence between some term structures is produced primarily, in the long run.

Chapter 6

Conclusion

Long term liabilities insurers and pension funds are faced with new challenges raised by the recent European solvability framework and by the forthcoming international accounting framework, IFRS4 Phase 2. Both frameworks require the computation of a present value of future commitments. Such a value, can only be produced through a discount factor that is deemed to be risk-free.

The present survey has been concerned with a specific methodology adopted by the European Insurance and Occupational Pensions Authority to produce risk-free interest rates. The Smith-Wilson method responds to the European regulation's desire to include liquid market data in a stable and smooth curve. However, the method presents several downturns that are difficult to cope with in a negative short term interest rate market. As a result, long term technical provision managers have requested to overthink the current Solvency directive.

In the first part of this thesis, we have investigated the features of the Smith-Wilson method. While the Smith-Wilson technique responds to the principles claimed by the current legislation in terms of the existing trade-off between smoothness and perfect fit, it presents different issues. The main problematic characteristic resulted the apriority and high level of the asymptotic rate. For this explicit reason, we have searched for methodologies that do not assume a fixed asymptotic rate. But still, capture the basic properties required by law. A recent study carried out in 2016, delivered two different redefinitions of the Smith-Wilson method with the purpose of producing a smoother curve and a market consistent asymptotic rate. As such, we have formally and empirically researched over all these methods. Among the models studied, we have also included in the survey the ECB's term structure technique: The Nelson, Siegel and Svensson model. The Nelson, Siegel and Svensson is a model that creates a smooth and stable curve. Nonetheless, it is exposed to fitting criticisms. Contrarily, the Smith-Wilson method perfectly fits all liquid points.

However, according to the methodology used the results undeniably support the market operators concerns. A 4.2% is indeed an inconsistent asymptotic rate for the euro currency that adversely affects long term expectations. Despite the fact that EIOPA is reducing the current UFR level to 3.7%, it still doesn't match with the present market situation, let alone its apriority that leaves sufficient room for political maneuvering.
An overperformance in terms of market consistency is gained through models that do not require a fixed asymptotic rate and still reach all liquid points. Yet, these methods posse a challenge in terms of the optimization of the speed of convergence and fall short of long term stability. We have argued that a given asymptotic rate governs the stability of the structure in the long term and incorporates macroeconomic assumptions. Adopting a moving UFR may well introduce undesired market volatility in the risk-free interest rate term structure. This could easily translate into a new trade-off between stability and market consistency that can be the subject of future investigation.
Additionally, regarding the findings around the market evolution, we have observed that while the present negative rate market situation produces a considerable gap between the fixed UFR Smith-Wilson method and more market consistent methods, the gap is far less significant in higher rates circumstances. If this is the case, the results reinforce the idea that the Smith-Wilson method is indeed a good response to EIOPA's and the market operators' demands. However, this study confirms the need to define proper objective criteria that create a market compliant long term equilibrium.

6.0.1 Further Research

This final paragraph is entirely dedicated to recommendations on further areas of research around the topic discussed in this thesis.
Issues have occurred when trying to implement methodologies related to the Smith-Wilson method with a non-constant UFR. A major problem was the catch-22 in the optimization of the UFR and the speed of convergence. An interesting research area could investigate on a methodology that optimizes both the velocity of of convergence and the asymptotic rate avoiding to define none of them ex ante. This will enhance the Smith-Wilson method's optimality.
The earlier mentioned "Consultation Paper on the methodology to derive the UFR and its implementation" published on April 2016 brought up a review of EIOPA's applied method against several existing UFR-methodologies: the Berrrie&Hibbert approach, the Dutch UFR, Swiss UFR and the International Association of Insurance Supervisors (IAIS) UFR. Despite the fact that all these methods have been discarded for being incompatible with the norms of the European legislation, a further study may evaluate the features

of these models. The Dutch UFR has been excluded for incorporating the term premia which apparently contradict article 47 of the Delegated Regulation (Directive 2015/35). However, the term premium or additional adjustments could be a valuable subject of survey.

Furthermore, we have discussed how EIOPA has chosen a different UFR level for 2017 through a re-dimensioning of the value of the weights and a re-definition of “mean” used to determine the real rate. A final proposal of survey could, indeed, equally analyse the real rate formulation, aiming at a more accurate evaluation of long term economical expectations.

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Appendix A

Smith-Wilson R Code

Listing A.1: R Code used to build the Smith Wilson Curve as descrived in EIOPA's Technical Documentation

```
#INPUTS
#vector with the swap rates (see EIOPA's TS)
swaps<-c(0.002, 0.00225, 0.003, 0.00425, 0.0055, 0.007,
         0.0085, 0.01, 0.0115, 0.01275, 0.014, 0.01475, 0.01575,
         0.0165, 0.017, 0.0175, 0.018, 0.01825, 0.0185, 0.01875)

#vector of cash flow
vec.cf<-c(0.002, 0.00225,0.003, 0.00425, 0.0055, 0.007,
         0.0085, 0.01, 0.0115, 0.01275, 0.014, 0.01475, 0.01575,
         0.0165, 0.017, 0.0175, 0.018, 0.01825, 0.0185, 0.01875)

#vector of maturities for observed swaps
maturities<-c(seq(1:20))

#vector of prices
prices<-c(rep(1, length(swaps)))

#time vectors needed to create the cash flow matrix
tj<-seq(1:length(swaps))

#vector maturities to LLP
t<-seq(1:length(vec.cf))
```

```
#last liquid point
LLP<-20

#START
#function that creates the cash flows of swaps
fCashFlowMatrix <- function(s=c(),tt,ttj,n) {
 M<-matrix(, length(ttj), length(tt))
  for(i in 1:length(ttj)) {
    M[i,]<-c(rep(s[i],n[i]-1), 1+s[i],
     rep(0, length(M[i,])-n[i]))
  }
  return(M)
}

#cash flow matrix
cf<-fCashFlowMatrix(swaps, t, tj, maturities)

#transposed matrix
tcf<-t(cf)

#function for calculating the Heart of Wilson in one point
fHeart.v.u<-function(v, u, a){
  heart=a*min(v,u)-exp(-a*max(v,u))*0.5*
     (exp(a*min(v,u))-exp(-a*min(v,u)))
  return(heart)
}

#Heart of Wilson for u as vector
fHeart.v<-function(v, u=c(), a){
  heart.v<-NULL
  for(i in 1:length(u)){
     heart.v[i]=fHeart.v.u(v, u[i], a)
  }
  return(heart.v)
}

#function for the derivative of the Heart of Wilson
```

```
fGuv<-function (v, u, a) {
  guv<-NULL
  if(v<=u){
    guv=a-a*(exp(-a*u))*cosh(a*v)
  }
  else guv=a*exp(-a*v)*(sinh(a*u))
  guv
}

#function for the derivative of the Heart of Wilson
#with u as vector
fGuv.v<-function(v, u=c(), a){
  guv.v<-NULL
  for(i in 1:length(u)){
        guv.v[i]=fGuv(v, u[i], a)
  }
  return(guv.v)
}

#function for the Wilson funtion in one point
fWilson<-function (v, u, ufr, alpha) {
  wilson=exp(-ufr*(v+u))*fHeart.v.u(v, u, alpha)
  return(wilson)
}

#function for the Wilson matrix
fWilsonMatrix<-function (tt=c(), ufr, alpha){
 W=matrix(, length(tt), length(tt))
  for(k in 1:length(tt)){
    Wv=c()
     for(i in 1:length(tt)){
       Wv[i]<-fWilson(tt[k], tt[i], ufr, alpha )
     }
    W[k,]<-Wv
  }
  return(W)
}
```

```
#Wilson Matrix calculus
UFR=log(1+0.042)
alpha=0.123760 #given
wilson<-fWilsonMatrix(t, UFR, alpha)

#function to calculate the parameter zeta
zeta.function<-function(X=matrix(), W=matrix(), pr, ufr, u){
  mu<-exp(-ufr*u)
  zeta<-(solve(X%*%W%*%t(X)))%*%(pr-X%*%mu)
  return(zeta)
}

#calculate parameters
zeta<-zeta.function(cf, wilson, prices, UFR, t)

#funtion to compute the quasi-constant k:
kfunction<-function(a, u=c(), ufr, X=matrix(), z){
  mu<-exp(-ufr*u)
  dmu<-diag(mu)
  Q<-dmu%*%t(X)
  k<-(1+a*u%*%Q%*%z)/((sinh(a*u))%*%Q%*%z)
  return(k)
}

#check k
k<-kfunction(alpha, t, UFR, cf, zeta) #OK

#check Qzeta
mu<-exp(-UFR*t)
dmu<-diag(mu)
Q<-dmu%*%t(cf)
Q%*%zeta #OK

#function for the convergence point
cpt.function<-function(llp){
  T<-max(llp+40,60)
```

```
  return(T)
}
T<-cpt.function(LLP)

#function for the convergence period
cpd.function<-function(llp){
  S<-max(40,60-llp)
  return(S)
}
S<-cpd.function(LLP)

#function for the upper forward intensity
f.function<-function(w, a, K, v){
  f=w+(a/(1-K*exp(a*v)))
  return(f)
}

#check
g.alpha.1<-abs(f.function(UFR, alpha, k, T)-UFR)

#function g.alpha
g.alpha.function<-function(a){
  wilson<-fWilsonMatrix(t, UFR, a)
  zeta<-zeta.function(cf, wilson, prices,UFR, t)
  k<-kfunction(a, t, UFR, cf, zeta)
  sg <- a/(abs(1-k*exp(a*T)))
  sg
}

#check
g.alpha.2<-g.alpha.function(alpha)

#euristic solution
g.alpha.function(0.05)
a.lower=0.05
while (g.alpha.function(a.lower)>=0.0001) {
  a.lower=a.lower+0.000001
```

```
}

#optimized alpha
a.lower->alpha.opt
alpha.opt #OK

#price function t given time v
fPresentValue<-function(ufr, v, u, q, z, alp) {
  pv<-exp(-ufr*v)*(1+fHeart.v(v, u, alp)%*%q%*%z)
  return(pv)
}

#spot intensity function
fYieldIntensity<-function(ufr, v, u, q, z, alp) {
  yi=(-log(fPresentValue(ufr, v, u, q, z, alp)))/v
  return(yi)
}

#yield intensity at time 0
one.vec<-rep(1, length(zeta))
y.zero<-UFR-alpha*one.vec%*%Q%*%zeta+alpha*
        (exp(-alpha*t)%*%Q%*%zeta)

#yield intensity
x1<-c(0,seq(1:120))
y1 <- c()
y1[1]<-y.zero*100
for (i in 1:120) {
  y1[i+1] <- c(fYieldIntensity(UFR, x1[i+1], t, Q, zeta, alpha)*
                100)
}

#annualized yield rate function
fAnnualRate<-function(ufr, v, u, q, z, alp){
  ar=(fPresentValue(ufr, v, u, q, z, alp))^(-1/v)-1
  return(ar)
}
```

```r
#annual rate at time 0
rate.zero<-exp(y.zero)-1

#annual rates
y2 <-c()
y2[1]<-rate.zero*100
for (i in 1:120) {
  y2[i+1] <- c(fAnnualRate(UFR, x1[i+1], t, Q, zeta, alpha)*100)
}

#forward intensity function
fForwardIntensity<-function(ufr, v, u, q, z, alp){
  fif=ufr-((fGuv.v(v, u, alp)%*%q%*%z)/(1+fHeart.v(v, u, alp)
        %*%q%*%z))
  return(fif)
}

#forward intensity
y3 <-c()
y3[1]<-y.zero*100
for (i in 1:120) {
  y3[i+1] <- c(fForwardIntensity(UFR, x1[i+1], t, Q, zeta, alpha)
                *100)
}

#plot
plot(x1, y3, type="l", pch=19,
     xlim=c(0, 120), ylim=c(-0.22, 2), xlab="Maturity",
     ylab="Yields (%) ", col="red")
lines(x1, y2, col="green")
lines(x1, y1, col="blue")
legend('bottomright', c("forward intensity", "annual rate",
           "yield to maturity"), bty="n", lty=c(1,1,1),
         col=c("red", "green", "blue" ))
grid()
#END
```

Appendix B

Alternative Methods R Code

Listing B.1: R Code used to determine the optimized UFR

```
#START
#check if the cash flow matrix is invertible
solve(cf) #OK

#calculate phi
phi<-solve(cf)%*%prices

#create function with the UFR as input
finf.function<-function(ufr){
  D<-diag(exp(-ufr*t))
  U<-diag(t)
  e<-rep(1,length(t))
  out=t(prices-cf%*%D%*%e)%*%solve(cf%*%D%*%Heart.of.Wilson
        %*%D%*%t(cf))%*%(cf%*%D%*%U)%*%(e+(Heart.of.Wilson%*%
        D%*%t(cf)%*%(solve(cf%*%D%*%Heart.of.Wilson%*%D%*%t(cf)))
        %*%(prices-cf%*%D%*%e)))
  retrun(out)
}

#get the root
library(rootSolve)#to use multroot function
UFR.OPT<-multiroot(finf.function, 0.04)$root

#simplified version of the function with the UFR as input
```

```
#only if the cash flow matrix is invertible
finf.function<-function(ufr){
  D<-diag(exp(ufr*t))
  U<-diag(t)
  e<-rep(1,length(t))
  out=t(D%*%phi-e)%*%solve(Heart.of.Wilson)%*%U%*%(D%*%phi)
  return(out)
}

#check
multiroot(finf.function, 0.04)$root
#END
```

Listing B.2: R Code used to determine the UFR according to the smoothess yield curve method

```
#START
#check if the cash flow matrix is invertible
solve(cf) #OK

#calculate phi
phi<-solve(cf)%*%prices
phi<-as.vector(phi)

#function to find the UFR
fUFR<-function(ph, u, a, HoW){
  yk=-log(ph)/u
  vec<-1/(a*u)
  tGk.j<-solve(HoW%*%diag(vec))
  vk<-c()
  for(i in 1:20){
    vk[i]<-sum(tGk.j[,i])
  }
  vk
  sumvk<-sum(vk)
  v0<-1-sumvk
  y0<--0.002562911#given
  ufr<-vk%*%yk+v0*y0
  return(ufr)
}

#optimal UFR
UFR.OPT<-fUFR(phi,t,alpha, Heart.of.Wilson)
UFR<-UFR.OPT
#END
```

Appendix C

Nelson, Siegel and Svensson Estimation and Curve R Code

Listing C.1: R Code used to build the NSS curve and to estimate the NSS parameters

```
#Input data
swaps.original<-c(-0.19,-0.15, -0.08, 0.01,0.13, 0.26,
                          0.39, 0.52, 0.64, 0.75)/100

#interpolated data
swaps<-c(-0.19,-0.15, -0.08, 0.01, 0.13, 0.26, 0.39,
        0.52, 0.64, 0.75, 0.875,0.93, 0.993333333, 1.056666667,
        1.12, 1.15, 1.18, 1.21, 1.24, 1.27)/100

#START
#BOOTSTRAPING
#calulate discount factor
v<-c()
v[1]<-1/(1+swaps[1])
v[2]<-(1-swaps[1]*v[1])/(1+swaps[1])
for(i in 3:length(swaps)){
  v[i]<-(1-swaps[i]*sum(v[1:(i-1)]))/(1+swaps[i])
}

#calculate yields
y<-c()
y[1]<-swaps[1]
```

```
y[2]<-((1+swaps[2])/(1-swaps[2]*v[1]))^(1/2)-1
for(k in 3:length(swaps)){
  y[k]<-((1+swaps[k])/(1-swaps[k]*sum(v[1:(k-1)])))^(1/k)-1
}

#maturities
tau<-c(seq(1:length(y)))

#function to create NSS values
NSS = function(tau, parameters){
  beta0 = parameters[1]
  beta1 = parameters[2]
  beta2 = parameters[3]
  beta3 = parameters[4]
  tau1 = parameters[5]
  tau2 = parameters[6]
  r1 = tau/tau1
  r2 = tau/tau2
  rr1 = 1/r1
  rr2 = 1/r2
  delta = beta0 + beta1*exp(-r1) + beta2*(r1)*exp(-r1) +
  beta3*r2*exp(-r2)
  h = beta0 + beta1*rr1*(1-exp(-r1)) + beta2*(rr1*(1-exp(-r1))
  - exp(-r1))
  +  beta3*(rr2*(1-exp(-r2)) - exp(-r2))
  i0 = which(tau==0)
  delta[i0] = beta0+beta1
  h[i0]     = beta0+beta1
  rate = exp(h/100) - 1
  v = exp(-h/100*tau)
  out = data.frame(delta=delta, h=h, rate=rate, v=v)
  return(out)
}

#ESTIMATION
#function to estimate the parameters
f.est = function(parameters) {
```

```
  par = parameters
  delta = array(0, dim=length(tau))
  for (i in 1:length(tau)){
    mod = NSS(tau[i], par)$tasso
    mkt = y[i]
    delta[i] = (mkt-mod)^2
  }
  out=sum(delta)
  return(out)
}

#constraints
par0 = c(1, -2.5 , 12, -16.405789, 1.2, 1.4)
plow = c(-5, -4, -100, -100, 0.1, 0.01)
pupp = c(400, 1,  100,  100, 10, 10)

#optimized parameters
est <- optim(par=par0, fn=f.est, method="L-BFGS-B",lower=plow,
upper=pupp)

#NSS parameters
beta0 = est$par[1]
beta1 = est$par[2]
beta2 = est$par[3]
beta3 = est$par[4]
tau1  = est$par[5]
tau2  = est$par[6]

#building the curve
#vector of parameters
par = c(beta0, beta1, beta2, beta3, tau1, tau2)

#extreme values
h0 = beta0+beta1
hinf = beta0

#vector of maturities to plot
```

```
tau1 = c(0,seq(1:120))

#data frame containing the yields and prices
outNSS1 <- NSS(tau1, par)

#plot rate curve
plot(tau1, outNSS1$rate*100, type="l",
     xlab="Time to Maturity",
     ylab="Spot rate ", col="blue")
grid()

#plot all three curves
plot(tau1, outNSS1$h, type="l", xlim=c(0, 120),
     ylim=c(-0.2, 2), xlab="Time to Maturity",
     ylab="Yields", col="blue")
lines(tau1, outNSS1$delta, col="mediumseagreen")
lines(tau1, outNSS1$rate*100, col="indianred4")
legend('bottomright', c("forward intensity", "spot rate",
     "annual rate"), bty="n", lty=c(1,1,1),
     col=c("mediumseagreen", "blue", "indianred4"))
grid()
#END
```