Technische Universität München

Department of Mathematics

Life insurance products with capital guarantees: Stackelberg equilibria between reinsurer and insurer

Master Thesis

by

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I hereby declare that this thesis is my own work and that no other sources have been used except those clearly indicated and referenced.

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Abstract

In this master thesis, we study two Stackelberg games between a reinsurance company and an insurance company. In both games, each party maximizes its expected utility of terminal wealth. However, the reinsurer is the leader and the insurer is the follower, i.e., the reinsurer knows for each of its actions the rational response of the insurer, which can be seen as an advantage. In academic literature, such games are mainly modeled on the whole level of aggregated claims and the reinsurance strategy is adjusted dynamically. However, in practice the reinsurance strategy is adjusted at discrete time points and is related to a specific product or product line. Therefore, our first Stackelberg game is an example from the existing literature. In the second Stackelberg game, we consider an innovative reinsurance-insurance problem in a more realistic scenario, where the reinsurance is part of the design of a single life insurance product with a capital guarantee and the reinsurance is only traded at the product settlement date without further adjustments during the investment period. Moreover, we model the situation when the reinsurance is not written directly on the insurer's portfolio but on a benchmark portfolio, which is highly correlated to it. We derive solutions to the optimization problem in semi-closed form by combining and generalizing non-standard portfolio optimization techniques. In the numerical studies, we find that in the Stackelberg equilibrium the reinsurer chooses the maximal reinsurance premium at which the insurer is buying the full amount of reinsurance. We also examine the sensitivity of the Stackelberg equilibrium with respect to changes in the market and product parameters.

Zusammenfassung

In dieser Masterarbeit untersuchen wir zwei Stackelberg-Spiele zwischen einem Rückversicherungsunternehmen und einem Versicherungsunternehmen. In beiden Spielen maximiert jede Partei ihren erwarteten Nutzen des Endvermögens. Allerdings ist der Rückversicherer das marktführende Unternehmen und der Versicherer das marktfolgende Unternehmen, d.h. der Rückversicherer kennt für jede seiner Aktionen die rationale Reaktion des Versicherers, was als Vorteil angesehen werden kann. In der wissenschaftlichen Literatur werden solche Spiele auf dem Gesamtschaden des Versicherungsunternehmens modelliert und die Rückversicherungsstrategie wird dynamisch angepasst. In der Praxis wird die Rückversicherungsstrategie jedoch zu diskreten Zeitpunkten angepasst und ist auf ein bestimmtes Produkt oder eine Produktlinie bezogen. Daher ist unser erstes Stackelberg-Spiel ein Beispiel aus der bestehenden Literatur. Im zweiten Stackelberg-Spiel betrachten wir ein innovatives Rückversicherungsproblem in einem realistischeren Szenario, bei dem die Rückversicherung Teil der Gestaltung eines einzelnen Lebensversicherungsprodukts mit Kapitalgarantie ist und nur zum Zeitpunkt der Produktabwicklung ohne weitere Anpassungen während der Investitionsperiode gehandelt wird. Des Weiteren modellieren wir die Situation, wenn die Rückversicherung nicht direkt auf das Portfolio des Versicherers gezeichnet wird, sondern auf ein Benchmark Portfolio, das stark mit diesem korreliert ist. Wir erhalten Lösungen für das Optimierungsproblem in halb-geschlossener Form, indem wir nicht-standardisierte Portfolio-Optimierungstechniken kombinieren und verallgemeinern. In den numerischen Studien finden wir, dass im Stackelberg-Gleichgewicht der Rückversicherer die maximale Rückversicherungsprämie wählt, zu der der Versicherer vollständig rückversichert. Wir untersuchen auch die Sensitivität des Stackelberg-Gleichgewichts in Bezug auf Änderungen der Markt- und Produktparameter.

Contents

Chapter 1

Introduction

Motivation

In the primary insurance business, an insurance company and a client agree on an insurance contract. The insurance contract contains the premium, which the client pays to the insurer, and the risk of the claim, which is transferred from the client to the insurer. In contrast, a reinsurance contract is created between a reinsurance company and an insurance company. Similar to the primary insurance business, the insurer pays a premium to the reinsurer and transfers a part of its risk to the reinsurer. The insurer uses reinsurance to reduce its actuarial risk, which comes from the uncertainty of the risk occurrence and the amount of the claim, and its financial risk, which comes from the investment in the financial market (cf. Bai et al. [2019]). In the reinsurance contract, the insurer chooses the amount of the risk that should be transferred to the reinsurer. This is called the reinsurance strategy of the insurer. In comparison, the reinsurer selects the premium, which is payed from the insurer to the reinsurer. This premium is called the reinsurance premium strategy. Often the reinsurer calculates its premium as the expected loss plus a safety loading. In this case, selecting the reinsurer's strategy is equivalent to selecting a safety loading. (cf. Albrecher et al. [2017])

The aim of the reinsurer and the insurer is to maximize the expected utility of their terminal surplus or terminal wealth. In practice, both parties need to agree in the reinsurance contract. If the reinsurer takes into account only its utility when creating a reinsurance contract, the reinsurance premium might be too high for the insurer to accept the contract and, therefore, the insurer may not be willing to buy as much reinsurance as the reinsurer expects. On the other hand, if the reinsurance premium is too low, the insurer chooses to buy reinsurance for more claims, but the net profit of this deal may be suboptimal for the reinsurer due to the low price. Hence, the utility of the reinsurer would be lower than expected. Accordingly, when considering the optimization problems of the reinsurer and the insurer separately, the optimal choice of one party might not necessarily be optimal for the other party and, therefore, the other party would not agree in the reinsurance contract (cf. Bai et al. [2019]). Due to that, it is important to consider the relationship between the reinsurer and the insurer when solving their investment-reinsurance optimization problems. Thus, we consider a game between a reinsurer and an insurer.

The insurance market consists of several thousands primary insurance companies and about 200 reinsurance companies (cf. Albrecher et al. [2017]). Hence, the reinsurance company has rather a monopoly position in the reinsurance contract (cf. Chen and Shen [2018]). In addition, the reinsurance company is often larger than the primary insurance company and acts international, whereas the insurer often acts on a national level (cf. Albrecher et al. [2017]). Therefore, the reinsurer has the ability to assess how the insurer will react, which results in a hierarchical information structure. Due to these asymmetries, the reinsurer "dominates" the insurer and the reinsurance contract can be considered as a hierarchical game between both parties. This is exactly the concept of Stackelberg games: Stackelberg games have a hierarchical structure, where the leader (the reinsurer) "dominates" the follower (the insurer). This means that the leader moves first and selects its strategy knowing the future optimal response of the follower, whereas the follower moves afterwards and chooses its strategy depending on the choice of the leader (cf. Osborne and Rubinstein [1994]).

In reality, an insurance company buys reinsurance on a single insurance product or business line. For example, the primary insurer ERGO offers the life insurance product "ERGO Rente Garantie". In this life insurance product, the insurer obtains a contribution payment from a representative client and in return, the representative client receives a guarantee of 80% up to 100% of its initial contribution at the end of the insurance contract. In return, the insurer invests the representative client's contribution in bonds and funds, and buys reinsurance from Munich Re (cf. Escobar-Anel et al. [2021]). Therefore, it is realistic to consider a Stackelberg game between a reinsurer and an insurer, where the insurer offers an insurance product to a representative client and buys reinsurance from the reinsurer.

Overview of literature on insurance-reinsurance Stackelberg games

Chen and Shen [2018] solve a dynamic reinsurance optimization problem of the reinsurer and the insurer in the framework of a Stackelberg game. The reinsurer aims to find its optimal reinsurance premium strategy while the insurer's goal is to find its optimal proportional reinsurance strategy. The reinsurance strategy and the reinsurance premium strategy are adjusted dynamically. Since the authors assume that all coefficients are stochastic, the so-called stochastic Hamilton-Jacobi-Bellman approach is introduced to solve the optimization problem. Bai et al. [2019] extend the model of Chen and Shen [2018] in the special case of deterministic, constant model coefficients (i.e., coefficients of the claim process, insurance premium, financial market and utility function). The researchers add a second insurance company that competes with the other insurance company, consider reinsurance-investment optimization problems and include time delay. The optimization problems are solved by the Hamilton-Jacobi-Bellman approach. Both papers (Chen and Shen [2018] and Bai et al. [2019]) work with a proportional reinsurance on the whole level of the aggregated claims. In Chapter 3, we give a more detailed overview of literature on insurance-reinsurance Stackelberg games.

Overview of literature on portfolio optimization

Escobar-Anel et al. [2021] focus on a product-based problem and consider an insurer that buys reinsurance to protect itself against the portfolio loss in a life insurance product with a capital guarantee. As in Chen and Shen [2018] and Bai et al. [2019], the reinsurance strategy is adjusted dynamically. In this paper, the researchers focus on the insurer's optimization problem without the interaction with the reinsurer, i.e., the researches did not consider a Stackelberg game between a reinsurer and the insurer. The reinsurance is modeled by a put option, where the underlying is given by a benchmark portfolio and the strike price by the capital guarantee. The benchmark portfolio is not equal to the insurer's portfolio but is highly correlated to it. For the insurer, the researchers consider a reinsurance-investment optimization problem with a Value-at-Risk and a no-short-selling constraints. In Chapter 4, we give a more detailed overview to Escobar-Anel et al. [2021].

Scope

To the best of our knowledge, the current literature on Stackelberg games in the context of insurance and reinsurance focuses on insurance companies, which buy reinsurance on the whole level of their aggregated claims and adjust their reinsurance strategy dynamically. As an example, we provide in this thesis a detailed overview of a special case of a Stackelberg game and its solution presented in Bai et al. [2019].

As mentioned before, in reality reinsurance is bought for potential losses within single insurance products or business lines, not on the whole company level.

In general, a reinsurance contract cannot be adjusted dynamically. At the beginning, the reinsurance company and the insurance company agree in a reinsurance contract, which can be adjusted at regular intervals, e.g., annually. Hence, it is more realistic to consider a situation, where the insurer buys reinsurance at the beginning without the possibility of dynamic adjustments.

To ensure the capital guarantee of the life insurance product, the insurer buys a reinsurance from the reinsurer. The reinsurance is modeled by a put option, where the strike price is given by the capital guarantee. The put option is a protection against the downside risk of the insurer's (product specific) investment portfolio. We assume that the underlying of the put option is not equal to the insurer's portfolio, since the insurer follows its individual investment strategy that the reinsurer might consider too risky or does not even know. Therefore, the reinsurer offers reinsurance on a standard portfolio, where the risks can be better assessed and hedged. For example, the asset manager of the insurer might overweight specific economy sectors, e.g. the technological stocks. However, the reinsurer agrees to provide protection only for a well-diversified portfolio replicating the performance of the whole economy, e.g. the DAX performance. (cf. Escobar-Anel et al. [2021])

Hence, the second Stackelberg game in the master thesis is modeled by a more realistic scenario than in the first Stackelberg game. According to the previous aspects, we assume that the insurer buys a reinsurance as a part of a single life insurance product with a capital guarantee. The reinsurance contract is modeled by a put option. We allow that the insurer can only buy reinsurance at the beginning of the reinsurance contract without future adjustments. The strike price of the put option is given by the capital guarantee and the underlying by a benchmark portfolio, which is not equal to the insurer's portfolio but has a high correlation with it. Therefore, the reinsurer can hedge its short position in the put option by investing in the benchmark portfolio. In addition, the reinsurer can invest in the same risky assets as the insurer.

Innovation

From a practical point of view, we extend the literature on Stackelberg games between a reinsurer and an insurer by setting up and solving a more realistic game, where reinsurance

- is written on potential losses of an insurance product related to a specific predetermined portfolio rather than the whole aggregated loss of an insurer, and
- is purchased at the beginning of the investment period and not dynamically traded at future points.

From a theoretical point of view, we extend the existing literature by solving novel optimization problems that are subproblems in the above described Stackelberg game.

We use the idea of backward induction to solve the Stackelberg game in two steps. First, we solve the optimization problem of the insurer for any possible strategy of the reinsurer. Afterwards, we find the optimal strategy of the reinsurer by solving the optimization problem of the reinsurer, knowing the optimal response of the insurer. In addition to the reinsurer's standard investment in the financial market, it has a fixed short position in a put option. Hence, we solve the optimization problem of the reinsurer by using an idea introduced in Korn and Trautmann [1999]. When solving the optimization problem of the insurer we are dealing with two peculiarities: a fixed-term investment (i.e., the insurer buys reinsurance only at the beginning of the contract) and a portfolio constraint (i.e., the insurer follows an individual investment strategy and is not willing to follow a standard reinsurable strategy). Therefore, we use a combination of the methods introduced in Desmettre and Seifried [2016] and Cvitanić and Karatzas [1992].

Thesis structure

Chapter 1 introduces the mathematical and stochastic preliminaries as well as the diffusion approximation of a claim process. It also summarizes the papers, which we use in Chapter 4 for finding the Stackelberg equilibrium. Chapter 2 gives a brief general introduction to Stackelberg games and describes the solution approach called backward induction. In Chapter 3, we solve a Stackelberg game between a reinsurer and an insurer, which is a special case of Bai et al. [2019]. We use the Hamilton-Jacobi-Bellman method to solve the problem and afterwards verify that the solution is indeed a solution to the Stackelberg game. In Chapter 4, we consider another Stackelberg game between a reinsurer and an insurer which is based on the framework and idea of Escobar-Anel et al. [2021]. First, we solve the optimization problem of the reinsurer and insurer theoretically. Afterwards, we analyze the solution to the Stackelberg game numerically and investigate its sensitivity with respect to the risk aversion of the reinsurer as well as the insurer, and with respect to model parameters influencing the fair price of the put option. Chapter 5 concludes the master thesis.

1.1 Mathematical Preliminaries

In this section, we state some basic mathematical preliminaries for the master thesis.

Definition 1.1. The inner product of the Hilbert space $\mathbb{R}^n$ is defined by

$$\langle x, y \rangle := x^\top y$$

and the norm by

$$\|x\| := \sqrt{\langle x, x \rangle}$$

for $x, y \in \mathbb{R}^n$.

Theorem 1.2 (Inverse of 2×2-matrix, p. 207 in Karpfinger and Stachel [2020])**.** Let A be a real-valued 2×2-matrix given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{R}$. Then the inverse is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Definition 1.3 (Gradient, p. 8 in Ulbrich and Ulbrich [2012])**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous differentiable function. The gradient of f with respect to $x \in \mathbb{R}^n$ is defined by

$$\nabla_x f(x) := \begin{pmatrix} f_{x_1}(x) \\ \vdots \\ f_{x_n}(x) \end{pmatrix} \in \mathbb{R}^n.$$

We denote by $f_{x_i}(x)$ $(= \frac{\partial}{\partial x_i} f(x))$ the derivative of f with respect to the ith component of x.

Theorem 1.4 (Characterizing a maximum, Theorem 16.6 in Forster [1976])**.** Let $f : [a, b] \to \mathbb{R}$ be a twice continuous differentiable function. If it holds for $x \in [a, b]$

(a) $f'(x) = 0$, and

(b) $f''(x) < 0$,

then f has a maximum at point x. We call (a) the first-order optimality condition (FOOC) and (b) the second-order optimality condition. If $f''(x) > 0$ instead of condition (b), then f has a minimum at point x.

Definition 1.5 (Convex set, Definition 6.1 in Ulbrich and Ulbrich [2012])**.** $X \subset \mathbb{R}^n$ is called a convex set if for all $x, y \in X$ and $\lambda \in [0, 1]$

$$\lambda x + (1 - \lambda)y \in X.$$

Definition 1.6 (Concave function, Definition 6.2 in Ulbrich and Ulbrich [2012])**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Then f is called

- convex if for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ it holds

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

- strictly convex if for all $x, y \in \mathbb{R}^n$ with $x \neq y$ and $\lambda \in (0, 1)$ it holds

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y).$$

- concave if $-f$ is convex, i.e., for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ it holds

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y).$$

- strictly concave if $-f$ is strictly convex, i.e., for all $x, y \in \mathbb{R}^n$ with $x \neq y$ and $\lambda \in (0, 1)$ it holds

$$f(\lambda x + (1 - \lambda)y) > \lambda f(x) + (1 - \lambda)f(y).$$

Theorem 1.7 (Convex functions, Theorem 6.3 in Ulbrich and Ulbrich [2012])**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. The function f is convex if and only if for all $x, y \in \mathbb{R}^n$ it holds

$$\nabla_x f(x)^\top (y - x) \leq f(y) - f(x),$$

where $\nabla_x f$ is the gradient of f with respect to x.

Remark. Since f is concave if $-f$ is convex, we have that f is concave if and only if for all $x, y \in \mathbb{R}^n$ it holds

$$\nabla_x f(x)^\top (y - x) \geq f(y) - f(x).$$

Definition 1.8 (Polynomially bounded, p. 7 in Desmettre and Seifried [2016])**.** A function $h : (0, \infty) \to \mathbb{R}$ is called polynomially bounded at 0 and ∞ if there exists $c, k \in (0, \infty)$ such that for all $y \in (0, \infty)$

$$|h(y)| \leq c\left(y + \frac{1}{y}\right)^k.$$

Definition 1.9 (Utility functions, Definition 5.1 in Korn [2014])**.** Let $U : (0, \infty) \to \mathbb{R}$ be a strictly concave, continuous differentiable and satisfies the Inada conditions, i.e.,

$$U'(0) := \lim_{x \downarrow 0} U'(x) = +\infty,$$
$$U'(\infty) := \lim_{x \to \infty} U'(x) = 0.$$

U is called a utility function.

Example 1.10 (Utility function)**.** The most common utility functions are:

- Logarithmic utility function: $U(x) = \ln(x)$ for $x \in (0, \infty)$. (cf. Example 5.2 in Korn [2014])
- Power utility function: $U(x) = \frac{1}{b}x^b$ for $x \in (0, \infty)$ and $b \in (-\infty, 1)\backslash\{0\}$. (cf. Example 5.2 in Korn [2014])
- Exponential utility function: $U(x) = -\frac{1}{\beta}e^{-\beta x}$ for $x \in (0, \infty)$ and $\beta > 0$. (cf. p. 919 in Chen and Shen [2018])

In the master thesis, we will use the exponential and power utility function.

Theorem 1.11 (Weierstrass, Corollary 2.32 in Aliprantis and Border [2006])**.** Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ a continuous function. Then there exist points $\underline{z}, \overline{z} \in [a, b]$ such that for all $z \in [a, b]$

$$f(\underline{z}) \leq f(z) \leq f(\overline{z}).$$

Theorem 1.12 (Berge's Maximum Theorem, Theorem 16.31 in Aliprantis and Border [2006])**.** Let $X, Y \subseteq \mathbb{R}$ be an interval and $c : X \to \mathbb{R}$, $f : X \times Y \to \mathbb{R}$ continuous functions. Define the value function $m : Y \to \mathbb{R}$ by

$$m(y) = \max_{x \in [0, c(y)]} f(x, y).$$

Assume that the maximum is unique (e.g. if f is strictly concave with respect to y). Then the function defined by

$$M(y) = \underset{x\in[0,c(y)]}{\arg\max}\ f(x,y)$$

is continuous.

1.2 Stochastic Preliminaries

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$, i.e., $\mathcal{F}_t \subset \mathcal{F}$ are σ-algebras and $\mathcal{F}_s \subset \mathcal{F}_t$ for $0 \leq s \leq t \leq T$ (cf. Definition 9.9 in Klenke [2013]).

Definition 1.13 (Distribution, Definition 1.103 and 13.21 in Klenke [2013])**.** (a) Let X be a real-valued random variable. The distribution function of X is given by $F : \mathbb{R} \to [0,1]$, $x \mapsto \mathbb{Q}(X \leq x)$.

(b) Let X and Y be two real-valued random variables with distribution functions F_X and F_Y, respectively. X and Y are called identically distributed (Notation: $X \overset{d}{=} Y$) if $F_X = F_Y$ holds.

(c) We say that the real-valued random variables X and Y are i.i.d. if X and Y are independent and identically distributed.

(d) Let $(X_n)_{n\in\mathbb{N}}$ and X be real-valued random variables with distribution functions F_n and F, respectively. We say that $(X_n)_{n\in\mathbb{N}}$ converges in distribution to X (Notation: $X_n \overset{n}{\to} X$ as $n \to \infty$) if it holds for all continuity points x of F

$$F_n(x) \to F(x).$$

Theorem 1.14 (Central Limit Theorem, Theorem 15.37 in Klenke [2013])**.** Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. real-valued random variables with $\mu := E[X_1]$ and $\sigma^2 := \mathrm{Var}(X_1) > 0$. Then for $S_n := \sum_{k=1}^n X_k$

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \overset{d}{\to} Z,$$

where $Z \sim \mathcal{N}(0,1)$, i.e., the distribution function of Z is given by

$$F_Z(x) = \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

for $x \in \mathbb{R}$.

Definition 1.15 (Poisson process, Definition 5.33 in Klenke [2013]). Let $N = (N(t))_{t\in[0,T]}$ be a $\mathbb{N}_0$-valued stochastic process. N is called a Poisson process with intensity $\lambda > 0$ if

(a) $N(0) = 0$ $\mathbb{Q}$-a.s.,

(b) N has independent increments, i.e., for all $0 = t_0 < \ldots < t_n$ the family $(N(t_i) - N(t_{i-1})_{i=1,\ldots,n}$ is independent,

(c) N has stationary and Poisson distributed increments, i.e.,

$$N(t) - N(s) \overset{d}{=} N(t-s) \sim \mathrm{Poi}(\lambda(t-s)),$$

where $X \sim \mathrm{Poi}(\lambda)$ means

$$\mathbb{Q}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!} \text{ for all } k \in \mathbb{N}_0.$$

Definition 1.16 (Adapted process, Definition 9.10 in Klenke [2013]). Let $X = (X(t))_{t\in[0,T]}$ be a stochastic process. X is called ($\mathbb{F}$-)adapted if $X(t)$ is $\mathcal{F}_t$-measurable for all $t \in [0,T]$.

Definition 1.17 (Progressively measurable, Definition 2.39 in Korn [2014]). Let $X = (X(t))_{t\in[0,T]}$ be a $\mathbb{R}^n$-valued stochastic process on $(\Omega, \mathcal{F}, \mathbb{Q})$. X is called progressively measurable if for any $t \in [0,T]$ the map

$$X : [0,t] \times \Omega \to \mathbb{R}^n,\ (s,\omega) \mapsto X(s,\omega)$$

is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t \times \mathcal{B}(\mathbb{R}^n)$-measurable.

Theorem 1.18 (Theorem 2.41 in Korn [2014]). If a stochastic process $X = (X(t))_{t\in[0,T]}$ is adapted and continuous, then X is progressively measurable.

Definition 1.19 (Martingale, Definition 2.12 in Korn [2014]). Let $X = (X(t))_{t\in[0,T]}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{Q})$. X is called a ($\mathbb{F}$-)martingale if

(a) X is $\mathbb{Q}$-adapted,

(b) $E[|X(t)|] < \infty$ for all $t \in [0,T]$, and

(c) for all $s,t \in [0,T]$ with $s \leq t$

$$E[X(t)|\mathcal{F}_s] = X(s) \ \mathbb{Q}\text{-a.s.}$$

Definition 1.20 (Brownian motion, Definition 2.6 in Korn [2014]). Let $W = (W(t))_{t\in[0,T]}$ be real-valued stochastic process on $(\Omega, \mathcal{F}, \mathbb{Q})$. W is called a (one-dimensional) Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$ if

(a) $W(0) = 0$ $\mathbb{Q}$-a.s.,

(b) the increments of W are stationary, i.e., for $s \geq t$

$$W(t) - W(s) \sim \mathcal{N}(0, t - s),$$

(c) the increments of W are independent, i.e., for $r \leq u \leq s < t$

$$W(t) - W(s) \text{ is independent of } W(u) - W(r).$$

The $\mathbb{R}^d$-valued stochastic process $W = (W_1(t), \ldots, W_d(t))^\top_{t\in[0,T]}$ is called a d-dimensional Brownian motion if the components W_i are independent (one-dimensional) Brownian motions.

Definition 1.21 (Itô-process, Definition 2.49 in Korn [2014]). Let $X = (X(t))_{t\in[0,T]}$ be an adapted stochastic process. X is called an Itô-process if it holds for all $t \in [0, T]$

$$dX(t) = \mu(t)dt + \sigma(t)dW(t)$$

where $X(0)$ is $\mathcal{F}_0$-measurable and $\mu = (\mu(t))_{t\in[0,T]}$, $\sigma = (\sigma_1(t), \ldots, \sigma_d(t))_{t\in[0,T]}$ are progressively measurable with

$$\int_0^t |\mu(s)|ds < \infty, \ \int_0^t |\sigma_i(s)|^2 ds < \infty \ \mathbb{Q}\text{-a.s.}$$

for all $t \in [0, T]$ and $i = 1, \ldots, d$.

Definition 1.22 (Quadratic covariance, Definition 2.51 in Korn [2014]). Let $X = (X(t))_{t\in[0,T]}, Y = (Y(t))_{t\in[0,T]}$ be two Itô-processes with

$$\begin{aligned} dX(t) &= \mu_X(t)dt + \sigma_X(t)dW(t), \\ dY(t) &= \mu_Y(t)dt + \sigma_Y(t)dW(t). \end{aligned}$$

The stochastic process $\langle X, Y\rangle = (\langle X, Y\rangle_t)_{t\in[0,T]}$ is called quadratic covariance of X and Y if

$$\langle X, Y\rangle_t = \sum_{i=1}^{d} \sigma_{Xi}(t)\sigma_{Yi}(t)dt.$$

$\langle X\rangle := \langle X, X\rangle$ is called the quadratic variance of X.

Theorem 1.23 (Itô's formula, Theorem 2.55 Korn [2014]). Let $X = (X(t))_{t\in[0,T]}$ be a Itô-process and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ a continuous differentiable function with respect to t and a twice continuous differentiable function with respect to x. Then

$$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2} f_{xx}(t, X(t))d\langle X\rangle_t.$$

Theorem 1.24 (Product rule, Corollary 2.56 in Korn [2014]). Let $X = (X(t))_{t\in[0,T]}, Y = (Y(t))_{t\in[0,T]}$ be two Itô-processes. Then

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + d\langle X, Y\rangle_t.$$

Definition 1.25 (Equivalent measures, Definition 2.1 in Zagst [2002]). Let $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ be two probability measures on $(\Omega, \mathcal{F})$.

(a) $\tilde{\mathbb{Q}}$ is called absolute continuous with respect to $\mathbb{Q}$ (Notation: $\tilde{\mathbb{Q}} \ll \mathbb{Q}$) if

$$\mathbb{Q}(A) = 0 \Rightarrow \tilde{\mathbb{Q}}(A) = 0 \;\forall\, A \in \mathcal{F}.$$

(b) $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ are called equivalent measures (Notation: $\tilde{\mathbb{Q}} \sim \mathbb{Q}$) if

$$\mathbb{Q} \ll \tilde{\mathbb{Q}} \text{ and } \tilde{\mathbb{Q}} \ll \mathbb{Q}.$$

Theorem 1.26 (Radon-Nikodym, Theorem 2.2 in Zagst [2002]). Let $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ be two probability measures on $(\Omega, \mathcal{F})$. Then $\tilde{\mathbb{Q}} \ll \mathbb{Q}$ if and only if there exists an integrable function $f \geq 0$ $\mathbb{Q}$-a.s. such that for all $A \in \mathcal{F}$

$$\tilde{\mathbb{Q}}(A) = \int_A f d\mathbb{Q}.$$

f is called the Radon-Nikodym derivative of $\tilde{\mathbb{Q}}$ with respect to $\mathbb{Q}$ (Notation: $f = \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}$).

Theorem 1.27 (Bayes Formula, Theorem 2.7 in Zagst [2002]). Let $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ be two probability measures on $(\Omega, \mathcal{F})$ and $f = \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}$ be a Radon-Nikodym derivative of $\tilde{\mathbb{Q}}$ with respect to $\mathbb{Q}$. Furthermore, let X be an integrable random variable on the probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{Q}})$ and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. Then it holds

$$\mathbb{E}_{\mathbb{Q}}[X \cdot f|\mathcal{G}] = \mathbb{E}_{\tilde{\mathbb{Q}}}[X|\mathcal{G}] \cdot \mathbb{E}_{\mathbb{Q}}[f|\mathcal{G}]$$

which is called the generalized version of Bayes formula.

Theorem 1.28 (Novikov's condition, Lemma 2.40 in Zagst [2002]). Let $W = (W_1(t), \dots, W_d(t))^\top_{t\in[0,T]}$ be a d-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$ and $\gamma = (\gamma(t))_{t\in[0,T]}$ a d-dimensional progressively measurable stochastic process on $(\Omega, \mathcal{F}, \mathbb{Q})$. Define for $t \in [0, T]$

$$Z(t) := \exp\bigg(-\frac{1}{2}\int_0^t \|\gamma(s)\|^2 ds - \int_0^t \gamma(s)^\top dW(s) \bigg).$$

If

$$\mathbb{E}\bigg[\exp\bigg(\frac{1}{2}\int_0^t \|\gamma(s)\|^2 ds \bigg) \bigg] < \infty$$

holds, then $(Z(t))_{t\in[0,T]}$ is a continuous martingale.

Theorem 1.29 (Girsanov's Theorem, Theorem 2.41 in Zagst [2002]). Let $W = (W_1(t), \dots, W_d(t))^\top_{t\in[0,T]}$ be a d-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$ and $\gamma = (\gamma(t))_{t\in[0,T]}$ a d-dimensional progressively measurable stochastic process such that the process $(Z(t))_{t\in[0,T]}$ defined by

$$Z(t) := \exp\bigg(-\frac{1}{2}\int_0^t \|\gamma(s)\|^2 ds - \int_0^t \gamma(s)^\top dW(s) \bigg)$$

is a martingale. Define

$$\tilde{W}(t) := W(t) + \int_0^t \gamma(s) ds.$$

Then $\tilde{W} = (\tilde{W}_1(t), \dots, \tilde{W}_d(t))_{t\in[0,T]}$ is a d-dimensional Brownian motion under the equivalent probability measure $\tilde{\mathbb{Q}}$ with Radon-Nikodym derivative $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = Z$.

Definition 1.30 (Strong solution of SDE, Definition 2.44 in Zagst [2002]). Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Furthermore, let $W = (W(t))_{t\geq 0}$ be a d-dimensional Brownian motion. We say that the n-dimensional stochastic process $Y = (Y(t))_{t\geq 0}$ is a strong solution to the stochastic differential equation (SDE)

$$\begin{aligned} dY(t) =& \mu(Y(t), t)dt + \sigma(Y(t), t)dW(t) \\ Y(0) =& y \in \mathbb{R}^n \end{aligned}$$

if Y fulfills

$$Y(t) = y + \int_0^t \mu(Y(s), s)ds + \int_0^t \sigma(Y(s), s)dW(s)$$

where $\mu : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^{n \times d}$ are progressively measurable stochastic processes satisfying

$$\int_0^t |\mu(Y(s), s)| ds < \infty \text{ and } \int_0^t \sigma_{ij}(Y(s), s)^2 ds < \infty \ \mathbb{Q}\text{-a.s.}$$

for all $t \geq 0$ and $i = 1, \ldots, n$, $j = 1, \ldots, d$.

Theorem 1.31 (Existence and Uniqueness, Theorem 2.45 in Zagst [2002]). Let the SDE be given by

$$\begin{aligned} dY(t) =& \mu(Y(t), t)dt + \sigma(Y(t), t)dW(t) \\ Y(0) =& y \in \mathbb{R}^n \end{aligned}$$

where $\mu : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^{n \times d}$ are continuous functions such that for all $t \geq 0$, $x, y \in \mathbb{R}^n$ and some constant $L > 0$ it holds

(a) Lipschitz condition:

$$\|\mu(x, t) - \mu(y, t)\|^2 + \|\sigma(x, t) - \sigma(y, t)\|^2 \leq L\|x - y\|^2.$$

(b) Growth condition:

$$\|\mu(y, t)\|^2 + \|\sigma(y, t)\|^2 \leq L^2(1 + \|y\|^2).$$

Then there exists a unique strong solution Y of the SDE and $C > 0$ (depending on L and $T > 0$) such that

$$\mathbb{E}[\|Y(t)\|^2] \leq C(1 + \|y\|^2)e^{Ct}$$

for all $t \in [0, T]$ and

$$\mathbb{E}[\sup_{t \in [0,T]} \|Y(t)\|^2] < \infty.$$

Definition 1.32 (Characteristic operator, Definition 2.46 in Zagst [2002]). Let Y be a unique strong solution to the SDE

$$\begin{aligned} dY(t) =& \mu(Y(t), t)dt + \sigma(Y(t), t)dW(t) \\ Y(0) =& y \in \mathbb{R}^n \end{aligned}$$

under the conditions (a) and (b) from Theorem 1.31. The operator $\mathcal{D}$ is called the characteristic operator for Y if

$$\mathcal{D}\Phi(y,t) := \Phi_t(y,t) + \sum_{i=1}^{n} \mu_i(y,t)\Phi_{y_i}(y,t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(y,t)\Phi_{y_i y_j}(y,t)$$

where $\Phi : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is twice continuous differentiable with respect to y, continuous differentiable with respect to t and

$$a_{ij}(y,t) := \sum_{k=1}^{d} \sigma_{ik}(y,t)\sigma_{jk}(y,t).$$

1.3 Approximation of the Claim Process

1.3.1 Classical Risk Process

In this section, we will introduce the classical risk process, based on the book Grandell [1991].

Definition 1.33 (Claim Process)**.** The claim process $C = (C(t))_{t\in[0,T]}$ is defined by

$$C(t) := \sum_{k=1}^{N(t)} X_k, \tag{1.1}$$

where

(i) $N = (N(t))_{t\in[0,T]}$ is a Poisson process with intensity $\lambda > 0$, and

(ii) $(X_k)_{k\in\mathbb{N}}$ is a sequence of independent identically distributed (i.i.d.) random variables with common distribution function F with $F(0) = 0$, mean $\mathbb{E}[X_1] = \mu$ and variance $Var(X_k) = \sigma^2$ for $k \in \mathbb{N}$.

The process N and the sequence $(X_k)_{k\in\mathbb{N}}$ are independent.

Definition 1.34 (Classical Risk Process, Definition 1)**.** The classical risk process $R = (R(t))_{t\in[0,T]}$ is defined by

$$R(t) := pt - C(t) = pt - \sum_{k=1}^{N(t)} X_k$$

where p is a positive real constant. We will choose $p := (1+\theta)\lambda\mu$ where $\theta > 0$ is a safety loading of the insurer.

Remark. Interpretation:

- $N(t)$ is the number of claims until time t.
- X_k denotes the k-th claim.
- $C(t)$ is the amount of claims until time t.
- p is the premium paid from the client to the insurance company.

1.3.2 Diffusion Approximation

In this section, we consider an approximation of the risk process by a Brownian motion. This approximation is called the diffusion approximation and was introduced in Grandell [1977] and Grandell [1991]. The idea is to consider a sequence of risk processes that converge in distribution to a Brownian motion with a drift. In the following, we give a heuristic solution to the approximation of the risk process.
First, we define for $n \in \mathbb{N}$

$$N_n(t) := \frac{N(nt) - \lambda nt}{\sqrt{n}},$$

where N is a Poisson process with intensity $\lambda > 0$. It holds

$$\lim_{t\to\infty} \frac{Var(N(t))}{t} = \lim_{t\to\infty} \frac{\lambda t}{t} = \lambda$$

and

$$\frac{N(nt)}{n} \to \lambda t, \tag{1.2}$$

as $n \to \infty$. For the sequence $(N_n)_{n\in\mathbb{N}}$ it holds

$$N_n(t) \xrightarrow{d} \sqrt{\lambda} W_1(t), \tag{1.3}$$

as $n \to \infty$, where W_1 is a one-dimensional Brownian motion. Next, we define for $m \in \mathbb{N}$

$$\bar{C}(m) := \sum_{k=1}^{m} X_k. \tag{1.4}$$

Then, $\mathbb{E}[\bar{C}(m)] = m\mathbb{E}[X_1] = m\mu$ and $Var(\bar{C}(m)) = mVar(X_1) = m\sigma^2$, since X_k are i.i.d.. From the Central Limit Theorem (CLT) it follows

$$\frac{\bar{C}(m) - m\mu}{\sqrt{m\sigma^2}} \xrightarrow{d} W_2(1), \tag{1.5}$$

as $m \to \infty$, where W_2 is a one-dimensional Brownian motion independent of W_1. Now, we define for $n \in \mathbb{N}$

$$C_n(t) := \frac{C(nt) - \mu\lambda nt}{\sqrt{n}},$$

where $(C(t))_{t\in[0,T]}$ is a claim process (cf. Definition 1.33). It follows from the Central Limit Theorem (CLT)

$$\begin{aligned} C_n(t) &= \frac{C(nt) - \mu\lambda nt}{\sqrt{n}} \qquad (1.6)\\ &\overset{(1.1),(1.4)}{=} \frac{\bar{C}(N(nt)) - \mu\lambda nt}{\sqrt{n}} \\ &= \sigma\sqrt{\frac{N(nt)}{n}}\frac{\bar{C}(N(nt)) - \mu N(nt)}{\sigma\sqrt{N(nt)}} + \mu\frac{N(nt) - \lambda nt}{\sqrt{n}} \\ &\underset{(1.2),(1.3),(1.5)}{\xrightarrow{d}} \sigma\sqrt{\lambda t}W_2(1) + \mu\sqrt{\lambda}W_1(t) \\ &\overset{d}{=} \sigma\sqrt{\lambda}W_2(t) + \mu\sqrt{\lambda}W_1(t) \\ &\overset{d}{=} \sqrt{\lambda(\sigma^2 + \mu^2)}W_3(t), \qquad (1.7) \end{aligned}$$

as $n \to \infty$, since $\sqrt{t}W_2(1) \overset{d}{=} W_2(t)$ and W_1, W_2 are independent. W_3 is a one-dimensional Brownian motion. Hence, for large n we have the approximation

$$C(nt) \approx \mu\lambda nt + \sqrt{\lambda(\sigma^2 + \mu^2)}W_3(nt). \tag{1.8}$$

Lastly, we define

$$R_n(t) := \frac{p_n nt - C(nt)}{\sqrt{n}},$$

where $p_n := (1 + \theta_n)\lambda\mu$ and $\theta_n := \frac{\theta}{\sqrt{n}}$ with $\theta > 0$. Then:

$$\begin{aligned} R_n(t) &= \frac{p_n nt - C(nt)}{\sqrt{n}} \\ &= \frac{p_n nt - \mu\lambda nt}{\sqrt{n}} - \frac{C(nt) - \mu\lambda nt}{\sqrt{n}} \end{aligned}$$

$$
\begin{aligned}
&\overset{\text{def. of } p_n}{=} \frac{(1+\theta_n)\mu\lambda nt - \mu\lambda nt}{\sqrt{n}} - \frac{C(nt) - \mu\lambda nt}{\sqrt{n}} \\
&= \frac{\theta_n \mu\lambda nt}{\sqrt{n}} - \frac{C(nt) - \mu\lambda nt}{\sqrt{n}} \\
&= \theta\mu\lambda t - \frac{C(nt) - \mu\lambda nt}{\sqrt{n}} \\
&\underset{(1.7)}{\xrightarrow{d}} \theta\mu\lambda t - \sqrt{\lambda(\sigma^2+\mu^2)}W_3(t) \\
&\overset{d}{=} \theta\mu\lambda t + \sqrt{\lambda(\mu^2+\sigma^2)}W_3(t).
\end{aligned}
$$

For large n we have the approximation

$$
\begin{aligned}
R_n(t) \approx& \theta\mu\lambda t + \sqrt{\lambda(\mu^2+\sigma^2)}W_3(t) \\
=& pt - \mu\lambda t + \sqrt{\lambda(\mu^2+\sigma^2)}W_3(t) \\
\underset{(1.8)}{\overset{d}{\approx}}& pt - C(t) \\
=& R(t),
\end{aligned}
$$

where $p := (1+\theta)\lambda\mu$. Hence, we can approximate the risk process R, which corresponds for n large enough to R_n, by a Brownian motion with drift, i.e.,

$$
dR(t) = pdt - \lambda\mu dt + \sqrt{\lambda(\mu^2+\sigma^2)}dW_3(t).
$$

Using the above SDE for the risk process and the relation $C(t) = pt - R(t)$, we obtain for the claim process

$$
dC(t) = \lambda\mu dt - \sqrt{\lambda(\mu^2+\sigma^2)}dW_3(t).
$$

1.4 Summary of Relevant Papers

In Chapter 4 we solve a Stackelberg game between a reinsurer and an insurer. To prove the solution of the optimization problem of the insurer, we use Cvitanić and Karatzas [1992] and Desmettre and Seifried [2016]. For the optimization problem of the reinsurer we use Korn and Trautmann [1999]. Therefore, we summarize in this section the relevant parts for the master thesis of these papers. Since we apply the papers in Chapter 4, we assume for simplicity that the financial market in this section is given as in Chapter 4. Hence, the financial market consists of one risk-free asset S_0 (also called bond) and two risky assets S_1, S_2 (also called stocks). The prices are given by

$$\begin{aligned} dS_0(t) =& S_0(t) r dt,\ S_0(0) = 1 \\ dS_1(t) =& S_1(t)(\mu_1 dt + \sigma_1 dW_1(t)),\ S_1(0) = s_1 > 0 \\ dS_2(t) =& S_2(t)(\mu_2 dt + \sigma_2(\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t))),\ S_2(0) = s_2 > 0 \end{aligned} \tag{1.9}$$

where $\rho \in [-1, 1]$, $r,\ \mu_1,\ \mu_2,\ \sigma_1,\ \sigma_2 > 0$ such that $\mu_1 > r, \mu_2 > r$ and $W(t) := (W_1(t), W_2(t))^\top$ is a 2-dimensional Brownian motion. We denote

$$\mathbb{1} := \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \mu := \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \text{ and } \sigma := \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2\rho & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}.$$

We write $\sigma = (\sigma_{ij})_{i,j=1,2}$. r is called the interest rate, μ is the yield rate and σ is the volatility matrix. The market price of risk is defined by

$$\gamma := \sigma^{-1}(\mu - r\mathbb{1}) \tag{1.10}$$

and the discount factor (also called pricing kernel) by

$$\tilde{Z}(t) := \exp\Big(- \Big(r + \frac{1}{2}\|\gamma\|^2 \Big) t - \gamma^\top W(t) \Big).$$

1.4.1 Korn and Trautmann [1999]

Korn and Trautmann [1999] consider an expected utility maximization problem of portfolios containing options and/or stocks. They use the martingale approach and (option) replicating strategies to solve an expected utility maximization problem of portfolios which contains options. In the following, we will summarize the important parts of the paper.

Let the wealth process of the investor be given by

$$\begin{aligned} dV^{v_0,\varphi}(t) =& \varphi_0(t) dS_0(t) + \varphi_1(t) dS_1(t) + \varphi_2(t) dS_2(t), \\ V^{v_0,\varphi}(0) =& v_0 > 0, \end{aligned} \tag{1.11}$$

where $\varphi = (\varphi_0, \varphi_1, \varphi_2)^\top$ is a trading strategy. The set of all admissible trading strategies for the wealth process (1.11) is given by

$$\Lambda := \{\varphi \text{ self-financing} |\ V^{v_0,\varphi}(t) \geq 0\ \mathbb{Q}\text{-a.s. } \forall t \in [0, T]\}. \tag{1.12}$$

A contingent claim B is a non-negative, $\mathcal{F}_T$-measurable random variable with $\mathbb{E}[B^n] < \infty$ for some $n > 1$.

Definition 1.35 (Replicating strategy, Definition 2.2 in Korn and Trautmann [1999])**.** A trading strategy $\varphi \in \Lambda$ is called a replicating strategy for the contingent claim B if

$$V^{v_0,\varphi}(T) = B \ \mathbb{Q}\text{-a.s.} \tag{1.13}$$

If the price of the contingent claim has a special form, then the replicating strategy can be given explicitly:

Theorem 1.36 (Replicating strategy, Theorem 2.6 in Korn and Trautmann [1999])**.** Suppose that the price of a contingent claim at time t is given by $f(t, S_1(t), S_2(t))$, where f is a continuous differentiable function with respect to t and a twice continuously differential function with respect to S_1 and S_2. Then the replicating strategy $\psi = (\psi_0, \psi_1, \psi_2)^\top$ is given by

$$\begin{aligned}\psi_0(t) =& \frac{f(t, S_1(t), S_2(t)) - \sum_{i=1}^2 \psi_i(t) S_i(t)}{S_0(t)}\\ \psi_i(t) =& \frac{d}{dS_i} f(t, S_1(t), S_2(t)), \ i = 1, 2.\end{aligned}$$

The price process $f(t, S_1(t), S_2(t))$ satisfies

$$\begin{aligned}df(t, S_1(t), S_2(t)) =& \left[r f(t, S_1(t), S_2(t)) + \sum_{i=1}^2 \psi_i(t) S_i(t)(\mu_i - r) \right] dt\\ &+ \sum_{i=1}^2 \psi_i(t) S_i(t) \sum_{j=1}^2 \sigma_{ij} dW_j(t).\end{aligned} \tag{1.14}$$

Portfolio Optimization Problem

We consider a portfolio optimization problem:

$$\sup_{\varphi \in \Lambda'} \mathbb{E}[U(V^{v_0,\varphi}(T))], \tag{P_S}$$

where U is a utility function and Λ' is the set of all admissible trading strategies $\varphi \in \Lambda$ such that

$$\mathbb{E}[U(V^{v_0,\varphi}(T))^-] < \infty^1.$$

This condition makes sure that the expectation in (P_S) exists. We denote by I the inverse function of U'.

[1] $x^- := \max\{-x, 0\}$

Theorem 1.37 (Theorem 3.1 in Korn and Trautmann [1999]). Assume that it holds for all $y \in (0, \infty)$

$$\mathbb{E}[\tilde{Z}(T)I(y\tilde{Z}(T))] < \infty.$$

Then there exists an optimal trading strategy $\varphi^* \in \Lambda'$, which solves the portfolio optimization problem (P_S) and the optimal terminal wealth is given by

$$V^{v_0,\varphi^*}(T) = I(y^*\tilde{Z}(T)),$$

where y^* is the Lagrange multiplier which solves the budget constraint

$$\mathbb{E}[\tilde{Z}(T)I(y\tilde{Z}(T))] = v_0.$$

The optimal wealth process is given by

$$V^{v_0,\varphi^*}(t) = \tilde{Z}(t)^{-1}\mathbb{E}[\tilde{Z}(T)V^{v_0,\varphi^*}(T)|\mathcal{F}_t]$$

for all $t \in [0, T]$.

Theorem 1.38 (Remark 3.2 in Korn and Trautmann [1999]). Let $\varphi^* \in \Lambda'$ be the optimal trading strategy to the portfolio optimization problem (P_S). Assume that the optimal wealth process is given by

$$V^{v_0,\varphi^*}(t) = g(t, W_1(t), W_2(t))$$

where $g : [0, T) \times \mathbb{R}^2 \to [0, \infty)$ is a function, which is continuous differentiable with respect to t and twice continuous differentiable with respect to W_1 and W_2 with $g(0, 0, 0) = v_0$. Then the optimal trading strategy φ^* is given by

$$\begin{aligned} \varphi_i^*(t) &= \frac{1}{S_i(t)}((\sigma^\top)^{-1}\nabla_x g(t, W_1(t), W_2(t)))_i, \ i = 1, 2 \\ \varphi_0^*(t) &= \frac{V^{v_0,\varphi^*}(t) - \sum_{i=1}^2 \varphi_i^*(t)S_i(t)}{S_0(t)} \end{aligned} \tag{1.15}$$

where $\nabla_x g$ is the gradient of g with respect to the last two components, i.e., W_1 and W_2.

Remark. The formula (1.15) differs from the formula in Remark 3.2 in Korn and Trautmann [1999] and in Theorem 5.12 in Korn [2014], i.e., we have $(\sigma^\top)^{-1}$ whereas they have σ^{-1}. The following two points justify the difference:

1. If we consider a power utility function, i.e., $U(x) = \frac{1}{b}x^b$ for $b \in (-\infty, 1)\backslash\{0\}$, then

we know that the optimal portfolio process is given by

$$\pi^* = \frac{1}{1-b}(\sigma\sigma^\top)^{-1}(\mu - r\mathbb{1}). \tag{1.16}$$

The optimal terminal wealth of the investor is given by

$$\begin{aligned} V^*(T) =& I(y^*\tilde{Z}(T)) \\ =& \left(\left(\frac{v_0}{E\left[\tilde{Z}(T)^{\frac{b}{b-1}}\right]}\right)^{b-1}\tilde{Z}(T)\right)^{\frac{1}{b-1}} \\ =& \frac{v_0}{E\left[\tilde{Z}(T)^{\frac{b}{b-1}}\right]}\tilde{Z}(T)^{\frac{1}{b-1}}, \end{aligned} \tag{1.17}$$

where I is the inverse function of U', i.e., $I(y) = y^{\frac{1}{b-1}}$, and y^* is the Lagrange multiplier, which solves the budget constraint

$$E[\tilde{Z}(T)I(y^*\tilde{Z}(T))] = v_0 \Leftrightarrow y^* = \left(\frac{v_0}{E\left[\tilde{Z}(T)^{\frac{b}{b-1}}\right]}\right)^{b-1}.$$

Hence, the optimal wealth process of the investor is given by

$$\begin{aligned} V^*(t) =& \tilde{Z}(t)^{-1}E[\tilde{Z}(T)V^*(T)|\mathcal{F}_t] \\ \overset{(a)}{=}& \tilde{Z}(t)^{-1}\mathbb{E}\left[\tilde{Z}(T)\frac{v_0}{E\left[\tilde{Z}(T)^{\frac{b}{b-1}}\right]}\tilde{Z}(T)^{\frac{1}{b-1}}\middle|\mathcal{F}_t\right] \\ =& \tilde{Z}(t)^{-1}\frac{v_0}{E\left[\tilde{Z}(T)^{\frac{b}{b-1}}\right]}\mathbb{E}\left[\tilde{Z}(T)^{\frac{b}{b-1}}\middle|\mathcal{F}_t\right] \\ \overset{(b)}{=}& v_0 \cdot \cancel{\tilde{Z}(t)^{-1}} \cancel{\exp\left(\left(r\frac{b}{b-1} + \frac{1}{2}\|\gamma\|^2\left[\frac{b}{b-1} - \frac{b^2}{(b-1)^2}\right]\right)T\right)} \\ & \times \exp\left(-\left(r\frac{b}{b-1} + \frac{1}{2}\|\gamma\|^2\left[\frac{b}{b-1} - \frac{b^2}{(b-1)^2}\right]\right)(\cancel{T} - t)\right)\tilde{Z}(T)^{\frac{\overset{1}{\cancel{b}}}{b-1}} \\ \overset{(c)}{=}& v_0 \cdot \exp\left(\left(r\frac{b}{b-1} + \frac{1}{2}\|\gamma\|^2\left[\frac{b}{b-1} - \frac{b^2}{(b-1)^2}\right]\right)t\right) \\ & \times \exp\left(-\left(r + \frac{1}{2}\|\gamma\|^2\right)\frac{1}{b-1}t - \frac{1}{b-1}\gamma^\top W(t)\right) \\ =& v_0 \cdot \exp\left(\left(r - \frac{1}{2}\|\gamma\|^2\left[\frac{b^2}{(b-1)^2} - 1\right]\right)t - \frac{1}{b-1}\gamma^\top W(t)\right) \\ =&: g(t, W_1(t), W_2(t)), \end{aligned}$$

where (a) follows from (1.17), (b) from Lemma A.1 with $\lambda = 0$ and $k = \frac{b}{b-1}$, and (c) from the definition of $\tilde{Z}$. Hence, the gradient of g with respect to the last two components is given by

$$\nabla_x g(t, W_1(t), W_2(t)) = -\frac{1}{b-1} g(t, W_1(t), W_2(t))\gamma = -\frac{1}{b-1} V^*(t)\gamma. \tag{1.18}$$

If we use the relation of the portfolio processes and trading strategies, i.e. (cf. Definition 2.63 in Korn [2014])

$$\pi_i^*(t) = \frac{S_i(t)}{V^*(t)} \varphi_i^*(t),$$

and Equation (1.15) with σ^{-1} instead of $(\sigma^\top)^{-1}$ like Korn and Trautmann [1999], then it follows for the portfolio process

$$\begin{aligned}
\pi^*(t) &= \frac{1}{V^*(t)} \sigma^{-1} \nabla_x g(t, W_1(t), W_2(t)) \\
&\overset{(*)}{=} \frac{1}{1-b} \sigma^{-1} \gamma \\
&= \frac{1}{1-b} \sigma^{-1} \sigma^{-1} (\mu - r\mathbb{1}) \\
&\neq (1.16)
\end{aligned}$$

where $(*)$ follows from (1.18).

2. Consider the proof of Theorem 5.12 on page 264 in Korn [2014]. If we compare the dW-term, then

$$\begin{aligned}
\nabla_x g(t, W_1(t), W_2(t))^\top dW(t) &\overset{!}{=} V^*(t)\pi(t)^\top \sigma(t) dW(t) \\
&\Leftrightarrow \\
\nabla_x g(t, W_1(t), W_2(t))^\top &= V^*(t)\pi(t)^\top \sigma(t) \\
&\Leftrightarrow \\
\frac{1}{V^*(t)} \nabla_x g(t, W_1(t), W_2(t))^\top \sigma(t)^{-1} &= \pi(t)^\top \\
&\Leftrightarrow \\
\frac{1}{V^*(t)} (\nabla_x g(t, W_1(t), W_2(t))^\top \sigma(t)^{-1})^\top &= \pi(t) \\
&\Leftrightarrow \\
\frac{1}{V^{v_0,\pi,c}(t)} (\sigma(t)^{-1})^\top \nabla_x f(t, W_1(t), W_2(t)) &= \pi(t) \\
&\Leftrightarrow \\
\frac{1}{V^*(t)} (\sigma(t)^\top)^{-1} \nabla_x g(t, W_1(t), W_2(t)) &= \pi(t).
\end{aligned}$$

Option Optimization Problem with Constraints in Stocks

Now, we consider a wealth process of the investor, where the investor can trade in options on the stocks and in the stocks. We assume that there exists two options on the stocks in our market. The price process of the first option is given by $f_1(t, S_1(t), S_2(t))$ and of the second option by $f_2(t, S_1(t), S_2(t))$. This means, the investor has a trading strategy $\xi(t) = (\xi_0(t), \xi_1(t), \xi_2(t))^\top$ in the options and a trading strategy $\zeta(t) = (\zeta_1(t), \zeta_2(t)))^\top$ in the stocks, i.e., the wealth process of the investor is given by

$$\begin{aligned} dV^{v_0,(\xi,\zeta)}(t) =& \xi_0(t)dS_0(t) + \xi_1(t)df_1(t, S_1(t), S_2(t)) + \xi_2(t)df_2(t, S_1(t), S_2(t)) \\ &+ \zeta_1(t)dS_1(t) + \zeta_2(t)dS_2(t), \\ V^{v_0,(\xi,\zeta)}(0) =& v_0 > 0. \end{aligned} \tag{1.19}$$

The aim of the investor is to maximize the expected utility of the terminal wealth (1.19) for a given trading strategy ζ in the stocks

$$\sup_{\xi \in \Lambda'} \mathbb{E}[U(V^{v_0,(\xi,\zeta)}(T))]. \tag{P_O}$$

Theorem 1.39 (Theorem 5.1 in Korn and Trautmann [1999])**.** Assume that the option prices f_1 and f_2 satisfy the requirements of Theorem 1.36 and for every $t \in [0, T)$ the matrix $\psi(t) = (\psi_{ij}(t))_{i,j=1,2}$ given by

$$\psi_{ij}(t) := \frac{d}{ds_j} f_i(t, S_1(t), S_2(t))$$

is regular. Then for a given trading strategy ζ in the stocks, there exists an optimal trading strategy ξ^* in the options for the optimization problem (P_O). The optimal terminal wealth (1.19) is given by

$$V^{v_0,(\xi^*,\zeta)}(T) = I(y^* \tilde{Z}(T)),$$

where y^* is the Lagrange multiplier, which solves the budget constraint

$$\mathbb{E}[\tilde{Z}(T) I(y \tilde{Z}(T))] = v_0.$$

The optimal trading strategy ξ^* is given by

$$\begin{aligned} \bar{\xi}^*(t) &= (\psi(t)^\top)^{-1}(\bar{\varphi}^*(t) - \zeta(t)), \\ \xi_0^*(t) &= \frac{V^{v_0,(\xi^*,\zeta)}(T) - \sum_{i=1}^2 (\xi_i^*(t) f_i(t, S_1(t), S_2(t)) + \zeta_i(t) S_i(t))}{S_0(t)}, \end{aligned}$$

where φ^* is the optimal trading strategy to the portfolio optimization problem (P_S) and $\bar{\xi}^*$, $\bar{\varphi}^*$ denotes the last two components of ξ^*, φ^*.

Remark. In Chapter 4, we need a similar proposition to Theorem 1.39. The difference will be that we want to determine the optimal trading strategy in the stocks given the trading strategy in the options, not vice versa. This is possible by Remark 5.2 in Korn and Trautmann [1999].

1.4.2 Desmettre and Seifried [2016]

In Desmettre and Seifried [2016], the authors consider an investor who invests dynamically in the financial market and can buy an additional fixed-term security only at time 0. The aim of the investor is to maximize its expected utility of terminal wealth plus the payoff from the fixed-term security. They use the so-called generalized martingale approach. In the following, we summarize the most relevant parts for the thesis from the paper.

The financial market is given by (1.9). The investor invests in the financial market, i.e., it has a relative portfolio process $\pi(t) = (\pi_1(t), \pi_2(t))^\top$, $t \in [0, T]$. Additional to the financial market, there exists a fixed-term security that has a stochastic payoff $P(T)$ under $\mathbb{Q}$ at time T. At time 0, the fixed-term security can be traded at a price of $P(0)$. Hence, the investor decides at time 0 how much it is willing to invest in the fixed-term security, i.e., it has a fixed-term investment ξ. This means, the investor pays $\xi P(0)$ at the beginning and receives therefore $\xi P(T)$ at the end of the investment horizon.

Hence, we define the wealth process of the investor by

$$\begin{aligned} dV^{v_0(\xi),\pi}(t) =& (1 - \pi_1(t) - \pi_2(t)) V^{v_0(\xi),\pi}(t) \frac{dS_0(t)}{S_0(t)} + \pi_1(t) V^{v_0(\xi),\pi}(t) \frac{dS_1(t)}{S_1(t)} \\ &+ \pi_2(t) V^{v_0(\xi),\pi}(t) \frac{dS_2(t)}{S_2(t)} \\ V^{v_0(\xi),\pi}(0) =& v_0 - \xi P(0) =: v_0(\xi) \end{aligned} \tag{1.20}$$

where $v_0 > 0$ is the initial wealth of the investor. At time T, the investor receives the sum

$$V^{v_0(\xi),\pi}(T) + \xi P(T).$$

To avoid that the investor is insolvent at time 0, the absolute position in the fixed-term security ξ must fulfill

$$v_0 - \xi P(0) \geq 0 \;\Leftrightarrow\; \xi \leq \frac{v_0}{P(0)} =: \xi^{\max}.$$

The optimization problem of the investor is given by

$$\sup_{(\xi,\pi)\in\Lambda} \mathbb{E}[U(V^{v_0(\xi),\pi}(T) + \xi P(T))], \tag{P}$$

where U is the utility function of the investor and Λ denotes the set of all admissible strategies, i.e.

$$\begin{aligned}\Lambda := \{(\xi,\pi) |\ \xi \in [0,\xi^{\max}], \pi \text{ is self-financing}, V^{v_0(\xi),\pi}(t) \geq 0 \ \mathbb{Q}\text{-a.s. } \forall t \in [0,T] \\ \text{and } \mathbb{E}[U(V^{v_0(0),\pi}(T) + \xi P(T))] < \infty \ \forall \xi > 0\}.\end{aligned}$$

The procedure of solving the optimization problem (P) is the following:

1. For a fixed $\xi \in [0,\xi^{\max}]$ find the optimal π^* by the generalized martingale method.
2. Find ξ^*.

First, let $\xi \in [0,\xi^{\max}]$ be fixed. Define the random utility function $\hat{U} : (0,\infty) \to \mathbb{R}$ by

$$\hat{U}(x) := U(x + \xi P(T)).$$

This function is random and $\mathcal{F}_T$-measurable, since P is a random variable and $\mathcal{F}_T$-measurable. This case is called spanned in Desmettre and Seifried [2016]. They also consider the general case (i.e., not necessarily the spanned case, which means that $\hat{U}$ is not necessarily $\mathcal{F}_T$-measurable), but we will only consider the spanned case.

Since U is a utility function, it is concave and differentiable. Therefore, $\hat{U}$ is concave and differentiable. The inverse of $\hat{U}'$ is denoted by $\hat{I}$ and is given by $\hat{I} : (0,\infty) \to [0,\infty)$ with

$$\hat{I}(y) = \max\{I(y) - \xi P(T), 0\},$$

where I is the inverse function of U'. The function $\hat{I}$ is bijective on the interval $(0, U'(\xi P(T))]$.

Theorem 1.40 (Optimal solution to (P), Corollary 3.4 in Desmettre and Seifried [2016])**.** Assume it holds for any $y \in (0,\infty)$

$$\mathbb{E}[\tilde{Z}(T) I(y\tilde{Z}(T))] < \infty \text{ and } \mathbb{E}[U(I(y\tilde{Z}(T)))] < \infty.$$

Then there exists a solution (ξ^*, π^*) to the optimization problem (P). The optimal terminal wealth of the investor is given by

$$V^{v_0(\xi^*),\pi^*}(T) = \max\{\hat{I}(y^*(\xi^*)\tilde{Z}(T)), 0\},$$

where $y^*(\xi)$ is the Lagrange multiplier that is to be found via the budget constraint

$$\mathbb{E}[\tilde{Z}(T)\hat{I}(y^*(\xi)\tilde{Z}(T))] = v_0 - \xi P(0).$$

Therefore, the optimal total terminal wealth is given by

$$V^{v_0(\xi^*),\pi^*}(T) + \xi^* P(T) = \max\{I(y^*(\xi^*)\tilde{Z}(T)), \xi^* P(T)\}.$$

The optimal absolute position in the fixed-term security is given by

$$\xi^* = \underset{\xi\in[0,\xi^{\max}]}{\arg\max}\ \nu(\xi),$$

where the function ν is defined by

$$\nu(\xi) := \mathbb{E}[U(\max\{I(y^*(\xi)\tilde{Z}(T)), \xi P(T)\})].$$

In the next theorem, we give an explicit formula for the optimal portfolio process π^*.

Theorem 1.41 (Optimal portfolio process, Theorem 4.1 and 4.3 in Desmettre and Seifried [2016])**.** Assume that the conditions of Theorem 1.40 are fulfilled. Furthermore, suppose that $\hat{I}$ and $\frac{d\hat{I}(y)}{dy}$ are polynomially bounded at 0 and ∞. Then the optimal portfolio process π^* of the optimization problem (P) given $\xi \in [0, \xi^{\max}]$ is given by

$$\pi^*(t)V^{v_0(\xi),\pi^*}(t) = -(\sigma^\top)^{-1}\gamma\tilde{Z}(t)^{-1}\mathbb{E}\left[\tilde{Z}(T)y^*(\xi)\tilde{Z}(T)\frac{d\hat{I}}{dy}(y^*(\xi)\tilde{Z}(T))|\mathcal{F}_t\right] \tag{1.21}$$

$\mathbb{Q}$-a.s. for all $t \in [0, T]$, where $y^*(\xi)$ is the Lagrange multiplier from Theorem (1.40) and γ the market price of risk given by (1.10).
In the special case, when U is given by a power utility function (i.e., $U(x) = \frac{1}{b}x^b$ for $b \in (-\infty, 1)\backslash\{0\}$), then the optimal portfolio process π^* has the form

$$\pi^*(t)V^{v_0(\xi),\pi^*}(t) = \pi^M(V^{v_0(\xi),\pi^*}(t) + \xi\tilde{Z}(t)^{-1}\mathbb{E}[\tilde{Z}(T)P(T)\mathbf{1}_{\{V^{v_0(\xi),\pi^*}(T)>0\}}|\mathcal{F}_t]), \tag{1.22}$$

where π^M is the Merton portfolio process given by

$$\pi^M = \frac{1}{1-b}(\sigma\sigma^\top)^{-1}(\mu - r\mathbb{1}).$$

Proof. We will only prove that the formula of the special case (1.22) follows from the formula of the general case (1.21). The proof is based on the proof of Theorem 4.3 in Desmettre and Seifried [2016], which is stated in Appendix B.
In the special case of a power utility function, the utility function U is for $x \in (0, \infty)$

defined by

$$U(x) := \frac{1}{b}x^b,$$

where $b \in (-\infty, 1)\backslash\{0\}$. Therefore, we have

$$U'(x) = x^{b-1} \text{ and } I(y) = y^{\frac{1}{b-1}}$$

for $x, y \in (0, \infty)$, where I is the inverse of U'. The random utility function $\hat{U}$ is for $x \in (0, \infty)$ defined by

$$\hat{U}(x) := U(x + \xi P(T)) = \frac{1}{b}(x + \xi P(T))^b.$$

Hence, we get

$$\hat{U}'(x) = U'(x + \xi P(T)) = (x + \xi P(T))^{b-1}$$

and

$$\begin{aligned}\hat{I}(y) &= \max\{I(y) - \xi P(T), 0\} \\ &= \max\{y^{\frac{1}{b-1}} - \xi P(T), 0\} \\ &= (y^{\frac{1}{b-1}} - \xi P(T))\mathbf{1}_{\{y<(\xi P(T))^{b-1}\}}\end{aligned} \tag{1.23}$$

for $x, y \in (0, \infty)$, where $\hat{I}$ is the inverse of $\hat{U}'$. It follows for $y \in (0, \infty)$

$$\begin{aligned}\frac{d\hat{I}(y)}{dy} &= \frac{dI(y)}{dy}\mathbf{1}_{\{y<(\xi P(T))^{b-1}\}} \\ &= \frac{1}{b-1}y^{\frac{1}{b-1}-1}\mathbf{1}_{\{y<(\xi P(T))^{b-1}\}}.\end{aligned} \tag{1.24}$$

Hence, we have for all $y \in (0, \infty)$

$$\begin{aligned}y\frac{d\hat{I}(y)}{dy} &\overset{(1.24)}{=} \frac{1}{b-1}y^{\frac{1}{b-1}}\mathbf{1}_{\{y<(\xi P(T))^{b-1}\}} \\ &\overset{(1.23)}{=} -\frac{1}{1-b}(\hat{I}(y) + \xi P(T)\mathbf{1}_{\{y<(\xi P(T))^{b-1}\}}).\end{aligned} \tag{1.25}$$

It follows

$$\pi^*(t)V^{v_0(\xi),\pi^*}(t) \overset{(a)}{=} -(\sigma^\top)^{-1}\gamma\tilde{Z}(t)^{-1}\mathbb{E}\left[\tilde{Z}(T)y^*(\xi)\tilde{Z}(T)\frac{d\hat{I}}{dy}(y^*(\xi)\tilde{Z}(T))|\mathcal{F}_t\right]$$

$$
\begin{aligned}
&\overset{(b)}{=} - (\sigma^\top)^{-1}\gamma \tilde{Z}(t)^{-1}\mathbb{E}\Big[\tilde{Z}(T)\Big(-\frac{1}{1-b}\Big)(\hat{I}(y^*(\xi)\tilde{Z}(T)) \\
&\quad + \tilde{Z}(T)\xi P(T)\mathbf{1}_{\{y^*(\xi)\tilde{Z}(T)<(\xi P(T))^{b-1}\}})|\mathcal{F}_t\Big] \\
&\overset{(c)}{=}\frac{1}{1-b}(\sigma\sigma^\top)^{-1}(\mu - r\mathbb{1})(\tilde{Z}(t)^{-1}\mathbb{E}[\tilde{Z}(T)V^*(T)|\mathcal{F}_t] \\
&\quad + \xi\tilde{Z}(t)^{-1}\mathbb{E}[\tilde{Z}(T)P(T)\mathbf{1}_{\{V^{v_0(\xi),\pi^*}(T)>0\}}|\mathcal{F}_t]) \\
&\overset{(d)}{=}\pi^M(V^{v_0(\xi),\pi^*}(t) + \xi\tilde{Z}(t)^{-1}\mathbb{E}[\tilde{Z}(T)P(T)\mathbf{1}_{\{V^{v_0(\xi),\pi^*}(T)>0\}}|\mathcal{F}_t]),
\end{aligned}
$$

where

- (a) follows from (1.21),
- (b) from (1.25) with $y = y^*(\xi)\tilde{Z}(T)$,
- (c) from $\hat{I}(y^*(\xi)\tilde{Z}(T)) = V^{v_0(\xi),\pi^*}(T)$ and $\gamma = \sigma^{-1}(\mu - r\mathbb{1})$, and
- (d) from $V^{v_0(\xi),\pi^*}(t) = \tilde{Z}(t)^{-1}\mathbb{E}[\tilde{Z}(T)V^{v_0(\xi),\pi^*}(T)|\mathcal{F}_t]$ and $\pi^M = \frac{1}{1-b}(\sigma\sigma^\top)^{-1}(\mu - r\mathbb{1})$.

□

1.4.3 Cvitanić and Karatzas [1992]

Cvitanić and Karatzas [1992] consider an investor who maximizes its utility of terminal wealth given that the portfolio process is constrained by a closed convex subset of $\mathbb{R}^2$. The authors construct a family of unconstrained optimization problems and find the unconstrained optimization problem that models the required constraint in the original market. With this, they set up a corresponding duality problem to the constrained optimization problem.

We consider an investor with wealth process

$$
\begin{aligned}
dV^{v_0,\pi}(t) =&(1 - \pi_1(t) - \pi_2(t))V^{v_0,\pi}(t)\frac{dS_0(t)}{S_0(t)} + \pi_1(t)V^{v_0,\pi}(t)\frac{dS_1(t)}{S_1(t)} \\
&+ \pi_2(t)V^{v_0,\pi}(t)\frac{dS_2(t)}{S_2(t)} \\
=&V^{v_0,\pi}(t)(r + \pi(t)^\top(\mu - r\mathbb{1}))dt + V^{v_0,\pi}(t)\pi(t)^\top\sigma dW(t), \\
V^{v_0,\pi}(0) =&v_0 > 0.
\end{aligned}
\tag{1.26}
$$

The set K is a closed, convex subset of $\mathbb{R}^2$, which constrains the portfolio process π. The investor solves the following problem

$$\sup_{\pi \in \Lambda} \mathbb{E}[U(V^{v_0,\pi}(T))], \tag{P}$$

where U is a utility function and Λ denotes the set of all admissible controls given by

$$\Lambda := \{\pi \text{ self-financing} |\ \pi(t) \in K \ \mathbb{Q}\text{-a.s. } \forall t \in [0,T],\ V^{v_0,\pi}(t) \geq 0\ \forall t \in [0,T] \\ \text{and } \mathbb{E}[U(V^{v_0,\pi}(T))^-] < \infty\}.$$

The optimization problem (P) is called the constrained optimization problem, since the portfolio process has the constraint K.

The support function of K is defined by $\delta : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ with

$$\delta(x) := \sup_{y \in K}(-x^\top y).$$

The barrier cone $\tilde{K}$ is given by

$$\tilde{K} := \{x \in \mathbb{R}^2 | \delta(x) < \infty\}.$$

It is assumed that the function δ is continuous on $\tilde{K}$ and is bounded from below.

In the following, we introduce the auxiliary market. For this, we define the set of $\mathbb{R}^2$-valued dual processes by

$$\mathcal{D} := \left\{ \lambda = (\lambda(t))_{t \in [0,T]} \text{ progressively measurbale} \middle| \ \mathbb{E}\left[\int_0^T \|\lambda(t)\|^2 dt\right] < \infty, \\ \mathbb{E}\left[\int_0^T \delta(\lambda(t)) dt\right] < \infty \right\}.$$

If $\lambda \in \mathcal{D}$, then $\lambda(t) \in \tilde{K}$ $\mathbb{Q}$-a.s. for all $t \in [0,T]$. For $\lambda \in \mathcal{D}$ we introduce the auxiliary market by

$$\begin{aligned} dS_0^\lambda(t) &= S_0^\lambda(t)(r + \delta(\lambda(t)))dt \\ dS_1^\lambda(t) &= S_1^\lambda(t)((\mu_1 + \lambda_1(t) + \delta(\lambda(t)))dt + \sigma_1 dW_1(t)) \\ dS_2^\lambda(t) &= S_2^\lambda(t)((\mu_2 + \lambda_2(t) + \delta(\lambda(t)))dt + \sigma_2(\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t))). \end{aligned}$$

We define

$$\gamma_\lambda(t) := \gamma + \sigma^{-1}\lambda(t),$$

$$\tilde{Z}_\lambda(t) := \exp\bigg(-\int_0^t (r+\delta(\lambda(s)))ds - \int_0^t \|\gamma_\lambda(s)\|^2 ds - \int_0^t \gamma_\lambda(s)^\top dW(s) \bigg).$$

The wealth process of the investor in the auxiliary market is given by

$$\begin{aligned} dV_\lambda^{v_0,\pi}(t) =& (1-\pi_1(t)-\pi_2(t))V_\lambda^{v_0,\pi}(t)\frac{dS_0^\lambda(t)}{S_0^\lambda(t)} + \pi_1(t)V_\lambda^{v_0,\pi}(t)\frac{dS_1^\lambda(t)}{S_1^\lambda(t)} \\ &+ \pi_2(t)V_\lambda^{v_0,\pi}(t)\frac{dS_2^\lambda(t)}{S_2^\lambda(t)} \\ =& V_\lambda^{v_0,\pi}(t)(r+\pi(t)^\top(\mu - r\mathbb{1}))dt + V_\lambda^{v_0,\pi}(t)\pi(t)^\top \sigma dW(t) \\ &+ \underbrace{V_\lambda^{v_0,\pi}(t)[\delta(\lambda(t)) + \pi(t)^\top\lambda(t)]dt}_{\text{additional term}} \\ V_\lambda^{v_0,\pi}(0) =& v_0 > 0. \end{aligned} \tag{1.27}$$

The optimization problem in the auxiliary market is given by

$$\sup_{\pi\in\Lambda_\lambda} \mathbb{E}[U(V_\lambda^{v_0,\pi}(T))], \tag{P_λ}$$

where Λ_λ is the set of all admissible strategies defined by

$$\Lambda_\lambda := \{\pi \text{ self-financing} | V_\lambda^{v_0,\pi}(t) \geq 0 \ \forall t\in[0,T] \text{ and } \mathbb{E}[U(V_\lambda^{v_0,\pi}(T))^-] < \infty\}.$$

The optimization problem (P_λ) is called an unconstrained optimization problem. We define

$$\mathcal{D}' := \{\lambda \in \mathcal{D} | \mathbb{E}[\tilde{Z}_\lambda(T) I(y\tilde{Z}_\lambda(T))] < \infty \ \forall y \in (0,\infty)\},$$

where I is the inverse function of U'. By the martingale method, for any $\lambda \in \mathcal{D}'$ there exists the optimal portfolio process $\pi_\lambda^* \in \Lambda_\lambda$ to the unconstrained optimization problem (P_λ). Furthermore, the optimal terminal wealth is given by

$$V_\lambda^{v_0,\pi_\lambda^*}(T) = I(y_\lambda^*\tilde{Z}_\lambda(T)),$$

where y_λ^* is the Lagrange multiplier that satisfies the budget constraint

$$\mathbb{E}[\tilde{Z}_\lambda(T) I(y\tilde{Z}_\lambda(T))] = v_0.$$

Theorem 1.42 (Proposition 8.3)**.** Assume that there exists $\lambda^* \in \mathcal{D}'$ such that the optimal portfolio process $\pi_{\lambda^*}^* \in \Lambda_\lambda$ to the unconstrained optimization problem (P_{λ^*}) fulfills

$$\pi_{\lambda^*}^*(t) \in K \ \mathbb{Q}\text{-a.s. } \forall t \in [0,T],$$

$$\delta(\lambda^*(t)) + \pi^*_{\lambda^*}(t)^\top \lambda^*(t) = 0.$$

Then $\pi^*_{\lambda^*} \in \Lambda$ and $\pi^*_{\lambda^*}$ is the optimal portfolio process to the constrained optimization problem (P).

Remark. We say that λ^* is optimal.

In Chapter 10, Cvitanić and Karatzas [1992] state four conditions, which characterize the optimal portfolio process in the constrained optimization problem (P), i.e., they can be used to determine the optimal portfolio process in the constrained optimization problem (P). The four conditions are related to a given $\lambda \in \mathcal{D}'$. One of the four conditions is given by the assumptions from Theorem (1.42). With one of the conditions, the authors could set up a dual problem to the constrained optimization problem (P). This is stated in Chapter 12 of Cvitanić and Karatzas [1992]. We will not go into more detail here, as it is not relevant for Chapter 4 in the thesis.

Cvitanić and Karatzas [1992] stated the above results for a general financial market with one risk-free and n risky assets. The parameters of the financial market are assumed to be stochastic processes. In Chapter 15 of Cvitanić and Karatzas [1992], they discussed the case of deterministic coefficients as in our case.

Proposition 1.43 (Example 15.1)**.** Assume that $\delta(x) = 0$ for all $x \in \tilde{K}$. Then,

$$\lambda^* = \arg\min_{x \in \tilde{K}} \|\gamma + \sigma^{-1} x\|^2$$

is optimal (i.e., it fulfills the assumptions from Theorem 1.42).

Chapter 2

Theory of Stackelberg Games

2.1 What is a Stackelberg Game?

A game describes an interaction of several players. In the following, we consider only a game between two players. A Stackelberg game is a game with a hierarchical structure. Therefore, we call one player the leader and the other the follower of the Stackelberg game. We assume that the leader can choose its strategy from an action set A_L and the follower from an action set A_F. Each player has a gain function[1] (also called a payoff function) $G_i : Q \to \mathbb{R}$ ($i \in \{L, F\}$) that models the preference of the leader/follower to a strategy in the action set $A = A_L \times A_F$. This means that not only the strategy of the leader influences the utility of the leader but also the chosen strategy of the follower, and vice versa. We denote a_L and a_F the strategies of the leader and follower, respectively. In a Stackelberg game, both player wants to maximize their utility function. The hierarchy in the game should be understood as follows: The leader of the Stackelberg game dominates the follower and therefore, the leader chooses its strategy first knowing the response of the follower and afterwards the follower selects its strategy depending on the selected strategy of the leader. Hence, the Stackelberg game is given by

$$\begin{aligned} &\max_{a_L \in A_L} \mathbb{E}[G_L(a_L, a_F^*)] \\ &\quad \text{s.t. } a_F^* \in \arg\max_{a_F \in A_F} \mathbb{E}[G_F(a_L, a_F)]. \end{aligned} \tag{SG}$$

(Osborne and Rubinstein [1994])

We use backward induction to solve a Stackelberg game. The idea of backward induction

[1] In the literature, the gain function is often called utility function. In the following chapters, a function is called a utility function if it is strictly concave, continuous differentiable and fulfills the Inada condtions

is to solve the optimization problem of the follower first and then solve backward the optimization problem of the leader. This means that the follower chooses first its strategy a_F^* for every $a_L \in A_L$. Hence, the follower selects a map $\alpha : A_L \to A_F$. Afterwards, the leader selects its strategy a_L^* knowing the map α of the follower, i.e., which strategy the follower chooses for every $a_L \in A_L$. We call $(a_L^*, \alpha(a_L^*))$ the Stackelberg equilibrium in the Stackelberg game. (Fudenberg and Tirole [1991], pp. 68-69)

Definition 2.1 (Stackelberg equilibrium, p. 68 in Fudenberg and Tirole [1991])**.** Assume that there exists a map $\alpha : A_L \to A_F$ such that for any $a_L \in A_L$ it holds

$$\mathbb{E}[G_F(a_L, \alpha(a_L))] = \max_{a_F \in A_F} \mathbb{E}[G_F(a_L, a_F)]$$

and there exists a_L^* such that

$$\mathbb{E}[G_L(a_L^*, \alpha(a_L^*))] = \max_{a_L \in A_L} \mathbb{E}[G_L(a_L, \alpha(a_L))].$$

Then, $(a_L^*, a_F^*) := (a_L^*, \alpha(a_L^*))$ is called the Stackelberg equilibrium to the Stackelberg game (SG).

Example 2.2 (p. 68 in Fudenberg and Tirole [1991])**.** We consider a Stackelberg game between two players who have gain functions

$$G_L(a_L, a_F) = [12 - (a_L + a_F)]a_L,$$
$$G_F(a_L, a_F) = [12 - (a_L + a_F)]a_F$$

and action sets $A_L = A_F = \mathbb{R}$. Since the gain functions of the players are non-random, the Stackelberg game is given by

$$\max_{a_L \in A_L} G_L(a_L, a_F^*)$$
$$\text{s.t. } a_F^* \in \arg\max_{a_F \in A_F} G_F(a_L, a_F).$$

In the Stackelberg game, the leader chooses its strategy first and the follower selects its strategy second given the strategy of the leader. We use the solution method of backward induction:

1. Let $a_L \in A_L$ be arbitrary. By the first-order optimality condition (FOOC), we have

$$\left.\frac{dG_F(a_L, a_F)}{da_F}\right|_{a_F = a_F^*} = 12 - a_L - 2a_F^* \overset{!}{=} 0$$

$$\Leftrightarrow$$

$$a_F^* = 6 - \frac{a_L}{2}.$$

Since a_F^* depends on the choice of the leader, we define the map $\alpha : A_L \to A_F$ by $\alpha(a_L) := 6 - \frac{a_L}{2}$ as the strategy choice of the follower. Since G_F is concave in a_F, FOOC yields the maximum.

2. For the optimization problem of the leader, we get

$$\begin{aligned}\max_{a_L \in A_L} G_L(a_L, \alpha(a_L)) &= \max_{a_L \in A_L} [12 - (a_L + \alpha(a_L))]a_L \\ &= \max_{a_L \in A_L} [12 - (a_L + 6 - \frac{a_L}{2})]a_L \\ &= \max_{a_L \in A_L} [6 - \frac{a_L}{2}]a_L.\end{aligned}$$

Hence, by FOOC, we get

$$\begin{aligned}\frac{d}{da_L}([6 - \frac{a_L}{2}]a_L)\bigg|_{a_L = a_L^*} &= 6 - a_L^* \overset{!}{=} 0 \\ &\Leftrightarrow \\ a_L^* &= 6.\end{aligned}$$

q_L^* is optimal, as G_L is concave in a_L.

3. The Stackelberg equilibrium is given by $(a_L^*, \alpha(a_L^*)) = (6, 3)$.

2.2 Existence of Solutions to Stochastic Dynamic Stackelberg Games

In this section, we discuss the existence of a solution to a stochastic dynamic Stackelberg game. The Stackelberg game consists of two players, the leader and the follower. Furthermore, we have a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ and a d-dimensional $\mathbb{F}$-Brownian motion $W = (W(t))_{t\in[0,T]}$ where $T > 0$. The state space $Y_L^{a_L,a_F}$ of the leader is given by

$$\begin{aligned}dY_L^{a_L,a_F}(t) =& \mu_L(t, Y_L^{a_L,a_F}(t), a_L(t), a_F(t))dt + \sigma_L(t, Y_L^{a_L,a_F}(t), a_L(t), a_F(t))^\top dW(t) \\ Y_L^{a_L,a_F}(0) =& y_{0L}\end{aligned} \tag{2.1}$$

where $y_{0L} \in \mathbb{R}$ and $\mu_L : [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\sigma_L : [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^d$ are functions. We call a_L the control of the leader, which is n-dimensional, and a_F the control of the follower, which is m-dimensional.

Analogous, we define the state space $Y_F^{a_L,a_F}$ of the follower by

$$\begin{aligned} dY_F^{a_L,a_F}(t) =& \mu_F(t, Y_F^{a_L,a_F}(t), a_L(t), a_F(t))dt + \sigma_F(t, Y_F^{a_L,a_F}(t), a_L(t), a_F(t))^\top dW(t) \\ Y_F^{a_L,a_F}(0) =& y_{0F} \end{aligned} \tag{2.2}$$

where $y_{0F} \in \mathbb{R}$ and $\mu_F : [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\sigma_F : [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^d$ are functions. The set of all admissible controls of the leader and follower are given by

$$\begin{aligned} \Lambda_L &= \{a_L(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)\}, \\ \Lambda_F &= \{a_F(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)\}. \end{aligned}$$

The expected utility (payoff function) of the leader is defined by

$$J_L(a_L(\cdot), a_F(\cdot)) = \mathbb{E}[G_L(Y_L^{a_L,a_F}(T))],$$

where $G_L : \mathbb{R} \to \mathbb{R}$ is a concave function, and the expected utility (payoff function) of the follower by

$$J_F(a_L(\cdot), a_F(\cdot)) = \mathbb{E}[G_F(Y_F^{a_L,a_F}(T))],$$

where $G_F : \mathbb{R} \to \mathbb{R}$ is a concave function. The Stackelberg game is given by

$$\begin{aligned} &\max_{a_L(\cdot)\in\Lambda_L} J_L(a_L(\cdot), a_F^*(\cdot)) \\ &\qquad \text{s.t. } a_F^*(\cdot) \in \arg\max_{a_F(\cdot)\in\Lambda_F} J_F(a_L(\cdot), a_F(\cdot)). \end{aligned}$$

Since the Stackelberg game is solved by backward induction, we first show under what conditions there exists an optimal control map $\alpha : [0,T] \times \Lambda_L \to \Lambda_F$, which solves the optimization problem of the follower

$$\max_{a_F(\cdot)\in\Lambda_F} J(a_L(\cdot), a_F(\cdot))$$

given $a_L(\cdot) \in \Lambda_L$. We have the following assumptions for the follower: For all $a_L \in \Lambda_L$

(AF1) the maps $(t, a_F) \mapsto \mu_F(t, y, a_L, a_F), (t, a_F) \mapsto \sigma_F(t, y, a_L, a_F)$ are continuous for all $y \in \mathbb{R}$,

(AF2) there exists a constant $L > 0$ such that the maps $y \mapsto \mu_F(t, y, a_L, a_F)$, $y \mapsto \sigma_F(t, y, a_L, a_F), y \mapsto G_F(y)$ are Lipschitz continuous (cf. Condition (a) in

Theorem 1.31) and it holds

$$\begin{aligned}|\mu_F(t,0,a_L,a_F)| &\leq L\\ \|\sigma_F(t,0,a_L,a_F)\| &\leq L\\ |G_F(0)| &\leq L\end{aligned}$$

for all $t \in [0,T]$ and $a_F \in \mathbb{R}^m$, and

(AF3) for all $(t,y) \in [0,T] \times \mathbb{R}$ the set

$$\{(\mu_F(t,y,a_L,a_F),(\sigma^\top\sigma)(t,y,a_L,a_F))|a_F \in \mathbb{R}^m\}$$

is convex in $\mathbb{R}^2$.

Theorem 2.3. Assume that for any $a_L \in \Lambda_L$ the assumptions (AF1)-(AF3) hold and $\max_{a_F(\cdot)\in\Lambda_F} J(a_L(\cdot),a_F(\cdot)) < +\infty$. Then there exists an optimal control $a_F^*(\cdot) \in \Lambda_F$ to the optimization problem of the follower.

Proof. The proof can be found in Yong and Zhou [1999], Theorem 5.3 in Chapter 2. □

Now, we assume that there exists a map $\alpha : [0,T] \times \Lambda_L \to \Lambda_F$ that solves the optimization problem of the follower. Therefore, the leader has to solve the optimization problem

$$\sup_{a_L(\cdot)\in\Lambda_L} J(a_L(\cdot),\alpha(\cdot,a_L(\cdot))).$$

Hence, we define the functions

$$\begin{aligned}\bar{\mu}_L(t,y,a_L) :=& \mu_L(t,y,a_L,\alpha(t,a_L))\\ \bar{\sigma}_L(t,y,a_L) :=& \sigma_L(t,y,a_L,\alpha(t,a_L))\end{aligned}$$

where $t \in [0,T]$, $y \in \mathbb{R}$ and $a_L \in \mathbb{R}^n$. We have the following assumptions for the leader:

(AL1) the maps $(t,a_L) \mapsto \bar{\mu}_L(t,y,a_L)$, $(t,a_L) \mapsto \bar{\sigma}_L(t,y,a_L)$ are continuous for all $y \in \mathbb{R}$,

(AL2) there exists a constant $L > 0$ such that the maps $y \mapsto \bar{\mu}_L(t,y,a_L)$, $y \mapsto \bar{\sigma}_L(t,y,a_L)$, $y \mapsto G_L(y)$ are Lipschitz continuous (cf. Condition (a) in Theorem 1.31) and

$$\begin{aligned}|\bar{\mu}_L(t,0,a_L)| &\leq L\\ \|\bar{\sigma}_L(t,0,a_L)\| &\leq L\end{aligned}$$

$$|G_L(0)| \leq L$$

for all $t \in [0,T]$ and $a_L \in \mathbb{R}^n$,

(AL3) for all $(t,y) \in [0,T] \times \mathbb{R}$ the set

$$\{(\bar{\mu}_L(t,y,a_L), (\bar{\sigma}^\top \bar{\sigma})(t,y,a_L)) | a_L \in \mathbb{R}^n\}$$

is convex in $\mathbb{R}^2$.

Theorem 2.4. Assume that the assumptions (AL1)-(AL3) hold and $\max_{a_L(\cdot)\in\Lambda_L} J(a_L(\cdot), \alpha(\cdot, a_L(\cdot))) < +\infty$. Then there exists an optimal control $a_L^*(\cdot) \in \Lambda_L$ to the optimization problem of the leader.

Proof. The proof can be found in Yong and Zhou [1999], Theorem 5.3 in Chapter 2. □

Remark. If the map $(t, a_L) \mapsto \alpha(t, a_L)$ is continuous, then the assumption (AL1) of the leader is equivalent to the assumption

(AL1') for all $a_F \in \mathbb{R}^m$ and $y \in \mathbb{R}$ the maps $(t, a_L) \mapsto \mu_L(t, y, a_L, a_F), (t, a_L) \mapsto \sigma_L(t, y, a_L, a_F)$ are continuous.

Remark. The search of solutions to a Stackelberg game (and to a stochastic control problem) depends on the problem. The assumptions (AF1)-(AF3) and (AL1)-(AL3) are only sufficient conditions to prove the existence of a solution to a Stackelberg game. There exists Stackelberg games which have a solution but not all assumptions (AF1)-(AF3) and (AL1)-(AL3) are fulfilled. In such cases, the existence of a solution needs to be proved by another way. For example in Chapter 3 and 4 of the master thesis, the assumptions (AF3) and (AL3) are not fulfilled but, as we will show there, there exists a solution to the Stackelberg games.

Example 2.5 (Assumption (AF3) is not fulfilled in Chapter 3)**.** For $(t,y) \in [0,T] \times [0,\infty)$, we denote $\mu_F(t,y,p_R,q,\pi_I), \sigma_F(t,y,p_R,q,\pi_I)$ by $\mu_F(q,\pi_I), \sigma_F(q,\pi_I)$ for notation convenience. We need to show that the set

$$\begin{aligned}\mathcal{S} :=& \{(\mu_F(q,\pi_I), (\sigma_F^\top \sigma_F)(q,\pi_I)) | (q,\pi_I) \in [0,1] \times \mathbb{R}\} \\ :=& \begin{pmatrix} \theta_I \mu - (1-q)(p_R - \mu) + y(r_I + (\tilde{\mu} - r_I)\pi_I) \\ \sigma^2 q^2 + y^2 \pi_I^2 \tilde{\sigma}^2 S_1(t)^{2\delta} \end{pmatrix}\end{aligned}$$

is convex in $\mathbb{R}^2$.
Let $s_1, s_2 \in \mathcal{S}$ and $\lambda \in [0,1]$. We show $\lambda s_1 + (1-\lambda)s_2 \in \mathcal{S}$.
Since $s_1, s_2 \in \mathcal{S}$, there exists $(q^{s_1}, \pi_I^{s_1}), (q^{s_2}, \pi_I^{s_2}) \in [0,1] \times \mathbb{R}$ such that

$$\begin{aligned}
s_1 =& (\mu_F(q^{s_1}, \pi_I^{s_1}), (\sigma_F^\top \sigma_F)(q^{s_1}, \pi_I^{s_1})) \\
=& \begin{pmatrix} \theta_I \mu - (1-q^{s_1})(p_R - \mu) + y(r_I + (\tilde{\mu} - r_I)\pi_I^{s_1}) \\ \sigma^2 (q^{s_1})^2 + y^2 (\pi_I^{s_1})^2 \tilde{\sigma}^2 S_1(t)^{2\delta} \end{pmatrix} \\
s_2 =& (\mu_F(q^{s_2}, \pi_I^{s_2}), (\sigma_F^\top \sigma_F)(q^{s_2}, \pi_I^{s_2})) \\
=& \begin{pmatrix} \theta_I \mu - (1-q^{s_2})(p_R - \mu) + y(r_I + (\tilde{\mu} - r_I)\pi_I^{s_2}) \\ \sigma^2 (q^{s_2})^2 + y^2 (\pi_I^{s_2})^2 \tilde{\sigma}^2 S_1(t)^{2\delta} \end{pmatrix}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \lambda s_1 + (1-\lambda) s_2 \\
=& \lambda \begin{pmatrix} \theta_I \mu - (1-q^{s_1})(p_R - \mu) + y(r_I + (\tilde{\mu} - r_I)\pi_I^{s_1}) \\ \sigma^2 (q^{s_1})^2 + y^2 (\pi_I^{s_1})^2 \tilde{\sigma}^2 S_1(t)^{2\delta} \end{pmatrix} \\
& + (1-\lambda) \begin{pmatrix} \theta_I \mu - (1-q^{s_2})(p_R - \mu) + y(r_I + (\tilde{\mu} - r_I)\pi_I^{s_2}) \\ \sigma^2 (q^{s_2})^2 + y^2 (\pi_I^{s_2})^2 \tilde{\sigma}^2 S_1(t)^{2\delta} \end{pmatrix} \\
=& \begin{pmatrix} \theta_I \mu - (1-(\lambda q^{s_1} + (1-\lambda) q^{s_2}))(p_R - \mu) + y(r_I + (\tilde{\mu} - r_I)(\lambda \pi_I^{s_1} + (1-\lambda)\pi_I^{s_2})) \\ \sigma^2 (\lambda (q^{s_1})^2 + (1-\lambda)(q^{s_2})^2) + y^2 (\lambda (\pi_I^{s_1})^2 + (1-\lambda)(\pi_I^{s_2})^2) \tilde{\sigma}^2 S_1(t)^{2\delta} \end{pmatrix} \notin \mathcal{S},
\end{aligned}$$

since

$$\begin{aligned}
(\lambda q^{s_1} + (1-\lambda) q^{s_2}))^2 &\neq (\lambda (q^{s_1})^2 + (1-\lambda)(q^{s_2})^2), \\
(\lambda \pi_I^{s_1} + (1-\lambda)\pi_I^{s_2})^2 &\neq (\lambda (\pi_I^{s_1})^2 + (1-\lambda)(\pi_I^{s_2})^2).
\end{aligned}$$

Therefore, $\mathcal{S}$ is not convex.
The proofs that (AL3) is not fulfilled in Chapter 3 and (AF3) and (AL3) are not fulfilled in Chapter 4 are analogous.

Chapter 3

Reinsurance of a Claim Process

3.1 Motivation and Paper Overview

A primary insurance company (also called insurance company or insurer) obtains insurance premiums from their clients to cover the clients' risks. A reinsurance company (also called reinsurer) is the insurer of insurance companies. Therefore, the reinsurer receives a reinsurance premium from the insurer and the insurer transfers its risk to the reinsurer. (Chen and Shen [2018])

A reinsurance contract is an agreement between a reinsurer and an insurer (Albrecher et al. [2017]). Therefore, we will consider a game between a reinsurer and an insurer to model the relationship between them. If we would only consider the optimal choice of one party, the other party might not agree to the contract as the choice is not necessarily optimal for them (Bai et al. [2019]). Hence, we consider a reinsurance-investment optimization problem for an insurer and reinsurer modeled by a game.

Since several thousands primary insurance companies and approximately 200 reinsurance companies exist (Albrecher et al. [2017]), the reinsurer has a monopoly position and dominates the insurer (Chen and Shen [2018]). Therefore, we consider a Stackelberg game between an insurer and reinsurer where the reinsurer is the leader and the insurer is the follower of the game.

To our best knowledge, the first paper where the optimal reinsurance is derived in the context of a Stackelberg game is Chen and Shen [2018]. The researchers consider one reinsurer (leader) and one insurer (follower). The insurer wants to reinsure its claim process which is modeled by the diffusion approximation (cf. Section 1.3). The reinsurer chooses a reinsurance premium strategy and the insurer a proportional reinsurance strategy. The aim of both parties is to maximize their expected utility of their terminal surplus. Chen and Shen [2018] started by solving the optimization problem in the general setting, i.e., general utility function and random coefficients. The surplus process of the reinsurer and the insurer is debited/credited by a random interest rate which can be different for the reinsurer and the insurer. Afterwards, they solve the special case with an exponential utility function with random model coefficients and lastly with deterministic, constant model coefficients (i.e., coefficients of the claim process, insurance premium, interest rate and utility functions)

Bai et al. [2019] extend the special case of the exponential utility function with deterministic, constant coefficients from Chen and Shen [2018]. Instead of considering only one insurer, they consider two insurers as the followers of the Stackelberg game. In addition to the Stackelberg game between the insurers and reinsurer, they have a non-zero sum game between the two insurers to model the competition in the insurance market. Another difference to the paper by Chen and Shen [2018] is that they do not only consider a reinsurance problem but a reinsurance-investment problem. Therefore, the insurance company reinsures its claims process through proportional reinsurance and invests in a financial market. The claim process of the insurer is modeled by the diffusion approximation (cf. Section 1.3). The reinsurer can also invest in the financial market. Hence, the reinsurer chooses a reinsurance premium strategy and an investment strategy, whereas the insurers choose their proportional reinsurance strategies and investment strategies. Furthermore, Bai et al. [2019] consider the surplus process with a time delay. Therefore, the aim of the reinsurer is to maximize the expected utility of terminal surplus with a time delay and the insurers maximize the expected utility of terminal surplus and relative preference (given by the difference of the surplus processes of the insurers) with a time delay.

Another paper about Stackelberg games in the context of insurance and reinsurance is from Chen and Shen [2019]. Again, they have one insurer (follower) and one reinsurer (leader). Compared to the papers before, the insurer wants to reinsure the claim process that is modeled by the Cramér-Lundberg model and not by the diffusion approximation. Another difference is that the reinsurance form is not necessarily proportional, i.e., the insurer chooses between any proportional and non-proportional reinsurance strategy. Hence, the reinsurer chooses a reinsurance premium strategy and the insurer a reinsurance strategy. The aim of the reinsurer and the insurer is to maximize their mean-variance cost functionals. Since they allow a general reinsurance form, they need to set a special reinsurance premium strategy to solve the Stackelberg game. For this reason, they consider that the reinsurance premium strategy is given by the variance premium principle and the expected value premium principle. Therefore, the reinsurer does not choose the reinsurance premium strategy but it chooses its safety loading of the reinsurance premium.

In the paper by Chen et al. [2020], they consider a multi-hierarchical Stackelberg game, i.e., they have one insurer and n reinsurers. The claim process is modeled by the Cramér-Lundberg model and the reinsurance form is general (i.e., proportional and non-proportional reinsurance is possible). The reinsurance premium strategy is given by the variance premium principle. Therefore, n reinsurers choose their safety loading strategies and the insurer as well as the first $n-1$ reinsurers choose their reinsurance strategies (i.e., the first $n-1$ reinsurers act as leader and follower for another party and therefore choose two strategies). The aim of the insurer and n reinsurers is to maximize their mean-variance payoff functionals.

Compared to all mentioned papers that consider a Stackelberg game between a reinsurer and an insurer, Asmussen et al. [2019] consider a Stackelberg game between two insurance companies to model their competition for customers. Both insurance companies have a deductible. The insurer with the lower deductible is the leader of the game. Both insurers choose a premium for the customers. The aim of the larger company is to maximize the difference between the reserves of the insurers, while the aim of the smaller company is to minimize the difference between the reserves of the insurers.

In this chapter, we state a Stackelberg game and find a solution, where the Stackelberg game is a special case of Bai et al. [2019]. As in Bai et al. [2019], the claim process is approximated by the diffusion approximation (cf. Section 1.3). The insurer wants to reinsure a part of its claim process and invests in a financial market, which is given by a CEV model. Hence, the insurer chooses a proportional reinsurance strategy and an investment strategy. In contrast, the reinsurer only chooses a reinsurance premium strategy and does not invest in the financial market, since we are only interested in the portfolio of the insurer. The aim of the insurer and the reinsurer is to maximize their expected utility of their terminal surplus.

The structure of this chapter is as follows: In Section 3.2 we state the framework and the Stackelberg game. The solution to the Stackelberg game is given in Section 3.3. The verification of the solution from Section 3.3 is proved in Section 3.4. In Section 3.5, we compare our result, which is a special case of Bai et al. [2019], to the framework and solution of Chen and Shen [2018] in detail.

3.2 Stackelberg Game

The framework and the Stackelberg game are special cases of the framework and the Stackelberg game in Bai et al. [2019].

3.2.1 Framework

Let $W = (W_1(t), W_2(t))$ be a two-dimensional Brownian motion. The financial market consists of a risk-free asset S_0 and a risky asset S_1. The dynamics of the risk-free asset S_0 are given by

$$dS_0(t) = r_I S_0(t)dt,\ S_0(0) = 1,$$

where $r_I > 0$ is an interest rate, and of the risky asset S_1 by

$$dS_1(t) = S_1(t)[\tilde{\mu}dt + \tilde{\sigma}S_1(t)^\delta dW_1(t)],\ S_1(0) = s_1 > 0,$$

where $\tilde{\mu} > 0$ with $\tilde{\mu} > r_I$, $\tilde{\sigma} > 0$ and $\delta \in \mathbb{R}$.

Remark. The financial market is given by a constant elasticity of variance (CEV) model with an elasticity parameter δ. The Black-Scholes financial market is a special case of the CEV model, i.e., if we set $\delta = 0$.

The claim process C of the insurer is given by a diffusion-type model (cf. Chapter 1.3)

$$dC(t) = \mu dt - \sigma dW_2(t),$$

where $\mu > 0,\ \sigma > 0$. The premium paid from a representative client to the insurer is determined by the expected value premium principle, i.e.,

$$p_I = (1 + \theta_I)\mu,$$

where $\theta_I > 0$ is the safety loading of the insurer.

3.2.2 Formulation of the Stackelberg Game

In the following, we state formally the Stackelberg game. The reinsurer is the leader and the insurer is the follower of the Stackelberg game. We assume that the insurer invests in the financial market with one risk-free asset S_0 and one risky asset S_1. In addition, the insurer chooses how much it is willing to reinsure from its claim process. Therefore, the insurer selects a portfolio process $\pi_I(t)$, $t \in [0, T]$, and a proportional reinsurance strategy $q(t)$, $t \in [0, T]$, i.e., which part of the loss is covered by the reinsurer. At time t, the insurer covers $q(t)$ of the claims whereas the reinsurer covers $1 - q(t)$ of the claims.
Furthermore, the reinsurer can only debit/credit its surplus process with an interest rate $r_R > 0$. Besides, it chooses the reinsurance premium which it receives from the insurer

for the reinsurance. Hence, the reinsurer chooses a reinsurance premium strategy $p_R(t)$, $t \in [0,T]$.

Optimization Problem of the Insurer (Follower)

The surplus process of the insurer $Y_I^{p_R,(q,\pi_I)}$ is given by

$$\begin{aligned} dY_I^{p_R,(q,\pi_I)}(t) =& \underbrace{(p_I - (1-q(t))p_R(t))dt}_{\text{Net premium}} - \underbrace{q(t)dC(t)}_{\text{claims covered by the insurer}} \\ &+ \underbrace{Y_I^{p_R,(q,\pi_I)}(t)(1-\pi_I(t))\frac{dS_0(t)}{S_0(t)}}_{\text{Investment in risk-free asset}} + \underbrace{Y_I^{p_R,(q,\pi_I)}(t)\pi_I(t)\frac{dS_1(t)}{S_1(t)}}_{\text{Investment in risky asset}} \\ =&(p_I - (1-q(t))p_R(t) - \mu q(t) + Y_I^{p_R,(q,\pi_I)}(t)(r_I + (\tilde{\mu} - r_I)\pi_I(t)))dt \\ &+ Y_I^{p_R,(q,\pi_I)}(t)\pi_I(t)\tilde{\sigma}S_1(t)^{\delta}dW_1(t) + \sigma q(t)dW_2(t), \\ Y_I(0) =& y_{0I} > 0. \end{aligned} \tag{3.1}$$

The aim of the insurer is to maximize its expected utility of its terminal surplus process, i.e.

$$\sup_{(q,\pi_I)\in\Lambda_I} \mathbb{E}[U_I(Y_I^{p_R,(q,\pi_I)}(T))]. \tag{3.2}$$

U_I is the utility function of the insurer given by an exponential utility function, i.e., for a relative risk aversion $\beta_I > 0$ of the insurer

$$U_I(x) := -\frac{1}{\beta_I}e^{-\beta_I x},$$

and Λ_I is the set of all admissible controls of the optimization problem (3.2) given by

$$\begin{aligned} \Lambda_I := \Bigg\{ &(q,\pi_I) \text{ progressively measurable} \Bigg| \, q(t) \in [0,1] \ \mathbb{Q}\text{-a.s. } \forall t \in [0,T], \\ &\mathbb{E}\left[\int_0^T |\pi_I(t)|^2 dt\right] < \infty \text{ and (3.1) has a unique strong solution } Y_I^{p_R,(q,\pi_I)}, \\ &\text{which is adapted, continuous and } \mathbb{E}\left[\sup_{t\in[0,T]} |Y_I^{p_R,(q,\pi_I)}(t)|^2\right] < \infty \Bigg\}. \end{aligned}$$

The value function of the insurer is defined by

$$\Phi^I(t,y,s) := \sup_{(q,\pi_I)\in\Lambda_I} \mathbb{E}_{t,y,s}[U_I(Y_I^{p_R,(q,\pi_I)}(T))]$$

$$= \sup_{(q,\pi_I)\in\Lambda_I} \mathbb{E}[U_I(Y_I^{p_R,(q,\pi_I)}(T))|\ \mathcal{F}_t,\ Y_I^{p_R,(q,\pi_I)}(t) = y,\ S_1(t) = s]. \tag{3.3}$$

Optimization Problem of the Reinsurer (Leader)

The surplus process of the reinsurer $Y_R^{p_R,q}$ is given by

$$\begin{aligned} dY_R^{p_R,q}(t) =& (\underbrace{(1-q(t))p_R(t)}_{\text{Premium}} + \underbrace{r_R Y_R^{p_R,q}(t)}_{\text{debit/credit}})dt - \underbrace{(1-q(t))dC(t)}_{\text{claims covered by reinsurer}} \\ =& ((1-q(t))(p_R(t)-\mu) + r_R Y_R^{p_R,q}(t))dt + \sigma(1-q(t))dW_2(t), \\ Y_R(0) =& y_{0R} > 0. \end{aligned} \tag{3.4}$$

The aim of the reinsurer is to maximize its expected utility of its terminal surplus process, i.e.

$$\sup_{p_R\in\Lambda_R} \mathbb{E}[U_R(Y_R^{p_R,q}(T))]. \tag{3.5}$$

U_R is the utility function of the reinsurer given by an exponential utility function, i.e., for a relative risk aversion $\beta_R > 0$ of the reinsurer

$$U_R(x) := -\frac{1}{\beta_R} e^{-\beta_R x},$$

and Λ_R is the set of all admissible controls of the optimization problem (3.5) given by

$$\begin{aligned} \Lambda_R := \Big\{ & p_R \text{ progressively measurable} \Big| \ p_R(t) \in [p_I, p]\ \mathbb{Q}\text{-a.s. } \forall t \in [0,T] \text{ and} \\ & \text{(3.4) has unique strong solution } Y_R^{p_R,q} \text{ which is adapteda and continuous with} \\ & \mathbb{E}\Big[\sup_{t\in[0,T]} |Y_R^{p_R,q}(t)|^2\Big] < \infty \Big\}, \end{aligned}$$

where $p := (1+\theta^{\max})\mu$ with $\theta^{\max} > \theta_I$ is the upper bound of the safety loading of the reinsurer. We bound the reinsurance premium p_R by the insurance premium p_I from below due to the fact that if the reinsurer covers 100% of the aggregated claims (i.e., the insurer pays the whole reinsurance premium p_R and not only a part) the reinsurance premium p_R has to be bigger or equal to the received insurance premium p_I for the claims. Otherwise, the insurer would buy as much reinsurance as possible and the reinsurer has to cover 100% of the claims. Since the reinsurance premium is less than the insurance premium, the insurer would make riskless profits.

The value function of the reinsurer is defined by

$$\begin{aligned}\Phi^R(t,y) &:= \sup_{p_R\in\Lambda_R} \mathbb{E}_{t,y}[U_R(Y_R^{p_R,q}(T))] \\ &= \sup_{p_R\in\Lambda_R} \mathbb{E}[U_R(Y_R^{p_R,q}(T))|\,\mathcal{F}_t,\ Y_R^{p_R,q}(t)=y]. \end{aligned} \tag{3.6}$$

Stackelberg Game

Definition 3.1 (Stackelberg Game)**.** The Stackelberg game is given by

$$\begin{aligned}&\sup_{p_R\in\Lambda_R} \mathbb{E}[U_R(Y_R^{p_R,q^*}(T))] \\ &\qquad \text{s.t. } (q^*(\cdot),\pi_I^*(\cdot)) \in \arg\max_{(q,\pi_I)\in\Lambda_I} \mathbb{E}[U_I(Y_I^{p_R,(q,\pi_I)}(T))]. \end{aligned} \tag{3.7}$$

Definition 3.2 (Stackelberg equilibrium, cf. Definition 2.1)**.** The solution $(p_R^*(\cdot),q^*(\cdot|p_R^*),\pi_I^*(\cdot|p_R^*))$ of the Stackelberg game (3.7) is called the Stackelberg equilibrium.

3.3 Solution to the Stackelberg Game

We define

$$\begin{aligned}\iota(t,p(t)) &:= \frac{p(t)-\mu}{\sigma^2\beta_I\psi^I(t)}, \\ \psi^I(t) &:= e^{r_I(T-t)}, \\ \psi^R(t) &:= e^{r_R(T-t)}, \\ u^I(t) &:= \begin{cases} -\frac{\tilde\gamma^2}{4\delta r_I}(1-e^{-2\delta r_I(T-t)}), & \delta\neq 0, \\ -\frac{1}{2}\tilde\gamma(T-t), & \delta=0, \end{cases} \\ M(t) &:= \frac{\beta_R\psi^R(t)+\beta_I\psi^I(t)}{2\beta_I\psi^I(t)+\beta_R\psi^R(t)}, \end{aligned} \tag{3.8}$$

where $\tilde\gamma := \frac{\tilde\mu-r_I}{\tilde\sigma}$ is the market price of risk.

Theorem 3.3 (Solution to the Stackelberg game, Theorem 1 in Bai et al. [2019])**.** The value function of the reinsurer is given by

$$\Phi^R(t,y_R) = -\frac{1}{\beta_R}e^{-\beta_R\psi^R(t)y_R+v^R(t)}$$

and of the insurer by

$$\Phi^I(t, y_I) = -\frac{1}{\beta_I} e^{-\beta_I \psi^I(t) y_I + u^I(t) s^{-2\delta} + v^I(t)},$$

where $v^I(\cdot)$ and $v^R(\cdot)$ are defined below, depending on four cases. The optimal investment strategy π_I^* of the insurer is given by

$$\pi_I^*(t) = \frac{1}{\beta_I \psi^I(t) y s^{2\delta}} \left[\frac{\tilde{\mu} - r_I}{\tilde{\sigma}^2} - 2\delta u^I(t) \right],$$

where $y = Y_I^{p_R^*,(q^*,\pi_I^*)}(t)$ and $s = S_1(t)$. We consider four different cases for the optimal reinsurance premium strategy p_R^* and reinsurance strategy q^*, and the functions v^I and v^R.

Case 1 Let $\iota(t, p_I) \geq 1$. Then

$$q^*(t) = 1 \text{ and } p_R^*(t) = p_R$$

for all $p_R \in [p_I, p]$. Furthermore, for $t \in [0, T]$

$$v^R(t) = 0$$

and

$$\begin{aligned} v^I(t) =& \frac{\beta_I \theta_I \mu}{r_I}(1 - \psi^I(t)) - \frac{\beta_I^2 \sigma^2}{4 r_I}(1 - \psi^I(t)^2) \\ &- \frac{\tilde{\gamma}^2 (2\delta + 1)\tilde{\sigma}^2}{4 r_I}(T - t) - \frac{(2\delta + 1)\tilde{\sigma}^2}{2 r_I} u^I(t). \end{aligned}$$

Case 2 Let $M(t) \leq \iota(t, p_I) < 1$. Then

$$p_R^*(t) = p_I \text{ and } q^*(t) = \iota(t, p_I).$$

Furthermore, it is

$$\begin{aligned} v^R(t) =& \frac{\beta_R (p_I - \mu)}{r_R}(1 - \psi^R(t)) - \frac{\beta_R (p_I - \mu)^2}{\sigma^2 \beta_I (r_R - r_I)} \left(1 - \frac{\psi^R(t)}{\psi^I(t)}\right) \\ &- \frac{\sigma^2 \beta_R^2}{4 r_I}(1 - \psi^R(t)^2) + \frac{\beta_R^2 (p_I - \mu)}{\beta_I (2 r_R - r_I)} \left(1 - \frac{\psi^R(t)^2}{\psi^I(t)}\right) \\ &- \frac{\beta_R^2 (p_I - \mu)^2}{4 \sigma^2 \beta_I^2 (r_R - r_I)} \left(1 - \left(\frac{\psi^R(t)}{\psi^I(t)}\right)^2\right) \end{aligned}$$

and

$$\begin{aligned}v^I(t) =&-\left(\frac{(p_I-\mu)^2}{2\sigma^2}+\frac{\tilde{\gamma}^2(2\delta+1)\tilde{\sigma}^2}{4r_I}\right)(T-t)\\&-\frac{(2\delta+1)\tilde{\sigma}^2}{2r_I}u^I(t).\end{aligned}$$

Case 3 Let $\iota(t,p_I) < M(t) < \iota(t,p)$. Then

$$p_R^*(t) = \mu + \sigma^2\beta_I\psi^I(t)M(t) \text{ and } q^*(t) = M(t).$$

Furthermore, it is

$$\begin{aligned}v^R(t) =&\sigma^2\beta_R\beta_I\int_T^t(1-M(\tau))M(\tau)\psi^R(\tau)\psi^I(\tau)d\tau\\&-\frac{\sigma^2\beta_R^2}{2}\int_T^t\psi^R(\tau)^2(1-M(\tau))^2d\tau\end{aligned}$$

and

$$\begin{aligned}v^I(t) =&\frac{\beta_I\theta_I\mu}{r_I}(1-\psi^I(t))-\sigma^2\beta_I^2\int_T^t\psi^I(\tau)^2M(\tau)d\tau\\&+\frac{\sigma^2\beta_I^2}{2}\int_T^t\psi^I(\tau)^2M(\tau)^2d\tau-\frac{\tilde{\gamma}^2(2\delta+1)\tilde{\sigma}^2}{4r_I}(T-t)\\&-\frac{(2\delta+1)\tilde{\sigma}^2}{2r_I}u^I(t).\end{aligned}$$

Case 4 Let $\iota(t,p) \le M(t)$. Then

$$p_R^*(t) = p \text{ and } q^*(t) = \iota(t,p).$$

Furthermore, it is

$$\begin{aligned}v^R(t) =&\frac{\beta_R(p-\mu)}{r_R}(1-\psi^R(t))-\frac{\beta_R(p-\mu)^2}{\sigma^2\beta_I(r_R-r_I)}\left(1-\frac{\psi^R(t)}{\psi^I(t)}\right)\\&-\frac{\sigma^2\beta_R^2}{4r_I}(1-\psi^R(t)^2)+\frac{\beta_R^2(p-\mu)}{\beta_I(2r_R-r_I)}\left(1-\frac{\psi^R(t)^2}{\psi^I(t)}\right)\\&-\frac{\beta_R^2(p-\mu)^2}{4\sigma^2\beta_I^2(r_R-r_I)}\left(1-\left(\frac{\psi^R(t)}{\psi^I(t)}\right)^2\right)\end{aligned}$$

and

$$v^I(t) =\frac{\beta_I((1+\theta_I)\mu-p)}{r_I}(1-\psi^I(t))$$

$$
\begin{aligned}
&- \left(\frac{(p-\mu)^2}{2\sigma^2} + \frac{\tilde{\gamma}^2(2\delta+1)\tilde{\sigma}^2}{4r_I} \right)(T-t) \\
&- \frac{(2\delta+1)\tilde{\sigma}^2}{2r_I} u^I(t).
\end{aligned}
$$

Proof. The proof is based on the proof of Theorem 1 in Bai et al. [2019]. It is stated in Appendix A.

We start with the **optimization problem of the insurer**. For this, we will use the HJB-approach. Let $p_R(\cdot) \in \Lambda_R$ be arbitrary. We set $x = (y, s)^\top$. The Hamilton-Jacobi-Bellman equation (HJB-equation) is given by

$$
\begin{aligned}
0 =& \sup_{(q(\cdot),\pi_I(\cdot))\in\Lambda_I} \mathcal{D}^I \Phi^I(t,x), \\
\Phi^I(T,x) =& U_I(y),
\end{aligned} \tag{3.9}
$$

where $\mathcal{D}^I$ is the characteristic operator (cf. Definition 1.32) of the insurer defined by

$$
\begin{aligned}
\mathcal{D}^I \Phi^I(t,x) :=& \Phi^I_t(t,x) + (p_I - p_R(t)(1-q(t)) - \mu q(t) + r_I y + (\tilde{\mu} - r_I)\pi_I(t) y)\Phi^I_y(t,x) \\
&+ \tilde{\mu} s \Phi^I_s(t,x) + \frac{1}{2}(\sigma^2 q(t)^2 + \tilde{\sigma}^2 \pi_I(t)^2 y^2 s^{2\delta})\Phi^I_{yy}(t,x) \\
&+ \tilde{\sigma}^2 \pi_I(t) y s^{2\delta+1} \Phi^I_{ys}(t,x) + \frac{1}{2}\tilde{\sigma}^2 s^{2\delta+2}\Phi^I_{ss}(t,x) \\
=& \Phi^I_t(t,x) + (\theta_I \mu - (p_R(t) - \mu)(1-q(t)) + r_I y + (\tilde{\mu} - r_I)\pi_I(t) y)\Phi^I_y(t,x) \\
&+ \tilde{\mu} s \Phi^I_s(t,x) + \frac{1}{2}(\sigma^2 q(t)^2 + \tilde{\sigma}^2 \pi_I(t)^2 y^2 s^{2\delta})\Phi^I_{yy}(t,x) \\
&+ \tilde{\sigma}^2 \pi_I(t) y s^{2\delta+1} \Phi^I_{ys}(t,x) + \frac{1}{2}\tilde{\sigma}^2 s^{2\delta+2}\Phi^I_{ss}(t,x),
\end{aligned}
$$

since

$$
\begin{aligned}
p_I - (1-q(t))p_R(t) - \mu q(t) =& (1+\theta_I)\mu - (1-q(t))p_R(t) - \mu q(t) \\
=& \theta_I \mu - (1-q(t))p_R(t) - (1-q(t))\mu \\
=& \theta_I \mu - (1-q(t))(p_R(t) - \mu).
\end{aligned}
$$

By the first-order optimality condition (FOOC), we take the first derivative of the characteristic operator with respect to $q(\cdot)$ and $\pi_I(\cdot)$ and set it equal to zero. Hence,

$$
\begin{aligned}
\frac{d\mathcal{D}^I \Phi^I(t,x)}{dq} &= \Phi^I_y(t,y,s)(p_R(t) - \mu) + \Phi^I_{yy}(t,y,s) q(t) \sigma^2 \overset{!}{=} 0 \\
\frac{d\mathcal{D}^I \Phi^I(t,x)}{d\pi_I} &= \Phi^I_y(t,y,s) y (\tilde{\mu} - r_I) + \Phi^I_{yy}(t,y,s) y^2 \pi_I(t) \tilde{\sigma}^2 s^{2\delta} \\
&\quad + \Phi^I_{ys}(t,y,s) \tilde{\sigma}^2 s^{2\delta+1} y \overset{!}{=} 0.
\end{aligned}
$$

It follows

$$\begin{aligned} q^*(t,p_R(t)) &= -\frac{\Phi^I_y(t,y,s)(p_R(t)-\mu)}{\Phi^I_{yy}(t,y,s)\sigma^2}, \\ \pi_I^*(t,p_R(t)) &= -\frac{\Phi^I_y(t,y,s)\cancel{y}(\tilde{\mu}-r_I)}{\Phi^I_{yy}(t,y,s)\tilde{\sigma}^2 s^{2\delta} y^{\cancel{2}}} - \frac{\Phi^I_{ys}(t,y,s)\cancel{\tilde{\sigma}^2} s^{\cancel{2\delta}+1}\cancel{y}}{\Phi^I_{yy}(t,y,s)\cancel{\tilde{\sigma}^2}\cancel{s^{2\delta}} y^{\cancel{2}}} \\ &= -\frac{\Phi^I_y(t,y,s)(\tilde{\mu}-r_I)}{\Phi^I_{yy}(t,y,s)\tilde{\sigma}^2 s^{2\delta} y} - \frac{\Phi^I_{ys}(t,y,s)s}{\Phi^I_{yy}(t,y,s)y}. \end{aligned}$$

Since we assume that $q^*(t,p_R(t)) \in [0,1]$, we have

$$q^*(t) := q^*(t,p_R(t)) = \min\left\{ \max\left\{ -\frac{\Phi^I_y(t,y,s)(p_R(t)-\mu)}{\Phi^I_{yy}(t,y,s)\sigma^2}, 0\right\}, 1\right\}$$

and, since π_I^* is independent of $p_R(\cdot)$ (i.e., the optimal investment strategy of the insurer does not depend on the reinsurance premium strategy of the reinsurer and the reinsurance strategy of the insurer), we can write

$$\pi_I^*(t) := \pi_I^*(t,p_R(t)) = -\frac{\Phi^I_y(t,y,s)(\tilde{\mu}-r_I)}{\Phi^I_{yy}(t,y,s)\tilde{\sigma}^2 s^{2\delta} y} - \frac{\Phi^I_{ys}(t,y,s)s}{\Phi^I_{yy}(t,y,s)y}.$$

To prove that the solutions are maxima of the characteristic operator, we need to show that the second-order optimality condition (SOOC) holds, i.e.

$$\begin{aligned} \frac{d^2\mathcal{D}^I\Phi^I(t,x)}{dq^2} &= \Phi^I_{yy}(t,y,s)\sigma^2 \overset{!}{<} 0, \\ \frac{d^2\mathcal{D}^I\Phi^I(t,x)}{d\pi_I^2} &= \Phi^I_{yy}(t,y,s)y^2\tilde{\sigma}^2 s^{2\delta} \overset{!}{<} 0. \end{aligned}$$

Since σ^2, y^2, $\tilde{\sigma}^2$ and $s^{2\delta}$ are non-negative, it is enough to show

$$\Phi^I_{yy}(t,y,s) < 0. \tag{3.10}$$

For the value function we have the ansatz

$$\Phi^I(t,y,s) = -\frac{1}{\beta_I}\exp(-\beta_I y \psi^I(t) + u^I(t)s^{-2\delta} + v^I(t)) \tag{3.11}$$

where the functions ψ^I, u^I and v^I are continuously differentiable with boundary conditions $\psi^I(T) = 1$ and $u^I(T) = v^I(T) = 0$. The boundary conditions for the functions ψ^I, u^I and v^I follows from the boundary condition (3.9). From the fact that β_I and the exponential function are positive, it follows that $\Phi^I(t,y,s) < 0$. With the ansatz for the value function

(3.11), we get

$$\begin{aligned}
\Phi^I_t(t,y,s) &= \Phi^I(t,y,s)[-\beta_I y \psi^I_t(t) + u^I_t(t)s^{-2\delta} + v^I_t(t)],\\
\Phi^I_y(t,y,s) &= \Phi^I(t,y,s)[-\beta_I \psi^I(t)],\\
\Phi^I_s(t,y,s) &= \Phi^I(t,y,s)[-2\delta u^I(t)s^{-2\delta-1}],\\
\Phi^I_{yy}(t,y,s) &= \Phi^I(t,y,s)\beta_I^2\psi^I(t)^2,\\
\Phi^I_{ss}(t,y,s) &= \Phi^I(t,y,s)[4\delta^2 u^I(t)^2 s^{-4\delta-2} + 2\delta(2\delta+1)u^I(t)s^{-2\delta-2}],\\
\Phi^I_{ys}(t,y,s) &= \Phi^I(t,y,s)2\delta\beta_I\psi^I(t)u^I(t)s^{-2\delta-1}.
\end{aligned}$$

For the optimal investment and reinsurance strategy of the insurer we get

$$\begin{aligned}
q^*(t,p_R(t)) &= \min\left\{\max\left\{-\frac{\Phi^I_y(t,y,s)(p_R(t)-\mu)}{\Phi^I_{yy}(t,y,s)\sigma^2},0\right\},1\right\}\\
&= \min\left\{\max\left\{\cancel{-}\frac{\cancel{\Phi^I(t,y,s)}(\cancel{-}\cancel{\beta_I}\,\cancel{\psi^I(t)})(p_R(t)-\mu)}{\cancel{\Phi^I(t,y,s)}\beta_I^{\cancel{2}}\psi^I(t)^{\cancel{2}}\sigma^2},0\right\},1\right\}\\
&= \min\left\{\max\left\{\frac{p_R(t)-\mu}{\beta_I\psi^I(t)\sigma^2},0\right\},1\right\}\\
&= \min\left\{\max\left\{\iota(t,p_R(t)),0\right\},1\right\},
\end{aligned}$$

where ι is defined by (cf. (3.8))

$$\iota(t,p_R) := \frac{p_R-\mu}{\sigma^2\beta_I\psi^I(t)},$$

and

$$\begin{aligned}
\pi^*_I(t) &= -\frac{\Phi^I_y(t,y,s)(\tilde{\mu}-r_I)}{\Phi^I_{yy}(t,y,s)\tilde{\sigma}^2 s^{2\delta}y} - \frac{\Phi^I_{ys}(t,y,s)s}{\Phi^I_{yy}(t,y,s)y}\\
&= \cancel{-}\frac{\cancel{\Phi^I(t,y,s)}(\cancel{-}\cancel{\beta_I}\,\cancel{\psi^I(t)})(\tilde{\mu}-r_I)}{\cancel{\Phi^I(t,y,s)}\beta_I^{\cancel{2}}\psi^I(t)^{\cancel{2}}s^{2\delta}y\tilde{\sigma}^2} - \frac{\cancel{\Phi^I(t,y,s)}2\delta\cancel{\beta_I}\,\cancel{\psi^I(t)}u^I(t)s^{-2\delta\cancel{-1}}\cancel{s}}{\cancel{\Phi^I(t,y,s)}\beta_I^{\cancel{2}}\psi^I(t)^{\cancel{2}}y}\\
&= \frac{1}{\beta_I\psi^I(t)ys^{2\delta}}\left(\frac{\tilde{\mu}-r_I}{\tilde{\sigma}^2} - 2\delta u^I(t)\right).
\end{aligned}$$

q^* and π^*_I maximize the characteristic operator, since the condition (3.10) is fulfilled:

$$\Phi^I_{yy}(t,y,s) = \Phi^I(t,y,s)\beta_I^2\psi^I(t)^2 < 0.$$

If we plug π^*_I and the derivatives of Φ^I in the HJB-equation, we get

$$0 = \Phi^I(t,y,s)\Bigg[-\beta_I\psi^I_t(t)y + u^I_t(t)s^{-2\delta} + v^I_t(t) - \beta_I\psi^I(t)[\theta_I\mu - (p_R(t)-\mu)(1-q^*(t,p_R(t)))]$$

$$
\begin{aligned}
&- \beta_I \psi^I(t) r_I y - \cancel{\beta_I \psi^I(t)}(\tilde{\mu} - r_I)\cancel{y} \frac{1}{\cancel{\beta_I \psi^I(t)}\cancel{y} s^{2\delta}} \left(\frac{\tilde{\mu} - r_I}{\tilde{\sigma}^2} - 2\delta u^I(t) \right) \\
&- 2\delta u^I(t) s^{-2\delta} \tilde{\mu} + \frac{1}{2} \beta_I^2 \psi^I(t)^2 q^*(t, p_R(t))^2 \sigma^2 \\
&+ \frac{1}{2} \cancel{\beta_I^2 \psi^I(t)^2 y^2} \tilde{\sigma}^2 \cancel{s^{2\delta}} \frac{1}{\cancel{\beta_I^2 \psi^I(t)^2 y^2}\, s^{\cancelto{2\delta}{4\delta}}} \left(\frac{\tilde{\mu} - r_I}{\tilde{\sigma}^2} - 2\delta u^I(t) \right)^2 \\
&+ 2\delta \cancel{\beta_I \psi^I(t)} u^I(t) \cancel{s^{-2\delta-1}} \tilde{\sigma}^2 \cancel{y} \cancel{s^{2\delta+1}} \frac{1}{\cancel{\beta_I \psi^I(t)}\cancel{y} s^{2\delta}} \left(\frac{\tilde{\mu} - r_I}{\tilde{\sigma}^2} - 2\delta u^I(t) \right) \\
&+ \frac{1}{\cancel{2}} \cancelto{2}{4}\, \delta^2 u^I(t)^2 s^{\cancelto{-2\delta}{-4\delta-2}} \tilde{\sigma}^2 \cancel{s^{2\delta+2}} + \frac{1}{\cancel{2}} 2\delta(2\delta + 1) u^I(t) \cancel{s^{-2\delta-2}} \tilde{\sigma}^2 \cancel{s^{2\delta+2}} \Bigg].
\end{aligned}
$$

Since $\Phi^I(t, y, s) < 0$, this is equivalent to

$$
\begin{aligned}
0 = &- \beta_I \psi_t^I(t) y + u_t^I(t) s^{-2\delta} + v_t^I(t) - \beta_I \psi^I(t) [\theta_I \mu - (p_R(t) - \mu)(1 - q^*(t, p_R(t)))] \\
&- \beta_I \psi^I(t) r_I y - s^{-2\delta} \tilde{\gamma}^2 + 2\delta s^{-2\delta} (\tilde{\mu} - r_I) u^I(t) \\
&- 2\delta u^I(t) s^{-2\delta} \tilde{\mu} + \frac{1}{2} \beta_I^2 \psi^I(t)^2 q^*(t, p_R(t))^2 \sigma^2 \\
&+ \frac{1}{2} s^{-2\delta} \tilde{\sigma}^2 \left(\frac{\tilde{\gamma}^2}{\tilde{\sigma}^2} - 4\delta u^I(t) \frac{\tilde{\mu} - r_I}{\tilde{\sigma}^2} + 4\delta^2 u^I(t)^2 \right) \\
&+ 2\delta s^{-2\delta} u^I(t) (\tilde{\mu} - r_I) - 4\delta^2 s^{-2\delta} u^I(t)^2 \tilde{\sigma}^2 \\
&+ 2\delta^2 u^I(t)^2 s^{-2\delta} \tilde{\sigma}^2 + \delta(2\delta + 1) u^I(t) \tilde{\sigma}^2,
\end{aligned}
$$

where $\tilde{\gamma} := \frac{\tilde{\mu} - r_I}{\tilde{\sigma}}$ is the market price of risk. It follows

$$
\begin{aligned}
0 = &- \beta_I \psi_t^I(t) y + u_t^I(t) s^{-2\delta} + v_t^I(t) - \beta_I \psi^I(t) [\theta_I \mu - (p_R(t) - \mu)(1 - q^*(t, p_R(t)))] \\
&- \beta_I \psi^I(t) r_I y - s^{-2\delta} \tilde{\gamma}^2 + \cancel{2\delta s^{-2\delta} \tilde{\mu} u^I(t)} - 2\delta s^{-2\delta} r_I u^I(t) \\
&\cancel{-2\delta u^I(t) s^{-2\delta} \tilde{\mu}} + \frac{1}{2} \beta_I^2 \psi^I(t)^2 q^*(t, p_R(t))^2 \sigma^2 \\
&+ \frac{1}{2} s^{-2\delta} \cancel{\tilde{\sigma}^2} \frac{\tilde{\gamma}^2}{\cancel{\tilde{\sigma}^2}} - \frac{1}{\cancel{2}} s^{-2\delta} \cancel{\tilde{\sigma}^2} \cancelto{2}{4}\, \delta u^I(t) \frac{\tilde{\mu} - r_I}{\cancel{\tilde{\sigma}^2}} + \frac{1}{\cancel{2}} s^{-2\delta} \tilde{\sigma}^2 \cancelto{2}{4}\, \delta^2 u^I(t)^2 \\
&+ 2\delta s^{-2\delta} u^I(t) (\tilde{\mu} - r_I) - 4\delta^2 s^{-2\delta} u^I(t)^2 \tilde{\sigma}^2 \\
&+ 2\delta^2 u^I(t)^2 s^{-2\delta} \tilde{\sigma}^2 + \delta(2\delta + 1) u^I(t) \tilde{\sigma}^2.
\end{aligned}
$$

That implies,

$$
\begin{aligned}
0 = &- \beta_I \psi_t^I(t) y + u_t^I(t) s^{-2\delta} + v_t^I(t) - \beta_I \psi^I(t) [\theta_I \mu - (p_R(t) - \mu)(1 - q^*(t, p_R(t)))] \\
&- \beta_I \psi^I(t) r_I y - \cancelto{\frac{1}{2}}{1}\, s^{-2\delta} \tilde{\gamma}^2 - 2\delta s^{-2\delta} r_I u^I(t) \\
&+ \frac{1}{2} \beta_I^2 \psi^I(t)^2 q^*(t, p_R(t))^2 \sigma^2 + \cancel{\frac{1}{2} s^{-2\delta} \tilde{\gamma}^2} \cancel{-2\delta s^{-2\delta} u^I(t) (\tilde{\mu} - r_I)} \cancel{+2\delta^2 s^{-2\delta} \tilde{\sigma}^2 u^I(t)^2}
\end{aligned}
$$

$$\begin{aligned}
&\cancel{+2\delta s^{-2\delta} u^I(t)(\mu - r_I)} \cancel{-4\delta^2 s^{-2\delta} u^I(t)^2 \tilde{\sigma}^2} \\
&\cancel{+2\delta^2 u^I(t)^2 s^{-2\delta} \tilde{\sigma}^2} + \delta(2\delta + 1)u^I(t)\tilde{\sigma}^2.
\end{aligned}$$

Hence, the HJB-equation is given by

$$\begin{aligned}
0 = &\Big[y(-\beta_I \psi_t^I(t) - \beta_I \psi^I(t) r_I) \Big] \\
&+ \Big[s^{-2\delta}(u_t^I(t) - 2\delta u^I(t) r_I - \frac{1}{2}\tilde{\gamma}^2) \Big] \\
&+ \Big[v_t^I(t) - \beta_I \psi^I(t)[\theta_I \mu - (p_R(t) - \mu)(1 - q^*(t, p_R(t)))] \\
&\quad + \frac{1}{2}\beta_I^2 \psi^I(t)^2 q^*(t, p_R(t))^2 \sigma^2 + \delta(2\delta + 1)u^I(t)\tilde{\sigma}^2 \Big].
\end{aligned}$$

In the next step, we take a closer look to the three terms in the HJB-equation above. We can solve these three terms separately, since the first term depends on y and not on s, the second term on s and not on y, and the third term is independent of y and s. Since y and s are not necessarily zero, it has to hold

$$0 = -\beta_I \psi_t^I(t) - \beta_I \psi^I(t) r_I \tag{3.12}$$

$$0 = u_t^I(t) - 2\delta u^I(t) r_I - \frac{1}{2}\tilde{\gamma}^2 \tag{3.13}$$

$$\begin{aligned}
0 = &v_t^I(t) - \beta_I \psi^I(t)[\theta_I \mu - (p_R(t) - \mu)(1 - q^*(t, p_R(t)))] \\
&+ \frac{1}{2}\beta_I^2 \psi^I(t)^2 q^*(t, p_R(t))^2 \sigma^2 + \delta(2\delta + 1)u^I(t)\tilde{\sigma}^2.
\end{aligned} \tag{3.14}$$

We start solving the Equation (3.12). It follows that we have the following homogeneous ODE with the boundary condition for the function ψ^I:

$$\begin{aligned}
\psi_t^I(t) &= -r_I \psi^I(t), \\
\psi^I(T) &= 1.
\end{aligned}$$

The solution to the homogeneous ODE is given by

$$\psi^I(t) = e^{r_I(T-t)}. \tag{3.15}$$

By (3.15), it holds that $\psi^I(t) > 0$ for all $t \in [0, T]$. Since we assume that for the reinsurance premium strategy p_R it has to hold $p_R(t) \in [p_I, p]$ for all $t \in [0, T]$, we get $p_R(t) - \mu \geq p_I - \mu = (1 + \theta_I)\mu - \mu = \theta_I \mu > 0$ for all $t \in [0, T]$ (θ_I and μ are positive). Therefore, $\iota(t, p_R(t)) > 0$, since $\beta_I > 0$ and $\sigma^2 > 0$. Hence, for the optimal reinsurance strategy q^* we have

$$\begin{aligned} q^*(t, p_R(t)) &= \min\{\max\{\iota(t, p_R(t)), 0\}, 1\} \\ &= \min\{\iota(t, p_R(t)), 1\}. \end{aligned}$$

Next, we consider the Equation (3.13). First, let $\delta \neq 0$. Then the function u^I is described by the following linear non-homogeneous ODE with boundary condition

$$\begin{aligned} u^I_t(t) &= 2\delta r_I u^I(t) + \frac{1}{2}\tilde{\gamma}^2 \\ u^I(T) &= 0. \end{aligned}$$

To get the solution of the linear non-homogeneous ODE, we use the integrating factor method. For the above equation, the integrating factor is given by

$$\tilde{u}^I(t) = e^{2\delta r_I t}.$$

Next, we multiply the linear non-homogeneous ODE with the reciprocal integrating factor

$$\begin{aligned} \tilde{u}^I(t)^{-1} u^I_t(t) &= \tilde{u}^I(t)^{-1}\left(2\delta r_I u^I(t) + \frac{1}{2}\tilde{\gamma}^2\right) \\ &\Leftrightarrow \\ e^{-2\delta r_I t} u^I_t(t) &= 2\delta r_I e^{-2\delta r_I t} u^I(t) + \frac{1}{2}\tilde{\gamma}^2 e^{-2\delta r_I t} \\ &\Leftrightarrow \\ e^{-2\delta r_I t} u^I_t(t) - 2\delta r_I e^{-2\delta r_I t} u^I(t) &= \frac{1}{2}\tilde{\gamma}^2 e^{-2\delta r_I t} \\ &\Leftrightarrow \\ \frac{d}{dt}\left(e^{-2\delta r_I t} u^I(t)\right) &= \frac{1}{2}\tilde{\gamma}^2 e^{-2\delta r_I t}. \end{aligned} \tag{3.16}$$

Since $u^I(T) = 0$, it follows

$$\begin{aligned} e^{-2\delta r_I t} u^I(t) &= e^{-2\delta r_I t} u^I(t) - e^{-2\delta r_I T} u^I(T) = \int_T^t \frac{d}{dt}\left(e^{-2\delta r_I \tau} u^I_t(\tau)\right) d\tau \\ &\overset{(3.16)}{=} \frac{1}{2}\tilde{\gamma}^2 \int_T^t e^{-2\delta r_I \tau} d\tau \\ &= -\frac{\tilde{\gamma}^2}{4\delta r_I}(e^{-2\delta r_I t} - e^{-2\delta r_I T}) \\ &\Leftrightarrow \end{aligned}$$

$$\begin{aligned}u^I(t) &= -\frac{\tilde{\gamma}^2}{4\delta r_I} e^{2\delta r_I t}(e^{-2\delta r_I t} - e^{-2\delta r_I T})\\ &= -\frac{\tilde{\gamma}^2}{4\delta r_I}(1 - e^{-2\delta r_I (T-t)}).\end{aligned}$$

If $\delta = 0$, then the function u^I is described by the ODE with boundary condition

$$u_t^I(t) = \frac{1}{2}\tilde{\gamma}^2 \tag{3.17}$$

$$u^I(T) = 0. \tag{3.18}$$

Hence, the solution is given by

$$\begin{aligned}u^I(t) &\overset{(3.18)}{=} u^I(t) - u^I(T) = \int_T^t u_t^I(\tau)d\tau\\ &\overset{(3.17)}{=} \int_T^t \frac{1}{2}\tilde{\gamma}^2 d\tau = -\frac{1}{2}\tilde{\gamma}^2(T-t).\end{aligned}$$

All in all, we have

$$u^I(t) = \begin{cases} -\frac{\tilde{\gamma}^2}{4\delta r_I}(1 - e^{-2\delta r_I (T-t)}), & \delta \neq 0,\\ -\frac{1}{2}\tilde{\gamma}^2(T-t), & \delta = 0.\end{cases} \tag{3.19}$$

Now, we will solve the Equation (3.14) for two cases:

Case 1: If $\iota(t, p_R(t)) \geq 1$, then $q^*(t, p_R(t)) = 1$ and we get for (3.14)

$$v_t^I(t) = \beta_I \psi^I(t)\theta_I \mu - \frac{1}{2}\beta_I^2 \psi^I(t)^2 \sigma^2 - \delta(2\delta+1)u^I(t)\tilde{\sigma}^2. \tag{3.20}$$

Since we have the boundary condition $v^I(T) = 0$, it follows

$$\begin{aligned}v^I(t) =& v^I(t) - v^I(T) = \int_T^t v_t^I(\tau)d\tau\\ \overset{(3.20)}{=}& \beta_I\theta_I\mu \int_T^t \psi^I(\tau)d\tau - \frac{1}{2}\beta_I^2\sigma^2 \int_T^t \psi^I(\tau)^2 d\tau - \delta(2\delta+1)\tilde{\sigma}^2 \int_T^t u^I(\tau)d\tau.\end{aligned}$$

We have

$$\begin{aligned}\int_T^t \psi^I(\tau)d\tau = \int_T^t e^{r_I(T-\tau)}d\tau\\ &= -\frac{1}{r_I}(e^{r_I(T-t)} - 1)\\ &= \frac{1}{r_I}(1 - \psi^I(t)),\end{aligned} \tag{3.21}$$

$$\begin{aligned}\int_T^t \psi^I(\tau)^2 d\tau =& \int_T^t e^{2r_I(T-\tau)} d\tau \\ =& \frac{1}{2r_I}(1-\psi^I(t)^2)\end{aligned} \tag{3.22}$$

and for $\delta \neq 0$

$$\begin{aligned}\int_T^t u^I(\tau) d\tau =& -\frac{\tilde{\gamma}^2}{4\delta r_I}\int_T^t (1-e^{-2\delta r_I(T-\tau)})d\tau \\ =& \frac{\tilde{\gamma}^2}{4\delta r_I}(T-t) - \frac{\tilde{\gamma}^2}{8\delta^2 r_I^2}(1-e^{-2\delta r_I(T-t)}) \\ =& \frac{\tilde{\gamma}^2}{4\delta r_I}(T-t) + \frac{1}{2\delta r_I}u^I(t).\end{aligned} \tag{3.23}$$

Hence, for $\delta \neq 0$ we have

$$\begin{aligned}v^I(t) \overset{(3.21),\,(3.22)}{\underset{\&\,(3.23)}{=}}& \frac{\beta_I\theta_I\mu}{r_I}(1-\psi^I(t)) - \frac{\beta_I^2\sigma^2}{4r_I}(1-\psi^I(t)^2) \\ &- \frac{\tilde{\gamma}^2(2\delta+1)\tilde{\sigma}^2}{4r_I}(T-t) - \frac{(2\delta+1)\tilde{\sigma}^2}{2r_I}u^I(t)\end{aligned}$$

and for $\delta = 0$

$$\begin{aligned}v^I(t) \overset{(3.21),\,(3.22)}{=}& \frac{\beta_I\theta_I\mu}{r_I}(1-\psi^I(t)) - \frac{\beta_I^2\sigma^2}{4r_I}(1-\psi^I(t)^2) \\ \overset{(3.19)}{=}& \frac{\beta_I\theta_I\mu}{r_I}(1-\psi^I(t)) - \frac{\beta_I^2\sigma^2}{4r_I}(1-\psi^I(t)^2) \\ &- \frac{\tilde{\gamma}^2(2\delta+1)\tilde{\sigma}^2}{4r_I}(T-t) - \frac{(2\delta+1)\tilde{\sigma}^2}{2r_I}u^I(t).\end{aligned}$$

All in all, we have

$$\begin{aligned}v^I(t) =& \frac{\beta_I\theta_I\mu}{r_I}(1-\psi^I(t)) - \frac{\beta_I^2\sigma^2}{4r_I}(1-\psi^I(t)^2) \\ &- \frac{\tilde{\gamma}^2(2\delta+1)\tilde{\sigma}^2}{4r_I}(T-t) - \frac{(2\delta+1)\tilde{\sigma}^2}{2r_I}u^I(t).\end{aligned}$$

Case 2: If $\iota(t,p_R(t)) < 1$, then $q^*(t,p_R(t)) = \iota(t,p_R(t))$ and we get for (3.14)

$$\begin{aligned}v_t^I(t) =& \beta_I\psi^I(t)\left[\theta_I\mu - (p_R(t)-\mu)\left(1-\frac{p_R(t)-\mu}{\beta_I\sigma^2\psi^I(t)}\right)\right] \\ &- \frac{1}{2}\beta_I^2\psi^I(t)^2\left(\frac{p_R(t)-\mu}{\beta_I\sigma^2\psi^I(t)}\right)^2\sigma^2 - \delta(2\delta+1)u^I(t)\tilde{\sigma}^2 \\ =& \beta_I\psi^I(t)\theta_I\mu - \beta_I\psi^I(t)(p_R(t)-\mu) + \cancel{\beta_I\psi^I(t)}\frac{(p_R(t)-\mu)^2}{\cancel{\beta_I\psi^I(t)}\sigma^2}\end{aligned}$$

$$
\begin{aligned}
&- \cancel{\beta_I^2\psi^I(t)^2}\sigma^2 \frac{1}{2}\frac{(p_R(t)-\mu)^2}{\cancel{\beta_I^2\psi^I(t)^2}\sigma^{\cancel{4}\,2}} - \delta(2\delta+1)\tilde{\sigma}^2 u^I(t) \\
=&\beta_I\psi^I(t)\theta_I\mu - \beta_I\psi^I(t)(p_R(t)-\mu) + \frac{(p_R(t)-\mu)^2}{\sigma^2} - \frac{1}{2}\frac{(p_R(t)-\mu)^2}{\sigma^2} \\
&- \delta(2\delta+1)\tilde{\sigma}^2 u^I(t) \\
=&\beta_I\psi^I(t)\theta_I\mu - \beta_I\psi^I(t)(p_R(t)-\mu) + \frac{(p_R(t)-\mu)^2}{2\sigma^2} - \delta(2\delta+1)\tilde{\sigma}^2 u^I(t). \quad (3.24)
\end{aligned}
$$

With the boundary condition $v^I(T) = 0$, it follows for $\delta \neq 0$

$$
\begin{aligned}
v^I(t) =& v^I(t) - v^I(T) = \int_T^t v_t^I(\tau)d\tau \\
\overset{(3.24)}{=}& \beta_I\theta_I\mu \int_T^t \psi^I(\tau)d\tau - \beta_I \int_T^t \psi^I(\tau)(p_R(\tau)-\mu)d\tau \\
&+ \int_T^t \frac{(p_R(\tau)-\mu)^2}{2\sigma^2}d\tau - \delta(2\delta+1)\tilde{\sigma}^2 \int_t^T u^I(\tau)d\tau \\
\overset{(3.21)\ \&\ (3.23)}{=}& \frac{\beta_I\theta_I\mu}{r_I}(1-\psi^I(t)) - \beta_I \int_T^t \psi^I(\tau)(p_R(\tau)-\mu)d\tau \\
&+ \int_T^t \frac{(p_R(\tau)-\mu)^2}{2\sigma^2}d\tau - \frac{\tilde{\gamma}^2(2\delta+1)\tilde{\sigma}^2}{4r_I}(T-t) \\
&- \frac{(2\delta+1)\tilde{\sigma}^2}{2r_I}u^I(t)
\end{aligned}
$$

and for $\delta = 0$

$$
\begin{aligned}
v^I(t) =& v^I(t) - v^I(T) = \int_T^t v_t^I(\tau)d\tau \\
\overset{(3.24)}{=}& \beta_I\theta_I\mu \int_T^t \psi^I(\tau)d\tau - \beta_I \int_T^t \psi^I(\tau)(p_R(\tau)-\mu)d\tau \\
&+ \int_T^t \frac{(p_R(\tau)-\mu)^2}{2\sigma^2}d\tau \\
\overset{(3.21)}{=}& \frac{\beta_I\theta_I\mu}{r_I}(1-\psi^I(t)) - \beta_I \int_T^t \psi^I(\tau)(p_R(\tau)-\mu)d\tau \\
&+ \int_T^t \frac{(p_R(\tau)-\mu)^2}{2\sigma^2}d\tau \\
\overset{(3.19)}{=}& \frac{\beta_I\theta_I\mu}{r_I}(1-\psi^I(t)) - \beta_I \int_T^t \psi^I(\tau)(p_R(\tau)-\mu)d\tau \\
&+ \int_T^t \frac{(p_R(\tau)-\mu)^2}{2\sigma^2}d\tau - \frac{\tilde{\gamma}^2(2\delta+1)\tilde{\sigma}^2}{4r_I}(T-t) - \frac{(2\delta+1)\tilde{\sigma}^2}{2r_I}u^I(t).
\end{aligned}
$$

All in all, we have

$$\begin{aligned} v^I(t) =& \frac{\beta_I \theta_I \mu}{r_I}(1-\psi^I(t)) - \beta_I \int_T^t \psi^I(\tau)(p_R(\tau)-\mu)d\tau \\ &+ \int_T^t \frac{(p_R(\tau)-\mu)^2}{2\sigma^2}d\tau - \frac{\tilde{\gamma}^2(2\delta+1)\tilde{\sigma}^2}{4r_I}(T-t) - \frac{(2\delta+1)\tilde{\sigma}^2}{2r_I}u^I(t). \end{aligned} \tag{3.25}$$

Now we consider the **optimization problem of the reinsurer**. Again, we will use the HJB-approach. The HJB-equation of the reinsurer is given by

$$\begin{aligned} 0 =& \sup_{p_R(\cdot)\in\Lambda_R} \mathcal{D}^R \Phi^R(t,y), \\ \Phi^R(T,y) =& U_R(y), \end{aligned} \tag{3.26}$$

where $\mathcal{D}^R$ is the characteristic operator of the reinsurer defined by

$$\begin{aligned} \mathcal{D}^R\Phi^R(t,y) :=& \Phi_t^R(t,y) + ((1-q^*(t,p_R(t)))(p_R(t)-\mu) + r_R y)\Phi_y^R(t,y) \\ &+ \frac{1}{2}\sigma^2(1-q^*(t,p_R(t)))^2\Phi_{yy}^R(t,y). \end{aligned}$$

For the value function we have the ansatz

$$\Phi^R(t,y) = -\frac{1}{\beta_R}\exp(-\beta_R y\psi^R(t) + v^R(t))$$

where ψ^R and v^R are continuously differentiable functions with boundary conditions $\psi^R(T) = 1$ and $v^R(T) = 0$, which follow from the boundary condition (3.26). The value function of the reinsurer Φ^R is negative, since β_R and the exponential function are positive. With the ansatz for the value function, we get

$$\begin{aligned} \Phi_t^R(t,y) &= \Phi^R(t,y)(-\beta_R y\psi_t^R(t) + v_t^R(t)), \\ \Phi_y^R(t,y) &= \Phi^R(t,y)(-\beta_R\psi^R(t)), \\ \Phi_{yy}^R(t,y) &= \Phi^R(t,y)\beta_R^2\psi^R(t)^2. \end{aligned}$$

Since Φ^R is negative, it follows for the HJB-equation

$$\begin{aligned} 0 = \Phi^R(t,y) \inf_{p_R(\cdot)\in\Lambda_R} \Big[& -\beta_R y\psi_t^R(t) + v_t^R(t) \\ & -\beta_R\psi^R(t)[(1-q^*(t,p_R(t)))(p_R(t)-\mu) + r_R y] \\ & + \frac{1}{2}\beta_R^2\psi^R(t)^2\sigma^2(1-q^*(t,p_R(t)))^2\Big] \end{aligned}$$

$$
\begin{aligned}
&= \Phi^R(t,y) \inf_{p_R(\cdot)\in\Lambda_R} \Bigg[\bigg(- \beta_R \psi_t^R(t) - \beta_R \psi^R(t) r_R \bigg) y \\
&\qquad + \bigg(v_t^R(t) - \beta_R \psi^R(t)(1-q^*(t,p_R(t)))(p_R(t)-\mu) \\
&\qquad + \frac{1}{2}\beta_R^2 \psi^R(t)^2 \sigma^2 (1-q^*(t,p_R(t)))^2 \bigg)\Bigg] \qquad (3.27)
\end{aligned}
$$

The first term of (3.27) depends on y and not on $p_R(\cdot)$ and the second term on $p_R(\cdot)$ and not on y. Hence, we can solve both parts separately

$$
\begin{aligned}
0 &= - \beta_R \psi_t^R(t) - \beta_R \psi^R(t) r_R, \qquad (3.28) \\
0 &= \inf_{p_R(\cdot)\in\Lambda_R} \Bigg[v_t^R(t) - \beta_R \psi^R(t)(1-q^*(t,p_R(t)))(p_R(t)-\mu) \\
&\quad + \frac{1}{2}\beta_R^2 \psi^R(t)^2 \sigma^2 (1-q^*(t,p_R(t)))^2 \Bigg]. \qquad (3.29)
\end{aligned}
$$

By (3.28), the function ψ^R is characterized by the homogeneous ODE with boundary condition

$$
\begin{aligned}
\psi_t^R(t) &= -r_R \psi^R(t), \\
\psi^R(T) &= 1.
\end{aligned}
$$

Hence,

$$\psi^R(t) = e^{r_R(T-t)}.$$

To find the optimal reinsurance premium strategy $p_R^*(\cdot)$ we consider different cases:

Case 1: If $\iota(t,p_R(t)) \geq 1$, then $q^*(t) := q^*(t,p_R(t)) = 1$ and (3.29) is given by

$$0 = \inf_{p_R(\cdot)\in\Lambda_R} \Big[v_t^R(t) \Big].$$

Hence, the function v^R is characterized by

$$
\begin{aligned}
v_t^R(t) &= 0, \\
v^R(T) &= 0.
\end{aligned}
$$

Therefore, $v^R(t) \equiv 0$ and the optimal reinsurance premium strategy is given by

$$p_R^*(t) \equiv p_R$$

for any $p_R \in [p_I, p]$.

Case 2: If $\iota(t, p_R(t)) < 1$, then $q^*(t, p_R(t)) = \iota(t, p_R(t)) = \frac{p_R(t)-\mu}{\sigma^2 \beta_I \psi^I(t)}$ and Equation (3.29) becomes

$$0 = \inf_{p_R(\cdot)\in\Lambda_R} \Bigg[v_t^R(t) - \beta_R \psi^R(t) \left(1 - \frac{p_R(t)-\mu}{\sigma^2 \beta_I \psi^I(t)}\right)(p_R(t) - \mu) + \frac{1}{2}\beta_R^2 \psi^R(t)^2 \sigma^2 \left(1 - \frac{p_R(t)-\mu}{\sigma^2 \beta_I \psi^I(t)}\right)^2 \Bigg].$$

Hence, we want to minimize the function $G : \Lambda_R \to \mathbb{R}$ given by

$$\begin{aligned} G(p_R) =& v_t^R(t) - \beta_R \psi^R(t) \left(1 - \frac{p_R(t)-\mu}{\sigma^2 \beta_I \psi^I(t)}\right)(p_R(t) - \mu) \\ &+ \frac{1}{2}\beta_R^2 \psi^R(t)^2 \sigma^2 \left(1 - \frac{p_R(t)-\mu}{\sigma^2 \beta_I \psi^I(t)}\right)^2. \end{aligned}$$

By the first-order optimality condition (FOOC), we get

$$\begin{aligned} \left.\frac{dG(p_R)}{dp_R}\right|_{p_R = p_R^*(t)} =& \frac{\beta_R \psi^R(t)}{\sigma^2 \beta_I \psi^I(t)}(p_R^*(t) - \mu) - \beta_R \psi^R(t)\left(1 - \frac{p_R^*(t)-\mu}{\sigma^2 \beta_I \psi^I(t)}\right) \\ &- \frac{\beta_R^2 \psi^R(t)^2 \cancel{2\sigma^2}}{\cancel{2\sigma^2} \beta_I \psi^I(t)}\left(1 - \frac{p_R^*(t)-\mu}{\sigma^2 \beta_I \psi^I(t)}\right) \\ =& \left(\frac{2\beta_R \psi^R(t)}{\sigma^2 \beta_I \psi^I(t)} + \frac{\beta_R^2 \psi^R(t)^2}{\sigma^2 \beta_I^2 \psi^I(t)^2}\right)(p_R^*(t) - \mu) \\ &- \left(\beta_R \psi^R(t) + \frac{\beta_R^2 \psi^R(t)^2}{\beta_I \psi^I(t)}\right) \\ \stackrel{!}{=}& 0 \end{aligned}$$

Hence,

$$\begin{aligned} p_R^*(t) - \mu =& \frac{\cancel{\beta_R \psi^R(t)}(\beta_I \psi^I(t) + \beta_R \psi^R(t))}{\cancel{\beta_I \psi^I(t)}} \times \frac{\sigma^2 \beta_I^{\cancel{2}} \psi^I(t)^{\cancel{2}}}{\cancel{\beta_R \psi^R(t)}(2\beta_I \psi^I(t) + \beta_R \psi^R(t))} \\ =& \sigma^2 \beta_I \psi^I(t) \frac{\beta_I \psi^I(t) + \beta_R \psi^R(t)}{2\beta_I \psi^I(t) + \beta_R \psi^R(t)} \\ =:& \ \sigma^2 \beta_I \psi^I(t) M(t). \end{aligned}$$

Since it has to hold that $p_R^*(t) \in [p_I, p]$ for all $t \in [0, T]$, we get

$$p_R^*(t) = \min\{\max\{\mu + \sigma^2 \beta_I \psi^I(t) M(t), p_I\}, p\}.$$

By the second condition of minimization, it has to hold

$$\frac{d^2G(p_R)}{dp_R^2}\bigg|_{p_R=p_R^*(t)} = \frac{2\beta_R\psi^R(t)}{\sigma^2\beta_I\psi^I(t)} + \frac{\beta_R^2\psi^R(t)^2}{\sigma^2\beta_I^2\psi^I(t)^2} \overset{!}{>} 0.$$

This is true, since $\beta_I, \beta_R > 0$ and ψ^I, ψ^R are positive functions.
We set $\tilde{p}(t) := \mu + \sigma^2\beta_I\psi^I(t)M(t)$ and therefore, $p_R^*(t) = \min\{\max\{\tilde{p}(t), p_I\}, p\}$. We consider three cases:

Case 2a: If $\tilde{p}(t) \leq p_I$, then

$$p_R^*(t) = p_I \text{ and therefore, } q^*(t) := q^*(t, p_R^*(t)) = \iota(t, p_I).$$

For Equation (3.29) it follows

$$\begin{aligned} v_t^R(t) =& \beta_R\psi^R(t)\left(1 - \frac{p_I - \mu}{\sigma^2\beta_I\psi^I(t)}\right)(p_I - \mu) \\ & - \frac{1}{2}\beta_R^2\psi^R(t)^2\sigma^2\left(1 - \frac{p_I - \mu}{\sigma^2\beta_I\psi^I(t)}\right)^2. \end{aligned} \tag{3.30}$$

By the boundary condition $v^R(T) = 0$, we get

$$\begin{aligned} v^R(t) =& v^R(t) - v^R(T) = \int_T^t v_t^R(\tau)d\tau \\ \overset{(3.30)}{=}& \beta_R(p_I - \mu)\int_T^t \psi^R(\tau)d\tau - \frac{\beta_R(p_I - \mu)^2}{\sigma^2\beta_I}\int_T^t \frac{\psi^R(\tau)}{\psi^I(\tau)}d\tau \\ & - \frac{1}{2}\sigma^2\beta_R^2\int_T^t \psi^R(\tau)^2 d\tau + \frac{\beta_R^2(p_I - \mu)}{\beta_I}\int_T^t \frac{\psi^R(\tau)^2}{\psi^I(\tau)}d\tau \\ & - \frac{\beta_R^2(p_I - \mu)^2}{2\sigma^2\beta_I^2}\int_T^t \left(\frac{\psi^R(\tau)}{\psi^I(\tau)}\right)^2 d\tau \end{aligned}$$

Analogous to (3.21) and (3.22), we have

$$\int_T^t \psi^R(\tau)d\tau = \frac{1}{r_R}(1 - \psi^R(t)), \tag{3.31}$$

$$\int_T^t \psi^R(\tau)^2 d\tau = \frac{1}{2r_R}(1 - \psi^R(t)^2) \tag{3.32}$$

and

$$\begin{aligned} \int_T^t \frac{\psi^R(\tau)}{\psi^I(\tau)}d\tau =& \int_T^t e^{(r_R - r_I)(T-\tau)}d\tau \\ =& \frac{-1}{r_R - r_I}(e^{(r_R - r_I)(T-t)} - 1) \end{aligned}$$

$$
=\frac{1}{r_R - r_I}\left(1 - \frac{\psi^R(t)}{\psi^I(t)}\right), \tag{3.33}
$$

$$
\begin{aligned}
\int_T^t \frac{\psi^R(\tau)^2}{\psi^I(\tau)} d\tau &= \int_T^t e^{(2r_R - r_I)(T-\tau)} d\tau \\
&= \frac{-1}{2r_R - r_I}(e^{(2r_R - r_I)(T-t)} - 1) \\
&= \frac{1}{2r_R - r_I}\left(1 - \frac{\psi^R(t)^2}{\psi^I(t)}\right),
\end{aligned} \tag{3.34}
$$

$$
\begin{aligned}
\int_T^t \frac{\psi^R(\tau)^2}{\psi^I(\tau)^2} d\tau &= \int_T^t e^{2(r_R - r_I)(T-\tau)} d\tau \\
&= \frac{-1}{2(r_R - r_I)}(e^{2(r_R - r_I)(T-t)} - 1) \\
&= \frac{1}{2(r_R - r_I)}\left(1 - \frac{\psi^R(t)^2}{\psi^I(t)^2}\right).
\end{aligned} \tag{3.35}
$$

For the function v^R it follows

$$
\begin{aligned}
v^R(t) =& \frac{\beta_R(p_I - \mu)}{r_R}(1 - \psi^R(t)) - \frac{\beta_R(p_I - \mu)^2}{\sigma^2 \beta_I (r_R - r_I)}\left(1 - \frac{\psi^R(t)}{\psi^I(t)}\right) \\
&- \frac{\sigma^2 \beta_R^2}{4r_I}(1 - \psi^R(t)^2) + \frac{\beta_R^2(p_I - \mu)}{\beta_I(2r_R - r_I)}\left(1 - \frac{\psi^R(t)^2}{\psi^I(t)}\right) \\
&- \frac{\beta_R^2(p_I - \mu)^2}{4\sigma^2 \beta_I^2 (r_R - r_I)}\left(1 - \left(\frac{\psi^R(t)}{\psi^I(t)}\right)^2\right).
\end{aligned}
$$

Since $p_I = (1 + \theta_I)\mu$, we get for the function v^I from (3.25)

$$
\begin{aligned}
v^I(t) &\overset{(3.25)}{=} \frac{\beta_I \theta_I \mu}{r_I}(1 - \psi^I(t)) - \beta_I \int_T^t \psi^I(\tau)(p_I - \mu) d\tau \\
&\quad + \int_T^t \frac{(p_I - \mu)^2}{2\sigma^2} d\tau - \frac{\tilde{\gamma}^2(2\delta + 1)\tilde{\sigma}^2}{4r_I}(T - t) \\
&\quad - \frac{(2\delta + 1)\tilde{\sigma}^2}{2r_I} u^I(t) \\
&\overset{(3.21)}{=} \frac{\beta_I((1 + \theta_I)\mu - p_I)}{r_I}(1 - \psi^I(t)) \\
&\quad - \left(\frac{(p_I - \mu)^2}{2\sigma^2} + \frac{\tilde{\gamma}^2(2\delta + 1)\tilde{\sigma}^2}{4r_I}\right)(T - t) \\
&\quad - \frac{(2\delta + 1)\tilde{\sigma}^2}{2r_I} u^I(t) \\
&= - \left(\frac{(p_I - \mu)^2}{2\sigma^2} + \frac{\tilde{\gamma}^2(2\delta + 1)\tilde{\sigma}^2}{4r_I}\right)(T - t) \\
&\quad - \frac{(2\delta + 1)\tilde{\sigma}^2}{2r_I} u^I(t).
\end{aligned}
$$

Case 2b: If $\tilde{p}(t) \geq p$, then

$$p_R^*(t) = p \text{ and therefore, } q^*(t) := q^*(t, p_R^*(t)) = \iota(t, p).$$

For Equation (3.29) it follows

$$\begin{aligned} v_t^R(t) =& \beta_R \psi^R(t) \left(1 - \frac{p-\mu}{\sigma^2 \beta_I \psi^I(t)}\right)(p-\mu) \\ &- \frac{1}{2}\beta_R^2 \psi^R(t)^2 \sigma^2 \left(1 - \frac{p-\mu}{\sigma^2 \beta_I \psi^I(t)}\right)^2. \end{aligned} \tag{3.36}$$

Hence,

$$\begin{aligned} v^R(t) =& v^R(t) - v^R(T) = \int_T^t v_t^R(\tau) d\tau \\ \overset{(3.36)}{=}& \beta_R(p-\mu) \int_T^t \psi^R(\tau) d\tau - \frac{\beta_R (p-\mu)^2}{\sigma^2 \beta_I} \int_T^t \frac{\psi^R(\tau)}{\psi^I(\tau)} d\tau \\ &- \frac{1}{2}\sigma^2 \beta_R^2 \int_T^t \psi^R(\tau)^2 d\tau + \frac{\beta_R^2 (p-\mu)}{\beta_I} \int_T^t \frac{\psi^R(\tau)^2}{\psi^I(\tau)} d\tau \\ &- \frac{\beta_R^2 (p-\mu)^2}{2\sigma^2 \beta_I^2} \int_T^t \left(\frac{\psi^R(\tau)}{\psi^I(\tau)}\right)^2 d\tau \\ \overset{(3.31)-(3.35)}{=}& \frac{\beta_R(p-\mu)}{r_R}(1 - \psi^R(t)) - \frac{\beta_R(p-\mu)^2}{\sigma^2 \beta_I (r_R - r_I)} \left(1 - \frac{\psi^R(t)}{\psi^I(t)}\right) \\ &- \frac{\sigma^2 \beta_R^2}{4 r_I}(1 - \psi^R(t)^2) + \frac{\beta_R^2 (p-\mu)}{\beta_I (2r_R - r_I)} \left(1 - \frac{\psi^R(t)^2}{\psi^I(t)}\right) \\ &- \frac{\beta_R^2 (p-\mu)^2}{4\sigma^2 \beta_I^2 (r_R - r_I)} \left(1 - \left(\frac{\psi^R(t)}{\psi^I(t)}\right)^2\right). \end{aligned}$$

For the function v^I we get from (3.25)

$$\begin{aligned} v^I(t) \overset{(3.25)}{=}& \frac{\beta_I \theta_I \mu}{r_I}(1 - \psi^I(t)) - \beta_I \int_T^t \psi^I(\tau)(p-\mu) d\tau \\ &+ \int_T^t \frac{(p-\mu)^2}{2\sigma^2} d\tau - \frac{\tilde{\gamma}^2 (2\delta+1)\tilde{\sigma}^2}{4 r_I}(T-t) \\ &- \frac{(2\delta+1)\tilde{\sigma}^2}{2 r_I} u^I(t) \\ \overset{(3.21)}{=}& \frac{\beta_I((1+\theta_I)\mu - p)}{r_I}(1 - \psi^I(t)) \\ &- \left(\frac{(p-\mu)^2}{2\sigma^2} + \frac{\tilde{\gamma}^2 (2\delta+1)\tilde{\sigma}^2}{4 r_I}\right)(T-t) \\ &- \frac{(2\delta+1)\tilde{\sigma}^2}{2 r_I} u^I(t). \end{aligned}$$

Case 2c: If $p_I < \tilde{p}(t) < p$, then

$$p_R^*(t) = \tilde{p}(t) = \mu + \sigma^2 \beta_I \psi^I(t) M(t)$$

and therefore,

$$q^*(t) := q^*(t, p_R^*(t)) = \frac{\cancel{\mu} + \cancel{\sigma^2 \beta_I \psi^I(t)} M(t) \cancel{- \mu}}{\cancel{\sigma^2 \beta_I \psi^I(t)}} = M(t).$$

For Equation (3.29) it follows

$$v_t^R(t) = \beta_R \psi^R(t)(1 - M(t))\sigma^2 \beta_I \psi^I(t) M(t) - \frac{1}{2}\beta_R^2 \psi^R(t)^2 \sigma^2 (1 - M(t))^2. \quad (3.37)$$

Hence,

$$\begin{aligned} v^R(t) =& v^R(t) - v^R(T) = \int_T^t v_t^R(\tau) d\tau \\ \overset{(3.37)}{=}& \sigma^2 \beta_R \beta_I \int_T^t (1 - M(\tau)) M(\tau) \psi^R(\tau) \psi^I(\tau) d\tau \\ &- \frac{\sigma^2 \beta_R^2}{2} \int_T^t \psi^R(\tau)^2 (1 - M(\tau))^2 d\tau. \end{aligned}$$

We get for the function v^I from (3.25)

$$\begin{aligned} v^I(t) \overset{(3.25)}{=}& \frac{\beta_I \theta_I \mu}{r_I}(1 - \psi^I(t)) - \beta_I \int_T^t \psi^I(\tau)(\tilde{p}(t) - \mu) d\tau \\ &+ \int_T^t \frac{(\tilde{p}(t) - \mu)^2}{2\sigma^2} d\tau - \frac{\tilde{\gamma}^2 (2\delta + 1)\tilde{\sigma}^2}{4 r_I}(T - t) \\ &- \frac{(2\delta + 1)\tilde{\sigma}^2}{2 r_I} u^I(t) \\ =& \frac{\beta_I \theta_I \mu}{r_I}(1 - \psi^I(t)) - \sigma^2 \beta_I^2 \int_T^t \psi^I(\tau)^2 M(\tau) d\tau \\ &+ \frac{\sigma^2 \beta_I^2}{2} \int_T^t \psi^I(\tau)^2 M(\tau)^2 d\tau - \frac{\tilde{\gamma}^2 (2\delta + 1)\tilde{\sigma}^2}{4 r_I}(T - t) \\ &- \frac{(2\delta + 1)\tilde{\sigma}^2}{2 r_I} u^I(t). \end{aligned}$$

□

3.4 Verification of the Solution

In this section, we verify that the solution in Section 3.3 obtained by the HJB-approach is indeed a solution to our Stackelberg game (3.7).

Theorem 3.4 (Verification Theorem, Theorem 2 in Bai et al. [2019]). The solutions $(p_R^*(\cdot), q^*(\cdot), \hat{\pi}_I^*(\cdot))$ obtained in Theorem 3.3 are admissible strategies, i.e., $p_R^*(\cdot) \in \Lambda_R$ and $(q^*(\cdot), \hat{\pi}_I^*(\cdot)) \in \Lambda_I$, and the optimal strategies to the Stackelberg game (3.7).

Proof. The proof is based on Lemma 1 and Theorem 2 in Bai et al. [2019]. They are stated in Appendix C and D.
First, we will show that the strategies are admissible, i.e., $p_R^*(\cdot) \in \Lambda_R$ and $(q^*(\cdot), \hat{\pi}_I^*(\cdot)) \in \Lambda_I$.
Reminder: The reinsurance premium strategy of the reinsurer in Theorem 3.3 is given by

$$p_R^*(t) = \min\{\max\{\mu + \sigma^2 \beta_I \psi^I(t) M(t), p_I\}, p\}.$$

- Since $p_R^*(\cdot)$ is non-random and continuous, we have that $p_R^*(\cdot)$ is progressively measurable and it holds $p_R^*(t) \in [p_I, p]$ for all $t \in [0, T]$.
- The optimal surplus process of the reinsurer is given by

$$dY_R^*(t) = [(1 - q^*(t))(p_R^*(t) - \mu) + r_R Y_R^*(t)]dt + \sigma(1 - q^*(t))dW_2(t). \tag{3.38}$$

  Set $b(t, y) := (1 - q^*(t))(p_R^*(t) - \mu) + r_R y$ and $\sigma(t, y) := \sigma(1 - q^*(t))$. Hence, for $y_1, y_2, y \in \mathbb{R}$ we get

$$\begin{aligned}
&\|b(t, y_1) - b(t, y_2)\| + \|\sigma(t, y_1) - \sigma(t, y_2)\| \\
&\quad = \|\cancel{(1 - q^*(t))(p_R^*(t) - \mu)} + r_R y_1 - \cancel{(1 - q^*(t))(p_R^*(t) - \mu)} - r_R y_2\| \\
&\qquad + \|\cancel{\sigma(1 - q^*(t))} - \cancel{\sigma(1 - q^*(t))}\| \\
&\quad = r_R \|y_1 - y_2\| \\
&\quad \le C \|y_1 - y_2\| \\
&\|b(t, y)\|^2 + \|\sigma(t, y)\|^2 \\
&\quad = ((1 - q^*(t))(p_R^*(t) - \mu) + r_R y)^2 + \sigma^2 (1 - q^*(t))^2 \\
&\quad \le 2(1 - q^*(t))^2 (p_R^*(t) - \mu)^2 + 2r_R^2 y^2 + \sigma^2 (1 - q^*(t))^2 \\
&\quad \le C^2 (1 + \|y\|^2)
\end{aligned}$$

  with $C := \max\{(1 - q^*(t))\sqrt{2(p_R^*(t) - \mu)^2 + \sigma^2}, \sqrt{2} r_R\}$. By the Existence and Uniqueness Theorem for SDEs (cf. Theorem 1.31), there exists a unique strong solution $Y_R^*(\cdot)$ to (3.38), which is continuous and adapted, such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |Y_R^*(t)|^2\Big] < \infty.$$

All in all, we have $p_R^* \in \Lambda_R$. Next, we will consider the strategies of the insurer.
Reminder: The optimal investment strategy and reinsurance strategy of the insurer are given by

$$\begin{aligned}
\pi_I^*(t) &= \frac{1}{\beta_I \psi^I(t) Y_I^*(t) S_1(t)^{2\delta}} \left[\frac{\tilde{\mu} - r_I}{\tilde{\sigma}^2} - 2\delta u^I(t)\right], \\
q^*(t) &= \min\{\frac{p_R^*(t) - \mu}{\sigma^2 \beta_I \psi^I(t)}, 1\}.
\end{aligned}$$

- $q^*(\cdot)$ is non-random and continuous, therefore $q^*(\cdot)$ is progressively measurable and it holds $q^*(t) \in [0,1]$ for all $t \in [0,T]$.
- The optimal surplus process of the insurer is given by

$$\begin{aligned}
dY_I^*(t) =& [\theta_I \mu - (1 - q^*(t))(p_R^*(t) - \mu) + Y_I^*(t)(r_I + (\tilde{\mu} - r_I)\pi_I^*(t))]dt \\
&+ Y_I^*(t)\pi_I^*(t)\tilde{\sigma} S_1(t)^\delta dW_1(t) + \sigma q(t) dW_2(t).
\end{aligned} \tag{3.39}$$

Let $s = S_1(t)$ and $c(t) := \frac{1}{\beta_I \psi^I(t) s^{2\delta}}\left[\frac{\tilde{\mu}-r_I}{\tilde{\sigma}^2} - 2\delta u^I(t)\right]$. We set

$$\begin{aligned}
b(t,y) :=& \theta_I \mu - (1 - q^*(t))(p_R^*(t) - \mu) + r_I y + (\tilde{\mu} - r_I)\pi_I^*(t) y \\
=& \theta_I \mu - (1 - q^*(t))(p_R^*(t) - \mu) + r_I y \\
&+ (\tilde{\mu} - r_I)\frac{1}{\beta_I \psi^I(t) \cancel{y} s^{2\delta}} \left[\frac{\tilde{\mu} - r_I}{\tilde{\sigma}^2} - 2\delta u^I(t)\right]\cancel{y} \\
=& \theta_I \mu - (1 - q^*(t))(p_R^*(t) - \mu) + r_I y + (\tilde{\mu} - r_I)c(t)
\end{aligned}$$

and

$$\begin{aligned}
\sigma(t,y) :=& (\sigma q^*(t), \pi_I^*(t) y s^\delta \tilde{\sigma})^\top \\
=& \left(\sigma q^*(t), \frac{1}{\beta_I \psi^I(t) \cancel{y} s^{2\delta}} \left[\frac{\tilde{\mu} - r_I}{\tilde{\sigma}^2} - 2\delta u^I(t)\right] \cancel{y} s^\delta \tilde{\sigma}\right)^\top \\
=& (\sigma q^*(t), c(t) s^\delta \tilde{\sigma})^\top.
\end{aligned}$$

Hence, for $y_1, y_2, y \in \mathbb{R}$

$$\begin{aligned}
&\|b(t,y_1) - b(t,y_2)\| + \|\sigma(t,y_1) - \sigma(t,y_2)\| \\
&\qquad = \| \cancel{\theta_I \mu - (1 - q^*(t))(p_R^*(t) - \mu)} + r_I y_1 + \cancel{(\tilde{\mu} - r_I)c(t)}
\end{aligned}$$

$$
\begin{aligned}
&\quad - \theta_I\mu - \cancel{(1-q^*(t))(p_R^*(t)-\mu)} - r_I y_2 - \cancel{(\tilde{\mu}-r_I)c(t)}\| \\
&\quad + \|\cancel{(\sigma q^*(t), c(t)s^\delta\tilde{\sigma})^\top} - \cancel{(\sigma q^*(t), c(t)s^\delta\tilde{\sigma})^\top}\| \\
&= r_I\|y_1-y_2\| \\
&\leq C\|y_1-y_2\|, \\
\|b(t,y)\|^2 &+ \|\sigma(t,y)\|^2 \\
&= (\theta_I\mu - (1-q^*(t))(p_R^*(t)-\mu) + r_I y + (\tilde{\mu}-r_I)c(t))^2 \\
&\quad + \sigma^2 q^*(t)^2 + c(t)^2 s^{2\delta}\tilde{\sigma}^2 \\
&\leq 2r_I^2 y^2 + 2(\theta_I\mu - (1-q^*(t))(p_R^*(t)-\mu) + (\tilde{\mu}-r_I)c(t))^2 \\
&\quad + \sigma^2 q^*(t)^2 + c(t)^2 s^{2\delta}\tilde{\sigma}^2 \\
&\leq C^2(1+\|y\|^2)
\end{aligned}
$$

with

$$C := \max\{\sqrt{2(\theta_I\mu - (1-q^*(t))(p_R^*(t)-\mu) + (\tilde{\mu}-r_I)c(t))^2 + \sigma^2 q^*(t)^2 + c(t)^2 s^{2\delta}\tilde{\sigma}^2}, \sqrt{2}r_I\}.$$

By the Existence and Uniqueness Theorem for SDEs (cf. Theorem 1.31), there exists an unique strong solution $Y_I^*(\cdot)$ to the SDE (3.39), which is continuous and adapted, such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|Y_I^*(t)|^2\Big] < \infty.$$

- Since S_1 and Y_I^* are adapted, we have that π_I^* is adapted. Furthermore, we have that $t \mapsto \pi_I^*(t)$ is continuous, since $\psi^I(\cdot)$ and $u^I(\cdot)$ are continuous functions. Therefore, π_I^* is progressively measurable (cf. Theorem (1.18)).
  Let $y = Y_I^*(t)$ and $s = S_1(t)$. Then, there exists a constant $\tilde{C} > 0$ such that

$$
\begin{aligned}
|\pi_I^*(t)| &= \left|\frac{1}{\beta_I\psi^I(t)ys^{2\delta}}\right|\left|\frac{\tilde{\mu}-r_I}{\tilde{\sigma}^2} - 2\delta u^I(t)\right| \\
&\leq \tilde{C}\left|\frac{1}{ys^{2\delta}}\right| < \infty,
\end{aligned}
$$

since $y, s > 0$ and the functions u^I and ψ^I are continuous on the compact set $[0,T]$ and therefore, bounded on $[0,T]$. Hence,

$$\mathbb{E}\Big[\int_0^T |\pi_I^*(t)|^2 dt\Big] \leq \tilde{C}^2 T\mathbb{E}\Big[\sup_{t\in[0,T]}\left|\frac{1}{ys^{2\delta}}\right|^2\Big] < \infty.$$

Now, we will prove that the optimal strategies q^* and π_I^* solve the optimization problem of the insurer (3.2) for any $p_R \in \Lambda_R$. To make it more manageable, we will write $\Phi^I(t)$ instead of $\Phi^I(t, Y_I^{p_R,(q,\pi_I)}(t), S_1(t))$. Let $\mathcal{D}^I$ be the characteristic operator of $Y_I^{p_R,(q,\pi_I)}$, i.e.,

$$\begin{aligned}\mathcal{D}^I\Phi^I(t) :=&\Phi_t^I(t) + \Phi_y^I(t)[\theta_I\mu - (p_R(t)-\mu)(1-q(t)) + Y_I^{p_R,(q,\pi_I)}(t)(r_I + (\tilde{\mu} - r_I)\pi_I(t))]\\ &+ \frac{1}{2}\Phi_{yy}^I(t)[q(t)^2\sigma^2 + \pi_I(t)^2\tilde{\sigma}^2 S_1(t)^{2\delta} Y_I^{p_R,(q,\pi_I)}(t)^2] + \Phi_s^I(t)\tilde{\mu}S_1(t)\\ &+ \frac{1}{2}\Phi_{ss}^I(t)\tilde{\sigma}^2 S_1(t)^{2\delta+2} + \Phi_{ys}^I(t)\pi_I(t)Y_I^{p_R,(q,\pi_I)}(t)\tilde{\sigma}^2 S_1(t)^{2\delta+1}.\end{aligned}$$

We set $M^I := \mathbb{R}_+ \times \mathbb{R}_+$ and take a sequence of bounded open sets $(M_n^I)_{n\in\mathbb{N}}$ with $M_n^I \subset M_{n+1}^I \subset M^I$ and $M^I = \bigcup_{n\in\mathbb{N}} M_n^I$. Furthermore, let τ_n be the exit time of $(Y_I^{p_R,(q,\pi_I)}(t), S_1(t))$ from M_n^I. We will prove that for all $n \in \mathbb{N}$

$$\mathbb{E}_{t,y,s}[\Phi^I(\tau_n \wedge T, Y_I(\tau_n \wedge T), S_1(\tau_n \wedge T))] < \infty$$

holds. By Ito's formula, we have

$$\begin{aligned}d(\Phi^I(t))^2 =&2\Phi^I(t)[\Phi_t^I(t)dt + \Phi_y^I(t)dY_I^{p_R,(q,\pi_I)}(t) + \Phi_s^I(t)dS_1(t)\\ &+ \frac{1}{2}\Phi_{yy}^I(t)\langle dY_I^{p_R,(q,\pi_I)}(t), dY_I^{p_R,(q,\pi_I)}(t)\rangle + \Phi_{ys}^I(t)\langle dY_I^{p_R,(q,\pi_I)}(t), dS_1(t)\rangle\\ &+ \frac{1}{2}\Phi_{ss}^I(t)\langle dS_1(t), dS_1(t)\rangle] + \Phi_y^I(t)^2\langle dY_I^{p_R,(q,\pi_I)}(t), dY_I^{p_R,(q,\pi_I)}(t)\rangle\\ &+ 2\Phi_y^I(t)\Phi_s^I(t)\langle dY_I^{p_R,(q,\pi_I)}(t), dS_1(t)\rangle + (\Phi_s^I(t))^2\langle dS_1(t), dS_1(t)\rangle\\ =&2\Phi^I(t)[\Phi_t^I(t) + \Phi_y^I(t)(\theta_I\mu + (p_R(t)-\mu)(1-q(t)) + Y_I^{p_R,(q,\pi_I)}(t)r_I\\ &+ Y_I^{p_R,(q,\pi_I)}(t)(\tilde{\mu} - r_I)\pi_I(t)) + \Phi_s^I(t)S_1(t)\tilde{\mu} + \frac{1}{2}\Phi_{yy}^I(t)(\sigma^2 q(t)^2\\ &+ Y_I^{p_R,(q,\pi_I)}(t)^2\pi_I(t)^2\tilde{\sigma}^2 S_1(t)^{2\delta}) + \Phi_{ys}^I(t)Y_I^{p_R,(q,\pi_I)}(t)\pi_I(t)\tilde{\sigma}^2 S_1(t)^{2\delta+1}\\ &+ \frac{1}{2}\Phi_{ss}^I(t)\tilde{\sigma}^2 S_1(t)^{2\delta+2}]dt + [\Phi_y^I(t)^2(\sigma^2 q(t)^2 + Y_I^{p_R,(q,\pi_I)}(t)^2\pi_I(t)^2\tilde{\sigma}^2 S_1(t)^{2\delta})\\ &+ 2\Phi_y^I(t)\Phi_s^I(t)Y_I^{p_R,(q,\pi_I)}(t)\pi_I(t)\tilde{\sigma}^2 S_1(t)^{2\delta+1} + \Phi_s^I(t)^2\tilde{\sigma}^2 S_1(t)^{2\delta+2}]dt\\ &+ 2\Phi^I(t)[\Phi_y^I(t)Y_I^{p_R,(q,\pi_I)}(t)\pi_I(t)\tilde{\sigma}S_1(t)^{\delta} + \Phi_s^I(t)\tilde{\sigma}S_1(t)^{\delta+1}]dW_1(t)\\ &+ 2\Phi^I(t)[\Phi_y^I(t)\sigma q(t)]dW_2(t)\\ =&2\Phi^I(t)\mathcal{D}^I\Phi^I(t)dt + [\Phi_y^I(t)^2(\sigma^2 q(t)^2 + Y_I^{p_R,(q,\pi_I)}(t)^2\pi_I(t)^2\tilde{\sigma}^2 S_1(t)^{2\delta})\\ &+ 2\Phi_y^I(t)\Phi_s^I(t)Y_I^{p_R,(q,\pi_I)}(t)\pi_I(t)\tilde{\sigma}^2 S_1(t)^{2\delta+1} + \Phi_s^I(t)^2\tilde{\sigma}^2 S_1(t)^{2\delta+2}]dt\\ &+ 2\Phi^I(t)[\Phi_y^I(t)Y_I^{p_R,(q,\pi_I)}(t)\pi_I(t)\tilde{\sigma}S_1(t)^{\delta} + \Phi_s^I(t)\tilde{\sigma}S_1(t)^{\delta+1}]dW_1(t)\\ &+ 2\Phi^I(t)\Phi_y^I(t)\sigma q(t)dW_2(t).\end{aligned}$$

If we insert $q^*(t)$ and $\pi_I^*(t)$ into the equation, we have by the proof of Theorem 3.3

$$\mathcal{D}^I\Phi^I(t)|_{(q^*(t),\pi_I^*(t))} = 0$$

and therefore,

$$\begin{aligned}
d(\Phi^I(t))^2 =& \Big[\Phi_y^I(t)^2(\sigma^2 q^*(t)^2 + \cancel{Y_I^{p_R,(q,\pi_I)}}(t)^2 \frac{1}{\beta_I^2\psi^I(t)^2\cancel{Y_I^{p_R,(q,\pi_I)}(t)^2}S_1(t)^{\cancelto{2\delta}{4\delta}}} \\
&\times \left[\frac{\tilde{\mu}-r_I}{\tilde{\sigma}^2} - 2\delta u^I(t)\right]^2 \tilde{\sigma}^2 \cancel{S_1(t)^{2\delta}}) \\
&+ 2\Phi_y^I(t)\Phi_s^I(t)\cancel{Y_I^{p_R,(q,\pi_I)}}(t)\frac{1}{\beta_I\psi^I(t)\cancel{Y_I^{p_R,(q,\pi_I)}}(t)\cancel{S_1(t)^{2\delta}}} \\
&\times \left[\frac{\tilde{\mu}-r_I}{\tilde{\sigma}^2} - 2\delta u^I(t)\right]\tilde{\sigma}^2 S_1(t)^{\cancel{2\delta}+1} \\
&+ \Phi_s^I(t)^2\tilde{\sigma}^2 S_1(t)^{2\delta+2}\Big]dt \\
&+ 2\Phi^I(t)\Big[\Phi_y^I(t)\cancel{Y_I^{p_R,(q,\pi_I)}}(t)\frac{1}{\beta_I\psi^I(t)\cancel{Y_I^{p_R,(q,\pi_I)}}(t)S_1(t)^{\cancel{2}\delta}} \\
&\times \left[\frac{\tilde{\mu}-r_I}{\tilde{\sigma}^2} - 2\delta u^I(t)\right]\tilde{\sigma}\cancel{S_1(t)^{\delta}} \\
&+ \Phi_s^I(t)\tilde{\sigma}S_1(t)^{\delta+1}\Big]dW_1(t) + 2\Phi^I(t)\Phi_y^I(t)\sigma q^*(t)dW_2(t).
\end{aligned}$$

Next, we will insert the formula for the value function, i.e., the formulas

$$\begin{aligned}
\Phi_y^I(t) =& -\Phi^I(t)\beta_I\psi^I(t) \\
\Phi_s^I(t) =& -\Phi^I(t)2\delta u^I(t)s^{-2\delta-1},
\end{aligned}$$

in the equation and get

$$\begin{aligned}
d(\Phi^I(t))^2 =& \Phi^I(t)^2\Big[\beta_I^2\psi^I(t)^2\sigma^2 q^*(t)^2 + \cancel{\beta_I^2\psi^I(t)^2}\tilde{\sigma}^2 S_1(t)^{-2\delta}\frac{1}{\cancel{\beta_I^2\psi^I(t)^2}}\left(\frac{\tilde{\mu}-r_I}{\tilde{\sigma}^2} - 2\delta u^I(t)\right)^2 \\
&+ 2\cancel{\beta_I\psi^I(t)}2\delta u^I(t)S_1(t)^{-2\delta\cancel{-1}}\frac{1}{\cancel{\beta_I\psi^I(t)}}\tilde{\sigma}^2\cancel{S_1(t)}\left(\frac{\tilde{\mu}-r_I}{\tilde{\sigma}^2} - 2\delta u^I(t)\right) \\
&+ 4\delta^2 u^I(t)^2 S_1(t)^{\cancelto{-2\delta}{-4\delta-2}}\tilde{\sigma}^2\cancel{S_1(t)^{2\delta+2}}\Big]dt \\
&- 2\Phi^I(t)^2\Big[\cancel{\beta_I\psi^I(t)}\frac{1}{\cancel{\beta_I\psi^I(t)}S_1(t)^{\delta}}\tilde{\sigma}\left(\frac{\tilde{\mu}-r_I}{\tilde{\sigma}^2} - 2\delta u^I(t)\right) \\
&+ 2\delta u^I(t)S_1(t)^{\cancelto{-\delta}{-2\delta-1}}\tilde{\sigma}\cancel{S_1(t)^{\delta+1}}\Big]dW_1(t) \\
&- 2\Phi^I(t)^2\beta_I\psi^I(t)\sigma q^*(t)dW_2(t) \\
=& \Phi^I(t)^2\left[\beta_I^2\psi^I(t)^2\sigma^2 q^*(t)^2 + \left(\frac{\tilde{\mu}-r_I}{\tilde{\sigma}S_1(t)^{\delta}}\right)^2\right]dt \\
&- 2\Phi^I(t)^2\beta_I\psi^I(t)\sigma q^*(t)dW_2(t) - 2\Phi^I(t)^2 S_1(t)^{-\delta}\frac{\tilde{\mu}-r_I}{\tilde{\sigma}}dW_1(t).
\end{aligned}$$

Define $h(\Phi^I(t)) := \Phi^I(t)^2$, $\Theta_1(t) := \beta_I^2\psi^I(t)^2\sigma^2 q^*(t)^2 + \big(\frac{\tilde{\mu}-r_I}{\tilde{\sigma}S_1(t)^\delta}\big)^2$, $\Theta_2(t) := -2\beta_I\psi^I(t)\sigma q^*(t)$ and $\Theta_3(t) := -2S_1(t)^{-\delta}\frac{\tilde{\mu}-r_I}{\tilde{\sigma}}$. Hence,

$$dh(\Phi^I(t)) = h(\Phi^I(t))[\Theta_1(t)dt + \Theta_2(t)dW_2(t) + \Theta_3(t)dW_1(t)].$$

$h(\Phi^I(t))$ is a geometric Brownian motion, i.e.

$$\begin{aligned}\Phi^I(t)^2 = \Phi^I(0)^2 \exp\Bigg(&\Theta_1(t)dt + \Theta_2(t)dW_2(t) - \frac{1}{2}\Theta_2(t)^2 dt \\ &+ \Theta_3(t)dW_1(t) - \frac{1}{2}\Theta_3(t)^2 dt\Bigg).\end{aligned} \tag{3.40}$$

We will show that $\big(\exp\big(\Theta_2(t)dW_2(t) - \frac{1}{2}\Theta_2(t)^2 dt\big)\big)_{t\in[0,T]}$ and $\big(\exp\big(\Theta_3(t)dW_2(t) - \frac{1}{2}\Theta_3(t)^2 dt\big)\big)_{t\in[0,\tau_n\wedge T]}$ are martingales with expectation 1:

1. $\big(\exp\big(\Theta_2(t)dW_2(t) - \frac{1}{2}\Theta_2(t)^2 dt\big)\big)_{t\in[0,T]}$ is a martingale: We have

$$\begin{aligned}\mathbb{E}\bigg[\exp\bigg(\frac{1}{2}\int_0^T |\Theta_2(t)|^2 dt\bigg)\bigg] &= \mathbb{E}\bigg[\exp\bigg(2\beta_I^2\sigma^2\int_0^T \psi^I(t)q^*(t)dt\bigg)\bigg] \\ &\le \mathbb{E}\bigg[\exp\bigg(2\beta_I^2\sigma^2\psi^I(0)T\bigg)\bigg] < \infty\end{aligned}$$

since $q^*(t) \le 1$ and $\psi^I(t) \le \psi^I(0)$ for all $t \in [0,T]$. Hence, the statement follows from Novikov's condition (cf. Theorem 1.28). Therefore,

$$\begin{aligned}&\mathbb{E}\bigg[\exp\bigg(\int_0^t \Theta_2(\tau)dW_2(\tau) - \frac{1}{2}\int_0^t \Theta_2(\tau)^2 d\tau\bigg)\bigg] \\ &\quad = \mathbb{E}\bigg[\exp\bigg(\int_0^0 \Theta_2(\tau)dW_2(\tau) - \frac{1}{2}\int_0^0 \Theta_2(\tau)^2 d\tau\bigg)\bigg] = 1.\end{aligned}$$

2. $\Bigg(\exp\big(\Theta_3(t)dW_2(t) - \frac{1}{2}\Theta_3(t)^2 dt\big)\Bigg)_{t\in[0,\tau_n\wedge T]}$ is a martingale: For $t \in [0,\tau_n \wedge T]$ we have

$$\mathbb{E}\bigg[\exp\bigg(\frac{1}{2}\int_0^t |\Theta_3(\tau)|^2 d\tau\bigg)\bigg] = \mathbb{E}\bigg[\exp\bigg(4\tilde{\gamma}^2\int_0^t S_1(\tau)^{-2\delta}d\tau\bigg)\bigg] < \infty$$

since S_1 is bounded on M_n^I and $\tilde{\gamma} := \frac{\tilde{\mu}-r_I}{\tilde{\sigma}} < \infty$. The statement follows from Novikov's condition (cf. Theorem 1.28). Therefore, for $t \in [0,\tau_n \wedge T]$

$$\mathbb{E}\bigg[\exp\bigg(\int_0^t \Theta_3(\tau)dW_2(\tau) - \frac{1}{2}\int_0^t \Theta_3(\tau)^2 d\tau\bigg)\bigg]$$

$$= \mathbb{E}\Big[\exp\Big(\int_0^0 \Theta_3(\tau)dW_2(\tau) - \frac{1}{2}\int_0^0 \Theta_3(\tau)^2 d\tau\Big)\Big] = 1.$$

For (3.40) follows with $t \in [0, \tau_n \wedge T]$

$$\mathbb{E}[\Phi^I(t)^2] = \mathbb{E}\Big[\Phi^I(0)^2 \exp\Big(\int_0^t \Theta_1(\tau)d\tau\Big)\Big]$$

and since $\Theta_1(t) < \infty$ ($Y_I^{p_R,(q,\pi_I)}$ and S_1 bounded on M_n^I) we have

$$\mathbb{E}_{t,y,s}\big[\Phi^I(\tau_n \wedge T)\big] = \mathbb{E}_{t,y,s}\big[\Phi^I(\tau_n \wedge T, Y_I(\tau_n \wedge T), S_1(\tau_n \wedge T))^2\big] < \infty.$$

Next, we will prove that the value function is exactly the function which solves the HJB-equation. Since τ_n is the exit time of $(Y_I^{p_R,(q,\pi_I)}(t), S_1(t))$ from the set M_n^I and $M^I = \bigcup_{n\in\mathbb{N}} M_n^I$, we have that $\tau_n \to T$ as $n \to \infty$. Let $(p_R(\cdot), q(\cdot), \pi_I(\cdot)) \in \Lambda_R \times \Lambda_I$. By Ito's formula we have

$$\begin{aligned} d\Phi^I(t) =& \mathcal{D}^I \Phi^I(t)dt + \Phi_y^I(t)q(t)\sigma dW_2(t) \\ &+ \Phi_y^I(t)Y_I^{p_R,(q,\pi_I)}(t)\pi_I(t)\tilde{\sigma}S_1(t)^\delta dW_1(t) + \Phi_s^I(t)\tilde{\sigma}S_1(t)^{\delta+1}dW_1(t). \end{aligned}$$

Hence, we have

$$\begin{aligned} \Phi^I(\tau_n \wedge T) = & \int_t^{\tau_n\wedge T} \mathcal{D}^I\Phi^I(\tau)d\tau + \int_t^{\tau_n\wedge T} \Phi_y^I(\tau)q(\tau)\sigma dW_2(\tau) \\ &+ \int_t^{\tau_n\wedge T} \Phi_y^I(\tau)Y_I(\tau)\pi_I(\tau)\tilde{\sigma}S_1(\tau)^\delta dW_1(\tau) \\ &+ \int_t^{\tau_n\wedge T} \Phi_s^I(\tau)\tilde{\sigma}S_1(\tau)^{\delta+1}dW_1(\tau). \end{aligned} \tag{3.41}$$

We will prove that $\big(\int_0^{\tau_n\wedge t} \Phi_y^I(\tau)q(\tau)\sigma dW_2(\tau)\big)_{t\in[0,T]}$, $\big(\int_0^{\tau_n\wedge t} \Phi_y^I(\tau)Y_I(\tau)\pi_I(\tau)\tilde{\sigma}S_1(\tau)^\delta dW_1(\tau)\big)_{t\in[0,T]}$ and $\big(\int_0^{\tau_n\wedge t} \Phi_s^I(\tau)\tilde{\sigma}S_1(\tau)^{\delta+1}dW_1(\tau)\big)_{t\in[0,T]}$ are square-integrable martingales with expectation 0:

1. $\big(\int_0^{\tau_n\wedge t} \Phi_y^I(\tau)q(\tau)\sigma dW_2(\tau)\big)_{t\in[0,T]}$ is a square-integrable martingale with expectation 0:

   For $t \in [0, \tau_n \wedge T]$ we have

   $$\mathbb{E}\Big[\int_0^t \Phi_y^I(\tau)^2 q(\tau)^2\sigma^2 d\tau\Big] \leq C\mathbb{E}\Big[\int_0^t \Phi^I(\tau)^2 d\tau\Big] < \infty$$

   since Φ^I is bounded on the set M_n^I, $q(t) \leq 1$, $\psi^I(t) \leq \psi^I(0)$ for all $t \in [0,T]$ and $C := \beta_I^2\sigma^2\psi^I(0)^2 < \infty$. It follows from Novikov's condition that $\big(\int_0^t \Phi_y^I(\tau)q(\tau)\sigma dW_2(\tau)\big)_{t\in[0,\tau_n\wedge T]}$

is a square-integrable martingale and therefore, $\big(\int_0^{\tau_n\wedge t}\Phi_y^I(\tau)q(\tau)\sigma dW_2(\tau)\big)_{t\in[0,T]}$ is a square-integrable martingale. Hence,

$$\mathbb{E}\bigg[\int_0^{\tau_n\wedge t}\Phi_y^I(\tau)q(\tau)\sigma dW_2(\tau)\bigg]=\mathbb{E}\bigg[\int_0^{\tau_n\wedge 0}\Phi_y^I(\tau)q(\tau)\sigma dW_2(\tau)\bigg]=0.$$

2. $\big(\int_0^{\tau_n\wedge t}\Phi_y^I(\tau)Y_I(\tau)\pi_I(\tau)\tilde{\sigma}S_1(\tau)^\delta dW_1(\tau)\big)_{t\in[0,T]}$ is a square-integrable martingale with expectation 0:
   For $t\in[0,\tau_n\wedge T]$ we have

$$\begin{aligned}&\mathbb{E}\bigg[\int_0^t\Phi_y^I(\tau)^2Y_I(\tau)^2\pi_I(\tau)^2\tilde{\sigma}^2S_1(\tau)^{2\delta}d\tau\bigg]\\&\qquad\le C\mathbb{E}\bigg[\int_0^t\Phi^I(\tau)^2Y_I(\tau)^2\pi_I(\tau)^2S_1(\tau)^{2\delta}d\tau\bigg]<\infty\end{aligned}$$

   since Φ^I, Y_I and S_1 are bounded on M_n^I, π_I is square-integrable and $C:=\beta_I^2\psi^I(0)^2\tilde{\sigma}^2<\infty$. It follows from Novikov's condition that $\big(\int_0^t\Phi_y^I(\tau)Y_I(\tau)\pi_I(\tau)\tilde{\sigma}S_1(\tau)^\delta dW_1(\tau)\big)_{t\in[0,\tau_n\wedge T]}$ is a square-integrable martingale and therefore, $\big(\int_0^{\tau_n\wedge t}\Phi_y^I(\tau)Y_I(\tau)\pi_I(\tau)\tilde{\sigma}S_1(\tau)^\delta dW_1(\tau)\big)_{t\in[0,T]}$ is a square-integrable martingale. Hence,

$$\begin{aligned}&\mathbb{E}\bigg[\int_0^{\tau_n\wedge t}\Phi_y^I(\tau)Y_I(\tau)\pi_I(\tau)\tilde{\sigma}S_1(\tau)^\delta dW_1(\tau)\bigg]\\&\qquad=\mathbb{E}\bigg[\int_0^{\tau_n\wedge 0}\Phi_y^I(\tau)Y_I(\tau)\pi_I(\tau)\tilde{\sigma}S_1(\tau)^\delta dW_1(\tau)\bigg]=0.\end{aligned}$$

3. $\big(\int_0^{\tau_n\wedge t}\Phi_s^I(\tau)\tilde{\sigma}S_1(\tau)^{\delta+1}dW_1(\tau)\big)_{t\in[0,T]}$ is a square-integrable martingale with expectation 0:
   For $t\in[0,\tau_n\wedge T]$ we have

$$\begin{aligned}\mathbb{E}\bigg[\int_0^t\Phi_s^I(\tau)^2\tilde{\sigma}^2S_1(\tau)^{2\delta+2}d\tau\bigg]&=\mathbb{E}\bigg[\int_0^t\Phi^I(\tau)^2u^I(\tau)^24\delta^2S_1(\tau)^{-4\delta-2}\tilde{\sigma}^2S_1(\tau)^{2\delta+2}d\tau\bigg]\\&\le C\mathbb{E}\bigg[\int_0^t\Phi^I(\tau)^2S_1(\tau)^{-2\delta}d\tau\bigg]<\infty\end{aligned}$$

   since Φ^I and S_1 are bounded on M_n^I and $C:=4\delta^2u^I(0)^2\tilde{\sigma}^2<\infty$. It follows from Novikov's condition that $\big(\int_0^t\Phi_s^I(\tau)\tilde{\sigma}S_1(\tau)^{\delta+1}dW_1(\tau)\big)_{t\in[0,\tau_n\wedge T]}$ is a square-integrable martingale and therefore, $\big(\int_0^{\tau_n\wedge t}\Phi_s^I(\tau)\tilde{\sigma}S_1(\tau)^{\delta+1}dW_1(\tau)\big)_{t\in[0,T]}$ is a square-integrable martingale. Hence,

$$\mathbb{E}\bigg[\int_0^{\tau_n\wedge t}\Phi_s^I(\tau)\tilde{\sigma}S_1(\tau)^{\delta+1}dW_1(\tau)\bigg]=\mathbb{E}\bigg[\int_0^{\tau_n\wedge 0}\Phi_s^I(\tau)\tilde{\sigma}S_1(\tau)^{\delta+1}dW_1(\tau)\bigg]=0.$$

It follows from Equation (3.41) that

$$\mathbb{E}_{t,y,s}[\Phi^I(\tau_n \wedge T)] = \Phi^I(t) + \mathbb{E}_{t,y,s}\left[\int_t^{\tau_n \wedge T} \mathcal{D}^I \Phi^I(\tau)d\tau\right]$$
$$\begin{cases} \leq \Phi^I(t) & \text{for all } (q(t), \pi_I(t)) \in \Lambda_I \\ = \Phi^I(t) & \text{if } (q(t), \pi_I(t)) = (q^*(t), \pi_I^*(t)) \end{cases},$$

since $\mathcal{D}^I\Phi^I(t)|_{(q(\cdot),\pi_I(\cdot))} \leq \mathcal{D}^I\Phi^I(t)|_{(q^*(\cdot),\pi_I^*(\cdot))} = 0$ for all $(q(\cdot), \pi_I(\cdot)) \in \Lambda_I$. Since

$$\mathbb{E}_{t,y,s}[|\Phi^I(\tau_n \wedge T)|^2] < \infty,$$

we have uniform integrability and therefore,

$$\begin{aligned}\sup_{(q(\cdot),\pi_I(\cdot))\in\Lambda_I} \mathbb{E}_{t,y,s}[U_I(Y_I(T))] &= \sup_{(q(\cdot),\pi_I(\cdot))\in\Lambda_I} \mathbb{E}_{t,y,s}[\Phi^I(T)] \\ &= \lim_{n\to\infty} \mathbb{E}_{t,y,s}[\Phi^I(\tau_n \wedge T)] \\ &\begin{cases} \leq \Phi^I(t) & \text{for all } (q(t), \pi_I(t)) \in \Lambda_I \\ = \Phi^I(t) & \text{if } (q(t), \pi_I(t)) = (q^*(t), \pi_I^*(t)) \end{cases}.\end{aligned}$$

Next, we do the verification of the optimization problem of the reinsurer, i.e., we prove that $p_R^*(\cdot)$ is the solution to the optimization problem of the reinsurer. Again, we will write $\Phi^R(t)$ instead of $\Phi^R(t, Y_R^{p_R,q}(t))$ to make it more manageable. We define

$$\mathcal{D}^R\Phi^R(t) := \Phi_t^R(t) + \Phi_y^R(t)[rY_R^{p_R,q}(t) + (1 - q^*(t))(p_R(t) - \mu)] + \frac{1}{2}\Phi_{yy}^R(t)(1 - q^*(t))^2\sigma^2.$$

Now, we set $M^R := \mathbb{R}_+$ and choose a sequence of bounded open set $(M_n^R)_{n\in\mathbb{N}}$ with $M_n^R \subset M_{n+1}^R \subset M^R$ and $M^R = \bigcup_{n\in\mathbb{N}} M_n^R$. Let τ_n be the exit time of $Y_R^{p_R,q}(t)$ from M_n^R. We will first prove for all $n \in \mathbb{N}$

$$\mathbb{E}_{t,y}[\Phi^R(\tau_n \wedge T, Y_R^{p_R,q}(\tau_n \wedge T))^2] < \infty.$$

By Ito's formula, we have

$$\begin{aligned} d(\Phi^R(t))^2 =& 2\Phi^R(t)[\Phi_t^R(t)dt + \Phi_y^R(t)dY_R^{p_R,q}(t) + \frac{1}{2}\Phi_{yy}^R(t)\langle dY_R^{p_R,q}(t), dY_R^{p_R,q}(t)\rangle] \\ &+ \frac{\cancel{1}}{\cancel{2}}\cancel{2}\Phi_y^R(t)^2\langle dY_R^{p_R,q}(t), dY_R^{p_R,q}(t)\rangle \\ =& 2\Phi^R(t)[\Phi_t^R(t)dt + \Phi_y^R(t)(rY_R^{p_R,q}(t) + (1 - q^*(t))(p_R(t) - \mu))dt \\ &+ \Phi_y^R(t)(1 - q^*(t))\sigma dW_2(t) + \frac{1}{2}\Phi_{yy}^R(t)(1 - q^*(t))^2\sigma^2 dt] \end{aligned}$$

$$
\begin{aligned}
&+ \Phi_y^R(t)^2(1-q^*(t))^2\sigma^2 dt \\
=&2\Phi^R(t)\mathcal{D}^R\Phi^R(t)dt + 2\Phi^R(t)\Phi_y^R(t)(1-q^*(t))\sigma dW_2(t) + \Phi_y^R(t)^2(1-q^*(t))^2\sigma^2 dt.
\end{aligned}
$$

It holds

$$\mathcal{D}^R\Phi^R(t)|_{p_R(t)=p_R^*(t)} = 0.$$

Hence, if we insert $p_R^*(\cdot)$ we get

$$d(\Phi^R(t))^2 = 2\Phi^R(t)\Phi_y^R(t)(1-q^*(t))\sigma dW_2(t) + \Phi_y^R(t)^2(1-q^*(t))^2\sigma^2 dt.$$

Next, we insert

$$\Phi_y^R(t) = -\Phi^R(t)\beta_R\psi^R(t),$$

into the equation and get

$$
\begin{aligned}
d(\Phi^R(t))^2|_{p_R(t)=p_R^*(t)} =&2\Phi^R(t)^2(-\beta_R\psi^R(t))(1-q^*(t))\sigma dW_2(t) \\
&+ \Phi^R(t)^2(\beta_R\psi^R(t))^2(1-q^*(t))^2\sigma^2 dt.
\end{aligned}
$$

With $h(\Phi^R(t)) := \Phi^R(t)^2$ we have

$$dh(\Phi^R(t)) = h(\Phi^R(t))[-2\beta_R\psi^R(t)(1-q^*(t))\sigma dW_2(t) + (\beta_R\psi^R(t))^2(1-q^*(t))^2\sigma^2 dt].$$

Hence, $h(\Phi^R(t))$ is a geometric Brownian motion and therefore, the solution is given by

$$h(\Phi^R(t)) = h(\Phi^R(0)) \exp(\Theta_1(t)dt + \Theta_2(t)dW_2(t) - \frac{1}{2}\Theta_2(t)^2 dt)$$

with $\Theta_1(t) := (\beta_R\psi^R(t))^2(1-q^*(t))^2\sigma^2$ and $\Theta_2(t) := -2\beta_R\psi^R(t)(1-q^*(t))\sigma$.
Now, we will prove that $(\exp(\Theta_2(t)dW_2(t) - \frac{1}{2}\Theta_2(t)^2 dt))_{t\in[0,T]}$ is a martingale with expectation 1. We have

$$
\begin{aligned}
\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \Theta_2(t)^2 dt\right)\right] =&\mathbb{E}\left[\exp\left(2\int_0^T \beta_R^2\psi^R(t)^2(1-q^*(t))^2\sigma^2 dt\right)\right] \\
=&\mathbb{E}\left[\exp\left(2\beta_R^2\sigma^2\int_0^T \psi^R(t)^2(1-q^*(t))^2 dt\right)\right] \\
\le&\mathbb{E}\left[\exp\left(2\beta_R^2\sigma^2\psi^R(0)^2 T\right)\right] \\
<&\infty,
\end{aligned}
$$

since $q^*(t) \geq 0 \Rightarrow 1 - q^*(t) \leq 1$ and $\psi^R(t) \leq \psi^R(0)$ for all $t \in [0, T]$. Hence, by Novikov's condition $(\exp(\Theta_2(t)dW_2(t) - \frac{1}{2}\Theta_2(t)^2dt))_{t\in[0,T]}$ is a martingale and therefore,

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(\int_0^t \Theta_2(\tau)dW_2(\tau) - \frac{1}{2}\int_0^t \Theta_2(\tau)^2 d\tau\Big)\Big] \\ = \mathbb{E}\Big[\exp\Big(\int_0^0 \Theta_2(\tau)dW_2(\tau) - \frac{1}{2}\int_0^0 \Theta_2(\tau)^2 d\tau\Big)\Big] = 1.\end{aligned}$$

It follows

$$\begin{aligned}\mathbb{E}[h(\Phi^R(t))] =& \mathbb{E}[h(\Phi^R(0))\exp(\Theta_1(t)dt + \Theta_2(t)dW_2(t) - \frac{1}{2}\Theta_2(t)^2dt)] \\ =& \mathbb{E}[h(\Phi^R(0))\exp(\Theta_1(t)dt)] < \infty,\end{aligned}$$

since for all $t \in [0, T]$

$$\begin{aligned}\exp\Big(\int_0^t \Theta_1(\tau)d\tau\Big) =& \exp\Big(\int_0^t (\beta_R\psi^R(\tau))^2(1 - q^*(\tau))^2\sigma^2 d\tau\Big) \\ =& \exp\Big(\beta_R^2\psi^R(0)^2\sigma^2 t\Big) < \infty.\end{aligned}$$

All in all, it holds for all $n \in \mathbb{N}$

$$\mathbb{E}_{t,y}[\Phi^R(\tau_n \wedge T, Y_R^{p_R,q}(\tau_n \wedge T))^2] < \infty.$$

Next, we will prove that the value function is exactly the function which solves the HJB-equation. Again, by Ito's formula we have

$$d\Phi^R(t) = \mathcal{D}^R\Phi^R(t)dt + \Phi_y^R(t)(1 - q^*(t))\sigma dW_2(t).$$

Hence,

$$\Phi^R(\tau_n \wedge T) = \Phi^R(t) + \int_t^{\tau_n\wedge T} \mathcal{D}^R\Phi^R(\tau)d\tau + \int_t^{\tau_n\wedge T} \Phi_y^R(\tau)(1 - q^*(\tau))\sigma dW_2(\tau).$$

We show that $\Big(\int_0^t \Phi_y^R(\tau)(1 - q^*(\tau))\sigma dW_2(\tau)\Big)_{t\in[0,\tau_n\wedge T]}$ is a martingale with expectation zero. For this, we will prove

$$\mathbb{E}\Big[\int_0^t \Phi_y^R(\tau)^2(1 - q^*(\tau))^2\sigma^2 d\tau\Big] < \infty.$$

For $t \in [0, \tau_n \wedge T]$

$$
\begin{aligned}
\mathbb{E}\bigg[\int_0^t \Phi_y^R(\tau)^2(1-q^*(\tau))^2\sigma^2 d\tau\bigg] =& \mathbb{E}\bigg[\beta_R^2\sigma^2\int_0^t \Phi^R(\tau)^2\psi^R(\tau)^2(1-q^*(\tau))^2 d\tau\bigg] \\
\leq& C\mathbb{E}\bigg[\int_0^t \Phi^R(\tau)^2 d\tau\bigg] < \infty
\end{aligned}
$$

since Φ^R is bounded on M_n^R and $C = \beta_R^2\sigma^2\psi^R(0)^2 < \infty$. Hence, $\big(\int_0^t \Phi_y^R(\tau)(1-q^*(\tau))\sigma dW_2(\tau)\big)_{t\in[0,\tau_n\wedge T]}$ is a square-integrable martingale and therefore, $\big(\int_0^{\tau_n\wedge t} \Phi_y^R(\tau)(1-q^*(\tau))\sigma dW_2(\tau)\big)_{t\in[0,T]}$ is a square-integrable martingale. It follows

$$
\mathbb{E}\bigg[\int_0^{\tau_n\wedge t} \Phi_y^R(\tau)(1-q^*(\tau))\sigma dW_2(\tau)\bigg] = \mathbb{E}\bigg[\int_0^0 \Phi_y^R(\tau)(1-q^*(\tau))\sigma dW_2(\tau)\bigg] = 0.
$$

Due to that, we have

$$
\begin{aligned}
\mathbb{E}_{t,y}[\Phi^R(\tau_n \wedge T)] =& \Phi^R(t) + \mathbb{E}_{t,y}\bigg[\int_t^{\tau_n\wedge T} \mathcal{D}^R\Phi^R(\tau)d\tau\bigg] \\
&\begin{cases} \leq \Phi^R(t) & \text{for all } p_R(\cdot) \in \Lambda_R \\ = \Phi^R(t) & \text{if } p_R(t) = p_R^*(t) \end{cases},
\end{aligned}
$$

since $\mathcal{D}^R\Phi^R(t)|_{p_R(t)} \leq \mathcal{D}^R\Phi^R(t)|_{p_R^*(t)} = 0$. Since $\mathbb{E}_{t,y}[\Phi^R(\tau_n \wedge T)^2] < \infty$, we have uniform integrability and therefore,

$$
\begin{aligned}
\sup_{p_R\in\Lambda_R} \mathbb{E}_{t,y}[U_R(Y_R^{p_R,q}(T))] =& \sup_{p_R\in\Lambda_R} \mathbb{E}_{t,y}[\Phi^R(T)] \\
=& \lim_{n\to\infty} \mathbb{E}_{t,y}[\Phi^R(\tau_n \wedge T)] \\
&\begin{cases} \leq \Phi^R(t) & \text{for all } p_R(\cdot) \in \Lambda_R \\ = \Phi^R(t) & \text{if } p_R(t) = p_R^*(t) \end{cases},
\end{aligned}
$$

since $\tau_n \to T$ as $n \to \infty$ by the construction of $(M_n^R)_{n\in\mathbb{N}}$. □

3.5 Comparison with Chen and Shen [2018]

Since the first paper regarding Stackelberg games in the context of insurance and reinsurance is Chen and Shen [2018], we compare the difference of our Stackelberg game (which is a special case of Bai et al. [2019]) with their paper.

Remark. Differences between the solution of Proposition 5.2 in Chen and Shen [2018] and the solution from Theorem 3.3:

1. Chen and Shen [2018] use the stochastic HJB-approach and in the appendix also the deterministic HJB-approach to calculate the solution of the Stackelberg game. In this master thesis, we only use the deterministic HJB-approach to find the solution.

2. In the solution of Theorem 3.3 we allow in addition that the insurer can invest in the financial market. Hence, the insurer selects a proportional reinsurance strategy and an investment strategy whereas the insurer in Chen and Shen [2018] only selects a proportional reinsurance strategy.

3. We study the same four cases as in Chen and Shen [2018] for the optimal reinsurance strategy and the reinsurance premium strategy. The solutions in the four cases are the same for the optimal reinsurance strategy and the reinsurance premium strategy.

4. The value function of the reinsurer is the same as in Chen and Shen [2018]. In contrast, the value function of the insurer has additional terms which result from the investment in the risky asset.

Chapter 4

Reinsurance of an Insurance Product

4.1 Motivation

In the previous chapter, we considered a Stackelberg game between a reinsurer and an insurer on the level of the whole surplus processes and aggregated risks. However, in reality, reinsurance agreements are mainly written for parts of the insurer's business lines or even for single products separately. Therefore, in this chapter we consider a Stackelberg game between a reinsurer and an insurer in the context of a life insurance product with a capital guarantee (cf. Escobar-Anel et al. [2021]).

Escobar-Anel et al. [2021] consider an insurer that sells a life insurance product with a capital guarantee to a representative client. The insurer invests dynamically in a risky asset, which is non-reinsurable, and reduces the downside risk by buying a put option on some benchmark fund from the reinsurer. The researchers determine the optimal dynamic investment and reinsurance strategy of the insurer with no-short-selling and Value-at-Risk constraints by maximizing the expected utility of the terminal wealth.

In contrast, we model the interaction between the reinsurer and the insurer in form of a Stackelberg game. Since approximately 200 reinsurance companies and several thousands insurance companies exist worldwide (cf. Albrecher et al. [2017]), the reinsurer has rather a monopoly position (cf. Chen and Shen [2018]). Due to the size of the reinsurance company and as it operates on an international level, the reinsurer is in a good position to assess the decision of the insurer. Therefore, in the Stackelberg game solved in this chapter, the reinsurer is the leader and the insurer the follower.

As in Escobar-Anel et al. [2021], we assume that the insurer follows an individual investment strategy that is riskier than what the reinsurer is willing to reinsure or not even known exactly to the reinsurer. Therefore, the reinsurance, which is modeled by a put option, is on some benchmark portfolio, which is not equal to the insurer's portfolio but highly correlated with it.

In contrast to Escobar-Anel et al. [2021] and Chapter 3 of this master thesis, we model the reinsurance contract as a fixed-term investment, not a dynamic one. The reason for this is that in general the reinsurer and the insurer agree on a reinsurance contract at the beginning and only adjust it at regular intervals, e.g., annually. Therefore, we assume that the insurer buys put options from the reinsurer only at the beginning of the investment period, and the reinsurer pays the reinsurance at the end of the reinsurance contract.

We model the following situation: The financial market contains one risk-free asset, one non-reinsurable fund and one reinsurable fund. We assume that the non-reinsurable fund and the reinsurable fund are correlated, but are not the same. The insurer's portfolio contains the risk-free asset and the non-reinsurable fund. Since the reinsurer is not willing to reinsure the insurer's portfolio, it sells a put option with the capital guarantee as the strike price and a benchmark portfolio as the underlying. The benchmark portfolio contains the risk-free asset and the reinsurable fund. Hence, it is highly correlated to the insurer's portfolio. In contrast, the reinsurer can invest in the risk-free asset as well as both the non-reinsurable and reinsurable fund. We allow the reinsurer to invest in the reinsurable fund so that it can hedge its put option position. It can also invest in the non-reinsurable fund, as reinsurer's are usually larger institutional investors than insurers, which is why their investment universe is usually broader than that of the insurers.
At the beginning of the insurance/reinsurance contract, the representative client pays an initial contribution to the insurer and the insurer buys reinsurance from the reinsurer. Hence, the reinsurer chooses a safety loading for the reinsurance premium and the insurer a reinsurance strategy, i.e., how much put options it is willing to buy from the reinsurer. At the end, the reinsurer pays the terminal payoff of the put options it sold to the insurer. The goal of the insurer is to find a reinsurance strategy and an investment strategy that maximize its expected utility of the total terminal wealth. The reinsurer wants to find a safety loading of the reinsurance premium and an investment strategy such that its expected utility of the total terminal wealth is maximized.

We extend the existing literature on Stackelberg games in the context of insurance and reinsurance by solving a novel Stackelberg game. The novelty of the game has several aspects: the reinsurance is on an individual life insurance product, is only traded at the beginning of the investment period and is not on the exact potential loss of the insurer's portfolio, but a correlated one.

Because the optimization problem of the insurer has a fixed-term investment in the put option as well as a portfolio constraint, standard methods cannot be applied. For the portfolio constraint, we use the method of auxiliary markets introduced in Cvitanić and Karatzas [1992]. To overcome the problem of the fixed-term investment, we apply the generalized martingale method, which is presented in Desmettre and Seifried [2016]. In contrast, the optimization problem of the reinsurer has an additional investment in a put option, which is known. We use the idea of replicating strategies as in Korn and Trautmann [1999] to solve the optimization problem of the reinsurer when an investment position in an option is fixed.

The chapter is organized as follows: In Section 4.2 we describe formally the financial market as well as the Stackelberg game between the reinsurer and the insurer. The optimal solution to the Stackelberg game is derived in Section 4.4. It is divided into the subsections devoted to the optimization problem of the insurer and optimization problem of the reinsurer in the context of the Stackelberg games, as the solution approaches differ. In Section 4.5, we apply the solution methods obtained in Section 4.4 to the case when both parties have power utility functions and conduct numerical studies.

4.2 Stackelberg Game

4.2.1 Framework

The financial market consists of one risk-free asset S_0 and two risky assets $S_1,\ S_2$. The dynamics of the risk-free asset S_0 is given by

$$dS_0(t) = S_0(t) r dt,\ S_0(t) = 1$$

and of the risky assets S_1 and S_2 by

$$\begin{aligned} dS_1(t) =& S_1(t)(\mu_1 dt + \sigma_1 dW_1(t)),\ S_1(0) = s_1 > 0, \\ dS_2(t) =& S_2(t)(\mu_2 dt + \sigma_2(\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t))),\ S_2(0) = s_2 > 0, \end{aligned}$$

where $W(t) = (W_1(t), W_2(t))^\top$ is a two-dimensional Brownian motion, $\rho \in [-1, 1]$ and $r,\ \mu_1,\ \mu_2,\ \sigma_1,\ \sigma_2$ are positive and deterministic constants such that $\mu_1 > r,\ \mu_2 > r$. We denote

$$\mathbb{1} := \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \mu := \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \text{ and } \sigma := \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2 \rho & \sigma_2 \sqrt{1-\rho^2} \end{pmatrix}.$$

The variable r is called the interest rate, μ the yield rate and σ the volatility matrix. The market price of risk is defined by

$$\gamma := \sigma^{-1}(\mu - r\mathbb{1})$$

and the discount factor (also called pricing kernel) by

$$\tilde{Z}(t) := \exp\Big(- \Big(r + \frac{1}{2}\|\gamma\|^2\Big)t - \gamma^\top W(t)\Big).$$

The risk-free asset can be interpreted as a bank account. We assume that the risky asset S_1 is a fund in which the insurer can invest but cannot get reinsured. Furthermore, the risky asset S_2 is a fund which is reinsurable but the insurer cannot invest in.
Let $\pi^{CM} \in [0,1]$. We model reinsurance as a put option on a benchmark portfolio that follows the strategy $\pi_B(t) = (0, \pi^{CM})^\top$ for all $t \in [0,T]$. The constant-mix strategy consists of investing $1 - \pi^{CM}$ of the wealth in S_0 and π^{CM} in S_2, so the dynamics of the constant-mix portfolio is given by

$$\begin{aligned} dV^{v_I,\pi_B}(t) =&(1-\pi^{CM})V^{v_I,\pi_B}(t)\frac{dS_0(t)}{S_0(t)} + \pi^{CM}V^{v_I,\pi_B}(t)\frac{dS_2(t)}{S_2(t)} \\ =&V^{v_I,\pi_B}(t)((r + \pi^{CM}(\mu_2 - r))dt + \pi^{CM}\sigma_2(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t))), \\ V^{v_I,\pi_B}(0) =&v_I, \end{aligned}$$

where $v_I > 0$ denotes the initial contribution of the representative client to the insurer. The insurer pays at time T a capital guarantee G_T to the representative client. We denote by P a put option with underlying V^{v_I,π_B} and strike price G_T the reinsurer sells to the insurer. Hence, the payoff of the put option P is given by

$$P(T) = (G_T - V^{v_I,\pi_B}(T))^+.$$[1]

The price of the put option P at time t is given by

$$P(t) = \tilde{Z}(t)^{-1}\mathbb{E}[\tilde{Z}(T)(G_T - V^{v_I,\pi_B}(T))^+|\mathcal{F}_t].$$

Lemma 4.1. The replicating strategy $\psi(t)$, $t \in [0,T]$, of the put option P is given by

$$\psi(t) = \Big(\frac{P(t) - \pi^{CM}V^{v_I,\pi_B}(t)(\Phi(d_+) - 1)}{S_0(t)}, 0, \frac{\pi^{CM}V^{v_I,\pi_B}(t)(\Phi(d_+) - 1)}{S_2(t)}\Big), \tag{4.1}$$

[1] $x^+ := \max\{x, 0\}$

where

$$d_+ := d_+(t, V^{v_I,\pi_B}(t)) := \frac{\ln\left(\frac{V^{v_I,\pi_B}(t)}{G_T}\right) + \left(r + \frac{1}{2}(\sigma_2\pi^{CM})^2\right)(T-t)}{\pi^{CM}\sigma_2\sqrt{T-t}}.$$

The dynamics of the put option P is given by

$$\begin{aligned} dP(t) =& [V^{v_I,\pi_B}(t)(\Phi(d_+)-1)\pi^{CM}(\mu_2 - r) + rP(t)]dt \\ &+ V^{v_I,\pi_B}(t)(\Phi(d_+)-1)\sigma_2\pi^{CM}(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)). \end{aligned} \tag{4.2}$$

Proof. The proof is stated in Appendix A. □

4.2.2 Formulation of the Stackelberg Game

The Stackelberg game consists of two optimization problems, the optimization problem of the insurer and the optimization problem of the reinsurer. As in Chapter 3, we assume that the reinsurer is the leader of the Stackelberg game and the insurer is the follower. We assume that the insurer invests continuously in the assets S_0 and S_1 and buys a specific number of put options from the reinsurer but only at time 0. Hence, the insurer chooses a portfolio process $\pi_I(t) = (\pi_{1I}(t), \pi_{2I}(t))^\top$, $t \in [0,T]$, such that $\pi_{2I}(t) \equiv 0$ $\mathbb{Q}$-a.s. for all $t \in [0,T]$, and a reinsurance strategy ξ_I, i.e., how many put options are bought at 0. Furthermore, we assume that the reinsurer invests continuously in the assets S_0, S_1 and S_2 and sells put options to the insurer at a price of $(1+\theta_R)P(0)$ at time 0, where θ_R is the safety loading of the reinsurer. Hence, the reinsurer chooses a portfolio process $\pi_R(t) = (\pi_{1R}(t), \pi_{2R}(t))^\top$, $t \in [0,T]$, and the safety loading θ_R.

Optimization Problem of the Insurer (Follower)

The wealth process of the insurer $V_I^{v_{I0}(\xi_I,\theta_R),\pi_I}$ is given by

$$\begin{aligned} dV_I^{v_{I0}(\xi_I,\theta_R),\pi_I}(t) =& (1-\pi_{1I}(t)-\pi_{2I}(t))V_I^{v_{I0}(\xi_I,\theta_R),\pi_I}(t)\frac{dS_0(t)}{S_0(t)} \\ &+ \pi_{1I}(t)V_I^{v_{I0}(\xi_I,\theta_R),\pi_I}(t)\frac{dS_1(t)}{S_1(t)} + \pi_{2I}(t)V_I^{v_{I0}(\xi_I,\theta_R),\pi_I}(t)\frac{dS_2(t)}{S_2(t)} \\ =& V_I^{v_{I0}(\xi_I,\theta_R),\pi_I}(t)(\pi_I(t)^\top(\mu - r\mathbb{1}) + r)dt + V_I^{v_{I0}(\xi_I,\theta_R),\pi_I}(t)\pi_I(t)^\top\sigma dW(t), \\ V_I^{\pi_I}(0) =& v_I - \xi_I(1+\theta_R)P(0) =: v_{I0}(\xi_I,\theta_R), \end{aligned} \tag{4.3}$$

where $v_I > 0$ is the initial wealth of the insurer and $\theta_R \in [0, \theta^{\max}]$ with $\theta^{\max} > 0$ is the safety loading of the reinsurer. The aim of the insurer is to maximize its expected utility of the terminal wealth plus the reinsurance payout, i.e.

$$\sup_{(\pi_I,\xi_I)\in\Lambda_I} \mathbb{E}[U_I(V_I^{v_{I0}(\xi_I,\theta_R),\pi_I}(T)+\xi_I P(T))], \tag{P_I}$$

where U_I is the utility function of the insurer and Λ_I is the set of all admissible strategies in the optimization problem (P_I) defined by

$$\begin{aligned}\Lambda_I := \{(\pi_I,\xi_I)|\ &\pi_I \text{ self-financing, } \pi_I(t)\in K\ \mathbb{Q}\text{-a.s. } \forall t\in[0,T],\ \xi_I\in[0,\xi^{\max}],\\ &V_I^{v_{I0}(\xi_I,\theta_R),\pi_I}(t)\geq 0\ \mathbb{Q}\text{-a.s. } \forall t\in[0,T] \text{ and}\\ &\mathbb{E}[U_I(V_I^{v_{I0}(\xi_I,\theta_R),\pi_I}(T)+\xi_I P(T))^-]<\infty]\}\end{aligned}$$

with $K := \mathbb{R}\times\{0\}$ and $\xi^{\max} := \min\left\{\bar{\xi}, \frac{v_I}{(1+\theta_R)P(0)}\right\}$ with $\bar{\xi}>0$. The set K is convex and describes the constraint on the portfolio process π_I, i.e., the insurer can only invest in the risky asset S_1 and not in S_2. We choose $\xi^{\max}$ in a way such that the initial wealth of the insurer is non-negative, i.e., the insurer is solvent at time 0. In addition, we allow that $\xi^{\max}$ can be limited by a constant $\bar{\xi}$ which is independent of θ_R. For example, we can choose $\bar{\xi}$ to be close to 1 to avoid that the insurer speculates with the reinsurance.

Optimization Problem of the Reinsurer (Leader)

For a given $\xi_I\in[0,\xi^{\max}]$, the wealth process of the reinsurer $V_R^{v_{R0}(\xi_I,\theta_R),\pi_R}$ is given by

$$\begin{aligned}dV_R^{v_{R0}(\xi_I,\theta_R),\pi_R}(t) =&(1-\pi_{1R}(t)-\pi_{2R}(t))V_R^{v_{R0}(\xi_I,\theta_R),\pi_R}(t)\frac{dS_0(t)}{S_0(t)}\\ &+\pi_{1R}(t)V_R^{v_{R0}(\xi_I,\theta_R),\pi_R}(t)\frac{dS_1(t)}{S_1(t)}+\pi_{2R}(t)V_R^{v_{R0}(\xi_I,\theta_R),\pi_R}(t)\frac{dS_2(t)}{S_2(t)},\\ V_R^{v_{R0}(\xi_I,\theta_R),\pi_R}(0) =&v_R+\xi_I(1+\theta_R)P(0) =: v_{R0}(\xi_I,\theta_R),\end{aligned} \tag{4.4}$$

where $v_R>0$ is the initial wealth of the reinsurer. The aim of the reinsurer is to maximize the expected utility of the terminal wealth less the reinsurance payout, i.e.

$$\sup_{(\pi_R,\theta_R)\in\Lambda_R} \mathbb{E}[U_R(V_R^{v_{R0}(\xi_I,\theta_R),\pi_R}(T)-\xi_I P(T))], \tag{P_R}$$

where U_R is the utility function of the reinsurer and Λ_R the set of all admissible strategies in the optimization problem (P_R) defined by

$$\begin{aligned}\Lambda_R := \{(\pi_R,\theta_R)|\ &\pi_R \text{ self-financing, } V_R^{v_{R0}(\xi_I,\theta_R),\pi_R}(t)\geq 0\ \mathbb{Q}\text{-a.s. } \forall t\in[0,T],\\ &\theta_R\in[0,\theta^{\max}] \text{ and } \mathbb{E}[U_R(V_R^{v_{R0}(\xi_I,\theta_R),\pi_R}(T)-\xi_I P(T))^-]<\infty\}\end{aligned}$$

with $\theta^{\max} > 0$. Since the reinsurer invests in the risky assets S_1 and S_2, we have no constraint on the portfolio process π_R of the reinsurer.

Remark. In Section 4.4.2, we will use the wealth process of the reinsurer via the trading strategy φ_R instead of the portfolio process π_R. We denote the wealth process by $V^{v_{R0}(\xi_I,\theta_R),\pi_R}$ if we consider the wealth process with respect to the portfolio process π_R and by $V^{v_{R0}(\xi_I,\theta_R),\varphi_R}$ with respect to the trading strategy φ_R. The relation between the trading strategy φ_R and the portfolio process π_R is given by (cf. Korn [2014])

$$\pi_{iR}(t) = \frac{S_i(t)}{V^{v_{R0}(\xi_I,\theta_R),\varphi_R}(t)}\varphi_{iR}(t),\ i = 1, 2.$$

Stackelberg Game

Definition 4.2 (Stackelberg Game)**.** The Stackelberg game between the reinsurer and insurer is given by

$$\begin{aligned} &\sup_{(\pi_R,\theta_R)\in\Lambda_R} \mathbb{E}[U_R(V_R^{v_{0R}(\xi_I^*(\theta_R),\theta_R),\pi_R}(T) - \xi_I^*(\theta_R)P(T))] \\ &\qquad \text{s.t. } (\xi_I^*(\theta_R), \pi_I^*(\cdot|\theta_R)) \in \arg\max_{(\xi_I,\pi_I)\in\Lambda_I} \mathbb{E}[U_I(V_I^{v_{0I}(\xi_I,\theta_R),\pi_I}(T) + \xi_I P(T))]. \end{aligned} \tag{SG}$$

Definition 4.3 (Stackelberg Equilibrium, cf. Definition 2.1)**.** The solution

$$(\pi_R^*(\cdot), \theta_R^*, \pi_I^*(\cdot|\theta_R^*), \xi_I^*(\theta_R^*))$$

of the Stackelberg game (SG) is called the Stackelberg equilibrium.

4.3 Comparison with Chapter 3

In this section, we emphasize the connection between the Stackelberg games in Chapter 3 and Chapter 4 and point out the differences between them.

In Chapter 3, we consider a Stackelberg game between a reinsurer and an insurer. The aim is to find an optimal reinsurance contract between the reinsurer and the insurer. The insurer has a claim process, which is approximated by a diffusion process. Therefore, the insurer has to cover dynamically the aggregated claims. The reinsurer offers a proportional reinsurance to the insurer, i.e., the insurer transfer a part of its risk to the reinsurer. In return, the reinsurer receives a reinsurance premium. Since the claims of the insurer occur dynamically, the insurer can adjust its proportional reinsurance strategy dynamically and pays dynamically a reinsurance premium to the reinsurer. Hence, the dynamics of the surplus process of the reinsurer is given by

$$
\begin{aligned}
dY_R^{p_R,q}(t) =& \underbrace{(1-q(t))p_R(t)dt}_{\text{Reinsurance premium}} - \underbrace{(1-q(t))dC(t)}_{\substack{\text{Payment of proportional}\\ \text{reinsurance to insurer}}} \\
&+ \underbrace{d\bar{V}_R(t)}_{\substack{\text{Investment portfolio}\\ \text{of the reinsurer}}} \\
Y_R^{p_R,q}(0) =& y_{0R} > 0
\end{aligned}
$$

and of the insurer by

$$
\begin{aligned}
dY_I^{p_R,(q,\pi_I)}(t) =& \underbrace{(p_I-(1-q(t))p_R(t))dt}_{\text{Net premium}} - \underbrace{dC(t)}_{\substack{\text{Payment of}\\ \text{claims to representative client}}} \\
&+ \underbrace{(1-q(t))dC(t)}_{\substack{\text{Payoff of proportional}\\ \text{reinsurance from reinsurer}}} + \underbrace{d\bar{V}_I(t)}_{\substack{\text{Investment portfolio}\\ \text{of the insurer}}} \\
Y_I^{p_R,(q,\pi_I)}(0) =& y_{0I} > 0.
\end{aligned}
$$

In comparison, in Chapter 4 we consider again a Stackelberg game between a reinsurer and an insurer. As in Chapter 3, the goal of Chapter 4 is to find an optimal reinsurance contract between the reinsurer and the insurer. In contrast to Chapter 3, we do not consider reinsurance on the whole level of aggregated claims of the insurer. In our case, the insurer offers a life insurance product to its clients with a capital guarantee and buys reinsurance for it. The representative client pays an initial contribution to the insurer and receives in return a capital guarantee at the end of the insurance contract. Therefore, the insurer has only one claim payment at the end of the contract. The reinsurer offers an excess-of-loss reinsurance[2] to the insurer. The retention of the excess-of-loss reinsurance is given by the terminal price of a benchmark portfolio, which is different to the insurer's portfolio but highly correlated. The capital guarantee of the representative client is the claim of the excess-of-loss reinsurance. Since, the claim occurs only once, the reinsurer only offers a one-time reinsurance payment at the end of the reinsurance contract. In the beginning, the insurer pays a one-time reinsurance premium to the reinsurer and decides on its reinsurance strategy. The reinsurance strategy of the insurer has no adjustment during the duration of the reinsurance contract. Hence, the terminal

[2]Excess-of-loss reinsurance is given by $(C-R)^+$, where C is the claim and R the retention. (cf. Albrecher et al. [2017])

surplus of the reinsurer is given by

$$\underbrace{V_R(T)}_{\substack{\text{Terminal value of investment} \\ \text{portfolio of reinsurer}}} - \xi_I \underbrace{\left(G_T - V^{v_{0I},\pi_B}(T)\right)^+}_{\substack{\text{Payment of excess-of-loss} \\ \text{reinsurance to insurer}}}.$$

and the initial wealth by

$$\underbrace{v_{0R}}_{\substack{\text{Initial wealth of} \\ \text{reinsurer before reinsurance}}} + \xi_I \underbrace{(1+\theta_R)P(0)}_{\substack{\text{Reinsurance premium with} \\ \text{safety loading } \theta_R \text{ from insurer}}},$$

where $P(0)$ is the fair price of the excess-of-loss reinsurance at time 0. For the insurer, the terminal surplus is given by

$$\underbrace{V_I(T)}_{\substack{\text{Terminal value of investment} \\ \text{portfolio of insurer}}} + \xi_I \underbrace{\left(G_T - V^{v_{0I},\pi_B}(T)\right)^+}_{\substack{\text{Payoff of excess-of-loss} \\ \text{reinsurance from reinsurer}}} - \underbrace{G_T}_{\substack{\text{Payment of claim} \\ \text{(i.e., guarantee) to representative client}}}$$

and the initial wealth by

$$\underbrace{v_{0I}}_{\substack{\text{Initial contribution} \\ \text{of representative client}}} - \xi_I \underbrace{(1+\theta_R)P(0)}_{\substack{\text{Reinsurance premium with} \\ \text{safety loading } \theta_R \text{ to reinsurer}}}.$$

To conclude, in both Stackelberg games the insurer wants to reinsure its claims. Therefore, the insurer buys reinsurance from the reinsurer. The main differences of the Stackelberg games in Chapter 3 and 4 are the following:

1. In Chapter 4, the occurrence of the claims and the reinsurance payment are one-time events at the end of the insurance and reinsurance contract. In contrast, the occurrence of the claims and the reinsurance payment are dynamic in Chapter 3.

2. In Chapter 4, the reinsurance premium payment and the selection of the reinsurance strategy is static at the beginning of the reinsurance contract. In Chapter 3, the reinsurance premium payment and the selection of the reinsurance strategy are dynamic.

3. In Chapter 3, the reinsurer offers a proportional reinsurance dynamically, whereas in Chapter 4, the reinsurer offers a static non-proportional reinsurance (excess-of-loss reinsurance) with a premium payment at the beginning and a reinsurance payment at the end of the reinsurance contract.

4.4 Solution of the Stackelberg Game

4.4.1 Optimization Problem of the Insurer (Follower)

Since we use the method of backward induction, we start with solving the optimization problem of the insurer (P_I). Therefore, let $\theta_R \in [0, \theta^{\max}]$ be arbitrary but fix. We use the generalized martingale method introduced by Desmettre and Seifried [2016]. In our case, the fixed-term security (i.e., the put option) is not spanned (i.e., it has not the same risk drivers as the liquid market). To circumvent conditional random utility functions, we treat the problem as a constrained optimization problem, i.e., we impose an explicit no trading constraint on S_2. To get an unconstrained optimization problem, and therefore a spanned fixed investment, we use auxiliary markets introduced by Cvitanić and Karatzas [1992].

Auxiliary Market

The financial market consisting of S_0, S_1 and S_2 is called the basic financial market $\mathcal{M}$. In the next step, we introduce the auxiliary market (cf. Cvitanić and Karatzas [1992] and Section 1.4.3). We set $K = \mathbb{R} \times \{0\}$, which is convex. Its support function is given by $\delta : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ with

$$\delta(x) := -\inf_{y \in K}(x^\top y) = -\inf_{y_1 \in \mathbb{R}}(x_1 y_1) = \begin{cases} 0, & \text{if } x_1 = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

The barrier cone of K is defined by

$$\tilde{K} := \{x \in \mathbb{R}^2 | \delta(x) < +\infty\} = \{x \in \mathbb{R}^2 | x_1 = 0\} = \{0\} \times \mathbb{R}.$$

For $x \in \tilde{K}$ we have $\delta(x) = 0$. Let the class of $\mathbb{R}^2$-valued dual processes given by

$$\mathcal{D} := \left\{ \lambda = (\lambda(t))_{t \in [0,T]} \text{ prog. measurable} \middle| \mathbb{E}\left[\int_0^T \|\lambda(t)\|^2 dt \right] < \infty,\ \mathbb{E}\left[\int_0^T \delta(\lambda(t)) dt \right] < \infty \right\}.$$

It holds for $\lambda \in \mathcal{D}$ that $\lambda(t) \in \tilde{K}$ $\mathbb{Q}$-a.s for all $t \in [0, T]$,i.e., $\lambda_1(t) = 0$ $\mathbb{Q}$-a.s. for all $t \in [0, T]$.
Let $\lambda \in \mathcal{D}$. We introduce the auxiliary market $\mathcal{M}_\lambda$ by

$$\begin{aligned} dS_0^\lambda(t) =& S_0^\lambda(t) r dt, \\ dS_1^\lambda(t) =& S_1^\lambda(t)(\mu_1 dt + \sigma_1 dW_1(t)), \end{aligned}$$

$$dS_2^\lambda(t) = S_2^\lambda(t)[(\mu_2 + \lambda_2(t))dt + \sigma_2(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t))],$$

since $\delta(\lambda(t)) = 0$ and $\lambda_1(t) = 0$ (cf. Section 1.4.3). We define the market price of risk and the discount factor (also called pricing kernel) in the auxiliary market by

$$\begin{aligned}
&\gamma_\lambda(t) := \gamma + \sigma^{-1}\lambda(t),\\
&\tilde{Z}_\lambda(t) := \exp\Big(- rt - \frac{1}{2}\int_0^t \|\gamma_\lambda(s)\|^2 ds - \int_0^t \gamma_\lambda(s)^\top dW(s)\Big).
\end{aligned}$$

For more details about the auxiliary market and ideas of Cvitanić and Karatzas [1992], see Section 1.4.3.

To apply the generalized martingale method introduced by Desmettre and Seifried [2016], we only need the stochastic payoff of the fixed-term security and the price at time 0. Therefore, we are only interested in the stochastic payoff at time T and the price at time 0 of the put option P. In the following, we interpret $P(T)$ as a random variable, which is $\mathcal{F}_T$-measurable and denote it by $P_\lambda(T)$ when working with the auxiliary market $\mathcal{M}_\lambda$.

Reminder: $P(0)$ denotes the fair price of the put option at time 0 in the basic market by risk-neutral valuations. In accordance to Desmettre and Seifried [2016], in the auxiliary market the price of the fixed-term security P with payoff $P(T)$ is given by $P(0)$, which is a pre-defined price, here the price of the put option in the basic market, and is not equal to the fair price of the put option in the auxiliary market $\mathbb{E}[\tilde{Z}_\lambda(T)P(T)]$. If we work in the auxiliary market, we denote the price of the fixed-term security by $P_\lambda(0)$, i.e., $P_\lambda(0) = P(0)$[3].

The wealth process of the insurer $V_\lambda^{v_{I0}(\xi_I,\theta_R),\pi_I}$ in $\mathcal{M}_\lambda$ is given by

$$\begin{aligned}
dV_\lambda^{v_{I0}^\lambda(\xi_I,\theta_R),\pi_I}(t) =&(1-\pi_{1I}(t)-\pi_{2I}(t))V_\lambda^{v_{I0}^\lambda(\xi_I,\theta_R),\pi_I}(t)\frac{dS_0^\lambda(t)}{S_0^\lambda(t)} \\
&+ \pi_{1I}(t)V_\lambda^{v_{I0}^\lambda(\xi_I,\theta_R),\pi_I}(t)\frac{dS_1^\lambda(t)}{S_1^\lambda(t)} + \pi_{2I}(t)V_\lambda^{v_{I0}^\lambda(\xi_I,\theta_R),\pi_I}(t)\frac{dS_2^\lambda(t)}{S_2^\lambda(t)} \\
=&V_\lambda^{v_{I0}^\lambda(\xi_I,\theta_R),\pi_I}(t)(\pi_I(t)^\top(\mu - r\mathbb{1}) + r)dt + V_\lambda^{v_{I0}^\lambda(\xi_I,\theta_R),\pi_I}(t)\pi_I(t)^\top\sigma dW(t) \\
&+ \underbrace{V_\lambda^{v_{I0}^\lambda(\xi_I,\theta_R),\pi_I}(t)\pi_I(t)^\top\lambda(t)dt}_{\text{additional term}}, \\
V_\lambda^{v_{I0}^\lambda(\xi_I,\theta_R),\pi_I}(0) =& \, v_I - \xi_I(1+\theta_R)P_\lambda(0) =: v_{I0}^\lambda(\xi_I,\theta_R).
\end{aligned} \tag{4.5}$$

[3]The price $P_\lambda(0)$ of the fixed-term security in the auxiliary market is pre-defined and does not equal to fair price of the put option in the auxiliary market $\mathbb{E}[\tilde{Z}_\lambda(T)P_\lambda(T)]$ (cf. Desmettre and Seifried [2016]), i.e., $P_\lambda(0) \neq \mathbb{E}[\tilde{Z}_\lambda(T)P(T)]$.

The unconstrained optimization problem of the insurer in $\mathcal{M}_\lambda$ is given by

$$\sup_{(\pi_I,\xi_I)\in\Lambda_I^\lambda} \mathbb{E}[U_I(V_\lambda^{v_{I0}^\lambda(\xi_I,\theta_R),\pi_I}(T)+\xi_I P_\lambda(T))], \tag{P_I^λ}$$

where

$$\begin{aligned}\Lambda_I^\lambda := \{(\pi_I,\xi_I)|\ &\pi_I \text{ self-financing, } V_\lambda^{v_{I0}^\lambda(\xi_I,\theta_R),\pi_I}(t)\geq 0 \ \mathbb{Q}\text{-a.s. } \forall t\in[0,T],\\ &\xi_I\in[0,\xi^{\max}] \text{ and } \mathbb{E}[U_I(V_\lambda^{v_{I0}^\lambda(\xi_I,\theta_R),\pi_I}(T)+\xi_I P(T))^-]<\infty\}.\end{aligned}$$

We denote by $(\pi_\lambda^*,\xi_\lambda^*)$ the solution to (P_I^λ), i.e.,

$$(\pi_\lambda^*,\xi_\lambda^*) = \arg\sup_{(\pi_I,\xi_I)\in\Lambda_I^\lambda} \mathbb{E}[U_I(V_\lambda^{v_{I0}^\lambda(\xi_I,\theta_R),\pi_I}(T)+\xi_I P_\lambda(T))].$$

Random utility function

As in Desmettre and Seifried [2016], we define the random utility function by

$$\hat{U}_I(x) := U_I(x+\xi_I P_\lambda(T))$$

for $x\in[0,\infty)$, where $\xi_I\in[0,\xi^{\max}]$. The utility function $\hat{U}_I$ is random, since $P_\lambda(T)$ is a random variable. Hence, $\hat{U}_I : [0,+\infty)\to[U_I(\xi_I P_\lambda(T)),+\infty)$ and $\hat{U}_I$ is continuously differentiable, strictly increasing and strictly concave. Therefore, it holds $\hat{U}_I' : [0,+\infty)\to(0,U_I'(\xi_I P_\lambda(T))]$ and

$$\hat{U}_I'(x) = U_I'(x+\xi_I P_\lambda(T)).$$

We denote the inverse function of $\hat{U}_I'$ by $\hat{I}_I : (0,+\infty)\to[0,+\infty)$. For $y\in(0,U_I'(\xi_I P_\lambda(T))]$ it is given by $I_I(y)-\xi_I P_\lambda(T)$, where I_I denotes the inverse of U_I'. For $y>U_I'(\xi_I P_\lambda(T))$ we set $\hat{I}(y):=0$. Hence, the random inverse function $\hat{I}_I$ is bijective on $(0,U_I'(\xi_I P_\lambda(T))]$.

Solution of the optimization problem

The procedure of solving the optimization problem of the insurer (P_I) is the following:

1. We solve for any $\lambda\in\mathcal{D}$ the unconstrained optimization problem of the insurer (P_I^λ) with the generalized martingale method introduced by Desmettre and Seifried [2016], i.e., we find $(\pi_\lambda^*,\xi_\lambda^*)$ for any $\lambda\in\mathcal{D}$.

2. If there exists a $\lambda^*\in\mathcal{D}$ such that $\pi_{\lambda^*}^*(t)\in K$ $\mathbb{Q}$-a.s. for all $t\in[0,T]$, then we can

prove that the solution $(\pi^*_{\lambda^*}, \xi^*_{\lambda^*})$ of the unconstrained optimization problem of the insurer (P_I^λ) is optimal for the optimization problem of the insurer (P_I).

Proposition 4.4 (Optimal solution to (P_I^λ))**.** Assume that for all $y \in (0, \infty)$

$$\mathbb{E}[\tilde{Z}_\lambda(T) I_I(y\tilde{Z}_\lambda(T))] < \infty \text{ and } \mathbb{E}[U_I(I_I(y\tilde{Z}_\lambda(T)))] < \infty$$

holds. Then, there exists a solution $(\pi^*_\lambda, \xi^*_\lambda)$ to the unconstrained optimization problem of the insurer (P_I^λ), where

$$\xi^*_\lambda = \underset{\xi \in [0, \xi^{\max}]}{\arg\max}\ \nu(\xi).$$

The function ν is given by

$$\nu(\xi_I) := \mathbb{E}[U_I(\max\{I_I(y^*(\xi_I)\tilde{Z}_\lambda(T)), \xi_I P_\lambda(T)\})],$$

where the Lagrange multiplier $y^* := y^*(\xi_I)$ is given by the budget constraint

$$\mathbb{E}[\tilde{Z}_\lambda(T)\hat{I}_I(y^*\tilde{Z}_\lambda(T))] = v_I - \xi_I(1 + \theta_R)P_\lambda(0).$$

The optimal terminal wealth $V^*_\lambda(T) := V_\lambda^{v^\lambda_{I0}(\xi^*_I, \theta_R), \pi^*_\lambda}(T)$ is given by

$$\hat{I}_I(y^*(\xi^*_\lambda)\tilde{Z}_\lambda(T)) = \max\{I_I(y^*(\xi^*_\lambda)\tilde{Z}_\lambda(T)) - \xi^*_\lambda P_\lambda(T), 0\}.$$

If U_I is a power utility function, i.e., $U_I(x) = \frac{1}{b_I}x^{b_I}$ with $b_I \in (-\infty, 0)\backslash\{0\}$, then the optimal portfolio process π^*_λ is given by

$$\pi^*_\lambda(t)V^*_\lambda(t) = \pi^M_\lambda(t)(V^*_\lambda(t) + \xi^*_\lambda \tilde{Z}_\lambda(t)^{-1}\mathbb{E}[\tilde{Z}_\lambda(T)P_\lambda(T)\mathbf{1}_{\{V^*_\lambda(T)>0\}}|\mathcal{F}_t])$$

where π^M_λ is the Merton portfolio process given by

$$\pi^M_\lambda(t) = \frac{1}{1 - b_I}(\sigma\sigma^\top)^{-1}(\mu + \lambda(t) - r\overrightarrow{1}).$$

Proof. This is exactly the statement of Theorem 1.40 and 1.41 in Section 1.4.2. For the proof of these theorems, see Desmettre and Seifried [2016]. □

Remark. If we choose $\xi^{\max} = \bar{\xi}$ with $\bar{\xi} < \frac{v_I}{(1+\theta_R)P(0)}$ for all $\theta_R \in [0, \theta^{\max}]$, then for the initial wealth of the insurer it follows

$$v_I - \xi_I(1 + \theta_R)P_\lambda(0) > 0$$

Since the random inverse function $\hat{I}_I$ is bijective on $(0, U_I'(\xi_I P_\lambda(T))]$ and given by $\hat{I}_I(y) = I_I(y) - \xi_I P_\lambda(T)$ for $y \in (0, U_I'(\xi_I P_\lambda(T))]$, we get

$$\hat{I}_I(y) > 0 \Leftrightarrow y \in (0, U_I'(\xi_I P_\lambda(T))). \tag{4.6}$$

It follows for the Lagrange multiplier y_I^* by the budget constraint

$$\begin{aligned}
\mathbb{E}[\tilde{Z}_\lambda(T)\hat{I}_I(y_I^*\tilde{Z}_\lambda(T))] &= \underbrace{v_I - \xi_I(1+\theta_R)P_\lambda(0)}_{>0} \\
&\overset{(a)}{\Leftrightarrow} \\
\hat{I}_I(y_I^*\tilde{Z}_\lambda(T)) &> 0 \\
&\overset{(b)}{\Leftrightarrow} \\
y_I^*\tilde{Z}_\lambda(T) &< U_I'(\xi_I P_\lambda(T)) \\
&\overset{(c)}{\Leftrightarrow} \\
I_I(y_I^*\tilde{Z}_\lambda(T)) &> \xi_I P_\lambda(T),
\end{aligned}$$

where (a) follows from $\tilde{Z}_\lambda(T) > 0$ $\mathbb{Q}$-a.s. and (b) from (4.6) and (c) from the fact that I_I is strictly decreasing. Hence, we get

$$\max\{I_I(y_I^*(\xi_I)\tilde{Z}_{\lambda^*}(T)), \xi_I P(T)\} = I_I(y_I^*(\xi_I)\tilde{Z}_{\lambda^*}(T)) \tag{4.7}$$

and it follows for the function ν

$$\begin{aligned}
\nu(\xi_I) =& \mathbb{E}[U_I(\max\{I_I(y_I^*(\xi_I)\tilde{Z}_{\lambda^*}(T)), \xi_I P(T)\})] \\
=& \mathbb{E}[U_I(I_I(y_I^*(\xi_I)\tilde{Z}_{\lambda^*}(T)))].
\end{aligned}$$

Hence, by Proposition 4.4 we know that under some conditions there exists an optimal solution $(\pi_\lambda^*, \xi_\lambda^*)$ to the unconstrained optimization problem of the insurer (P_I^λ). In the next proposition, we show how the solutions of the optimization problem of the insurer (P_I) and the unconstrained optimization problem of the insurer (P_I^λ) are linked.

Proposition 4.5 (Optimal solution to (P_I))**.** Suppose that there exists $\lambda^* \in \mathcal{D}$ such that for the optimal solution $(\pi_{\lambda^*}^*, \xi_{\lambda^*}^*)$ to $(P_I^{\lambda^*})$ we have $\pi_{\lambda^*}^*(t) \in K$ $\mathbb{Q}$-a.s. for all $t \in [0, T]$. Then $(\pi_{\lambda^*}^*, \xi_{\lambda^*}^*)$ is optimal for the optimization problem of the insurer (P_I).

Proof. The proof is based on the proof of Proposition 8.3 in Cvitanić and Karatzas [1992]. The proof consists of two parts:

1. We fix $\xi_I \in [0, \xi^{\max}]$ and prove that $\pi_{\lambda^*}^*$ is optimal for (P_I) given the fixed ξ_I.

2. We prove that $\xi^*_{\lambda^*}$ is optimal for (P_I) given the optimal portfolio process $\pi^*_I \equiv \pi^*_{\lambda^*}$.

1. For any $\xi_I \in [0, \xi^{\max}]$ and $\lambda \in \mathcal{D}$ it holds

$$\begin{aligned} v^\lambda_{I0}(\xi_I, \theta_R) &= v_I - \xi_I(1+\theta_R)P_\lambda(0) \\ &\overset{(*)}{=} v_I - \xi_I(1+\theta_R)P(0) \\ &= v_{I0}(\xi_I, \theta_R). \end{aligned}$$

where $(*)$ follows from the fact that $P_\lambda(0) = P(0)$. Hence, for the initial wealth it holds

$$V_\lambda^{v^\lambda_{I0}(\xi_I,\theta_R),\pi_I}(0) = V_I^{v_{I0}(\xi_I,\theta_R),\pi_I}(0). \tag{4.8}$$

Let $\xi_I \in [0, \xi^{\max}]$ be fixed. Furthermore, let π_I such that $(\pi_I, \xi_I) \in \Lambda_I$, i.e., it holds $\pi_I(t) \in K$ $\mathbb{Q}$-a.s. for all $t \in [0,T]$. Hence, for all $\lambda \in \mathcal{D}$ and $t \in [0,T]$ we have $\pi_I(t)^\top \lambda(t) = 0$, whence

$$V_\lambda^{v^\lambda_{I0}(\xi_I,\theta_R),\pi_I}(t) = V_I^{v_{I0}(\xi_I,\theta_R),\pi_I}(t) \geq 0 \text{ a.s.}, \tag{4.9}$$

where the equality follows from $\pi_I(t)^\top \lambda(t) = 0$ and Equation (4.8), and

$$\mathbb{E}[U_I(V_\lambda^{v^\lambda_{I0}(\xi_I,\theta_R),\pi_I}(T) + \xi_I P_\lambda(T))^-] = \mathbb{E}[U_I(V_I^{v_{I0}(\xi_I,\theta_R),\pi_I}(T) + \xi_I P(T))^-] < \infty,$$

where the equality follows from $P_\lambda(T) = P(T)$ and the Equation (4.8). Therefore, $(\pi_I, \xi_I) \in \Lambda_I^\lambda$. It follows $\Lambda_I \subset \Lambda_I^\lambda$ and

$$\begin{aligned} &\sup_{\pi_I:(\pi_I,\xi_I)\in\Lambda_I} \mathbb{E}[U_I(V_I^{v_{I0}(\xi_I,\theta_R),\pi_I}(T) + \xi_I P(T))] \\ &\overset{(a)}{=} \sup_{\pi_I:(\pi_I,\xi_I)\in\Lambda_I} \mathbb{E}[U_I(V_\lambda^{v^\lambda_{I0}(\xi_I,\theta_R),\pi_I}(T) + \xi_I P_\lambda(T))] \\ &\overset{(b)}{\leq} \sup_{\pi_I:(\pi_I,\xi_I)\in\Lambda_I^\lambda} \mathbb{E}[U_I(V_\lambda^{v^\lambda_{I0}(\xi_I,\theta_R),\pi_I}(T) + \xi_I P_\lambda(T))], \end{aligned} \tag{4.10}$$

where (a) follows from the Equation (4.9) and $P_\lambda(T) = P(T)$, and (b) from $\Lambda_I \subset \Lambda_I^\lambda$. Let $\lambda^* \in \mathcal{D}$ and the optimal portfolio process $\pi^*_{\lambda^*}$ of the unconstrained optimization problem of the insurer (P_I^λ) given a fixed ξ_I such that $(\pi^*_{\lambda^*}, \xi_I) \in \Lambda_I^\lambda$ and $\pi^*_{\lambda^*}(t) \in K$ $\mathbb{Q}$-a.s. for all $t \in [0,T]$. Then for all $t \in [0,T]$ it holds

$$V_{\lambda^*}^{v^{\lambda^*}_{I0}(\xi_I,\theta_R),\pi^*_{\lambda^*}}(t) \overset{(4.9)}{=} V_I^{v_{I0}(\xi_I,\theta_R),\pi^*_{\lambda^*}}(t). \tag{4.11}$$

Hence, $(\pi^*_{\lambda^*}, \xi_I) \in \Lambda_I$ and

$$\begin{aligned}
&\sup_{\pi_I:(\pi_I,\xi_I)\in\Lambda_I^{\lambda^*}} \mathbb{E}[U_I(V_{\lambda^*}^{v_{0I}^{\lambda^*}(\xi_I,\theta_R),\pi_I}(T) + \xi_I P_{\lambda^*}(T))] \\
&\quad \overset{(a)}{=} \mathbb{E}[U_I(V_{\lambda^*}^{v_{0I}^{\lambda^*}(\xi_I,\theta_R),\pi^*_{\lambda^*}}(T) + \xi_I P_{\lambda^*}(T))] \qquad (4.12)\\
&\quad \overset{(b)}{=} \mathbb{E}[U_I(V_I^{v_{0I}(\xi_I,\theta_R),\pi^*_{\lambda^*}}(T) + \xi_I P(T))] \\
&\quad \overset{(c)}{\leq} \sup_{\pi_I:(\pi_I,\xi_I)\in\Lambda_I} \mathbb{E}[U_I(V_I^{v_{0I}(\xi_I,\theta_R),\pi_I}(T) + \xi_I P(T))], \qquad (4.13)
\end{aligned}$$

where (a) follows from the definition of $\pi^*_{\lambda^*}$, (b) from the Equation (4.11) and $P_{\lambda^*}(T) = P(T)$, and (c) from $(\pi^*_{\lambda^*}, \xi_I) \in \Lambda_I$.
All in all, we have

$$\begin{aligned}
&\mathbb{E}[U_I(V_{\lambda^*}^{v_{0I}^{\lambda^*}(\xi_I,\theta_R),\pi^*_{\lambda^*}}(T) + \xi_I P_{\lambda^*}(T))] \\
&\overset{(a)}{=} \sup_{\pi_I:(\pi_I,\xi_I)\in\Lambda_I^{\lambda^*}} \mathbb{E}[U_I(V_{\lambda^*}^{v_{0I}^{\lambda^*}(\xi_I,\theta_R),\pi_I}(T) + \xi_I P_{\lambda^*}(T))] \\
&\overset{(b)}{=} \sup_{\pi_I:(\pi_I,\xi_I)\in\Lambda_I} \mathbb{E}[U_I(V_I^{v_{0I}(\xi_I,\theta_R),\pi_I}(T) + \xi_I P(T))],
\end{aligned}$$

where (a) follows from the Equation (4.12) and (b) from the inequalities (4.10) and (4.13). Therefore, $\pi^*_{\lambda^*}$ is optimal for the optimization problem of the insurer (P_I) given a fixed ξ_I.

2. Now, let $\xi^*_{\lambda^*}$ be the optimal fixed term reinsurance (i.e., number of puts) in the unconstrained optimization problem of the insurer $(P_I^{\lambda^*})$ given $\pi^*_{\lambda^*} \in K$. It holds

$$\{\xi_I |\ (\pi^*_{\lambda^*}, \xi_I) \in \Lambda_I^{\lambda^*}\} = [0, \xi^{\max}] = \{\xi_I |\ (\pi^*_{\lambda^*}, \xi_I) \in \Lambda_I\}. \qquad (4.14)$$

Then

$$\begin{aligned}
&\mathbb{E}[U_I(V_I^{v_{0I}(\xi_I^*,\theta_R),\pi^*_{\lambda^*}}(T) + \xi^*_{\lambda^*} P(T))] \\
&\qquad \overset{(a)}{=} \mathbb{E}[U_I(V_{\lambda^*}^{v_{0I}^{\lambda^*}(\xi_I^*,\theta_R),\pi^*_{\lambda^*}}(T) + \xi^*_{\lambda^*} P_{\lambda^*}(T))] \\
&\qquad \overset{(b)}{=} \sup_{\xi_I:(\pi^*_{\lambda^*},\xi_I)\in\Lambda_I^{\lambda^*}} \mathbb{E}[U_I(V_{\lambda^*}^{v_{0I}^{\lambda^*}(\xi_I,\theta_R),\pi^*_{\lambda^*}}(T) + \xi_I P_{\lambda^*}(T))] \\
&\qquad \overset{(c)}{=} \sup_{\xi_I:(\pi^*_{\lambda^*},\xi_I)\in\Lambda_I} \mathbb{E}[U_I(V_I^{v_{0I}(\xi_I,\theta_R),\pi^*_{\lambda^*}}(T) + \xi_I P(T))],
\end{aligned}$$

where (a) follows from $P(T) = P_{\lambda^*}(T)$ and Equation (4.11), (b) from the definition of ξ_I^* and (c) from $P_{\lambda^*}(T) = P(T)$ and Equations (4.9) and (4.14). Therefore, $\xi^*_{\lambda^*}$ is optimal for the optimization problem of the insurer (P_I) given the portfolio process $\pi^*_{\lambda^*} \in K$.

All in all, we have $(\pi^*_{\lambda^*}, \xi^*_{\lambda^*})$ is optimal for the optimization problem of the insurer (P_I). $\square$

4.4.2 Optimization Problem of the Reinsurer (Leader)

In this section, we solve the reinsurer's (leader's) optimization problem. For that we adapt the approach proposed by Korn and Trautmann [1999]. Our aim is to find the optimal safety loading θ^*_R and the optimal portfolio process π^*_R of the optimization problem (P_R). In this approach, we work a lot with trading strategies (instead of the relative portfolio processes). As mentioned in Section 4.2.2, the trading strategy φ_R of the reinsurer is linked to the portfolio process π_R of the reinsurer as follows (cf. Korn [2014]): For $i = 1, 2$ and $t \in [0, T]$

$$\pi_{iR}(t) = \frac{S_i(t)}{V_R^{v_{R0}(\xi^*_I(\theta_R),\theta_R),\pi_R}(t)} \varphi_{iR}(t).$$

Therefore, we use the notation $V_R^{v_{I0}(\xi_I,\theta_R),\varphi_R}$ for the wealth process of the reinsurer with respect to the trading strategy φ_R. We assume that the insurer has the optimal strategies $(\xi^*_I(\theta_R), \pi^*_I(t|\theta_R, \xi^*_I(\theta_R)))$ depending on $\theta_R \in [0, \theta^{\max}]$. Hence, the optimization problem of the reinsurer is given by

$$\sup_{(\varphi_R,\theta_R)\in\Lambda_R^{\varphi}} \mathbb{E}[U_R(V_R^{v_{0R}(\xi^*_I(\theta_R),\theta_R),\varphi_R}(T) - \xi^*_I(\theta_R)P(T))], \qquad (P_R^{\varphi_R})$$

where Λ_R^{φ} is the set of all admissible strategies (φ_R, θ_R) defined by

$$\begin{aligned}\Lambda_R^{\varphi} := \{(\varphi_R, \theta_R)|\ &\varphi_R \text{ self-financing, } V_R^{v_{0R}(\xi^*_I(\theta_R),\theta_R),\varphi_R}(t) \geq 0 \ \mathbb{Q}\text{-a.s. } \forall t \in [0, T],\\ &\theta_R \in [0, \theta^{\max}] \text{ and } \mathbb{E}[U_R(V_R^{v_{0R}(\xi^*_I(\theta_R),\theta_R),\varphi_R}(T) - \xi^*_I(\theta_R)P(T))^-] < \infty\}.\end{aligned}$$

The procedure of solving the optimization problem $(P_R^{\varphi_R})$ is the following:

1. We fix $\theta_R \in [0, \theta^{\max}]$, find the optimal trading strategy φ^*_R and convert it to π^*_R.
2. We find the optimal safety loading $\theta^*_R \in [0, \theta^{\max}]$.

Let $\theta_R \in [0, \theta^{\max}]$ be fixed. Our aim is to solve the optimization problem

$$\sup_{\varphi_R:(\varphi_R,\theta_R)\in\Lambda_R} \mathbb{E}[U_R(V_R^{v_{0R}(\xi^*_I(\theta_R),\theta_R),\varphi_R}(T) - \xi^*_I(\theta_R)P(T))]. \qquad (P_R^{\theta_R,\varphi})$$

As mentioned at the beginning of this section, we adapt the method introduced in Korn and Trautmann [1999] to solve the optimization problem $(P_R^{\theta_R,\varphi})$. In particular, we will use Theorem 5.1 and adjust it as mentioned in Remark 5.2 in Korn and Trautmann [1999]. Therefore, we define a new wealth process of the reinsurer with investment in the assets S_0, S_1 and S_2, and additionally in the put option P. The wealth process $V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R,\xi)}$ is given by

$$\begin{aligned} dV_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R,\xi)}(t) =&\varphi_{0R}(t)dS_0(t) + \varphi_{1R}(t)dS_1(t) + \varphi_{2R}(t)dS_2(t) + \xi(t)dP(t), \\ V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R,\xi)}(0) =&v_R + \xi_I^*(\theta_R)\theta_R P(0) =: \bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R). \end{aligned} \tag{4.15}$$

Remark. Note that $\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R)$ is not equal to $v_{0R}(\xi_I^*(\theta_R),\theta_R)$:

$$\begin{aligned} \bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R) =&v_R + \xi_I^*(\theta_R)\theta_R P(0) \\ =&v_R + \xi_I^*(\theta_R)(1+\theta_R)P(0) - \xi_I^*(\theta_R)P(0) \\ =&v_{0R}(\xi_I^*(\theta_R),\theta_R) - \xi_I^*(\theta_R)P(0). \end{aligned}$$

Since

$$V_R^{\bar{v}_{R0},(\varphi_R,\xi)}(T) = \varphi_{0R}(T)S_0(T) + \varphi_{1R}(T)S_1(T) + \varphi_{2R}(T)S_2(T) + \xi(T)P(T)$$

and the reinsurer has a short put position $-\xi_I^*(\theta_R)$, the optimization problem $(P_R^{\theta_R,\varphi})$ is equivalent to the optimization problem given by

$$\begin{aligned} &\sup_{\varphi_R\in\Lambda_R^{\theta_R,(\varphi_R,\xi)}} \mathbb{E}[U_R(V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R,\xi)}(T))] \\ &\text{s.t. } \xi(t) \equiv -\xi_I^*(\theta_R) \text{ for all } t\in[0,T]. \end{aligned} \tag{$P_R^{\theta_R,(\varphi_R,\xi)}$}$$

$\Lambda_R^{\theta_R,(\varphi_R,\xi)}$ is the set of all admissible strategies φ_R to the optimization problem $(P_R^{\theta_R,(\varphi_R,\xi)})$, i.e.,

$$\begin{aligned} \Lambda_R^{\theta_R,(\varphi_R,\xi)} := \{\varphi_R \text{ self-financing} |\ & V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R,\xi)}(t) \geq 0 \ \mathbb{Q}\text{-a.s. } \forall t\in[0,T] \\ &\text{and } \mathbb{E}[U_R(V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R,\xi)}(T))^-] < \infty\}. \end{aligned}$$

For the following proposition, we need to introduce the portfolio optimization problem of the reinsurer (see Section 1.4.1). The portfolio optimization problem of the reinsurer is given by

$$\sup_{\zeta_R\in\Lambda_R^S} \mathbb{E}[U_R(V_R^{\bar{v}_{R0}(\xi_I^*(\theta_R),\theta_R),\zeta_R}(T))] \tag{P_R^S}$$

where $V_R^{\bar{v}_{R0}(\xi_I^*(\theta_R),\theta_R),\zeta_R}$ is the wealth process of the reinsurer with respect to the trading strategy ζ_R and with initial wealth $\bar{v}_{R0}(\xi_I^*(\theta_R),\theta_R)$ instead of $v_{R0}(\xi_I^*(\theta_R),\theta_R)$. Λ_R^S is the set of all admissible strategies in the portfolio optimization problem (P_R^S):

$$\begin{aligned}\Lambda_R^S := \{\zeta_R \text{ self-financing}|\ & V^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),\zeta_R}(t) \geq 0\ \forall t \in [0,T] \\ & \text{and } \mathbb{E}[U_R(V^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),\zeta_R}(T))^-] < \infty\}.\end{aligned}$$

Proposition 4.6 (Optimal solution to $(P_R^{\varphi_R})$)**.**

(a) There exists an optimal trading strategy φ_R^* to the optimization problem $(P_R^{\theta_R,(\varphi_R,\xi)})$. The optimal terminal wealth in the optimization problem $(P_R^{\theta_R,(\varphi_R,\xi)})$ is given by

$$V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(T) = I_R(y_R^*(\theta_R)\tilde{Z}(T))$$

where $y_R^* \equiv y_R^*(\theta_R)$ is the Lagrange multiplier determined by

$$\mathbb{E}[\tilde{Z}(t)I_R(y_R^*\tilde{Z}(T))] = v_R + \xi_I^*(\theta_R)\theta_R P(0).$$

I_R is the inverse function of U_R'.

(b) Let ψ be the replicating strategy given by (4.1) and ζ_R^* the optimal trading strategy of the portfolio optimization problem (P_R^S). Then, the optimal trading strategy φ_R^* to the optimization problem $(P_R^{\theta_R,(\varphi_R,\xi)})$ (and therefore the optimal trading strategy to the optimization problem $(P_R^{\varphi_R})$) given the constraint $\xi(t) \equiv -\xi_I^*(\theta_R)$ in the put option P is given by

$$\begin{aligned}\varphi_{0R}^*(t) &= \frac{V^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(t) - \sum_{i=1}^2 \varphi_i^*(t)S_i(t) + \xi_I^*(\theta_R)P(t)}{S_0(t)} \\ \varphi_{1R}^*(t) &= \zeta_{1R}^*(t) \\ \varphi_{2R}^*(t) &= \zeta_{2R}^*(t) + \psi_2(t)\xi_I^*(\theta_R).\end{aligned}$$

Remark. Compared to the solution method of the optimization problem of the insurer in Section 4.4.1, we do not consider random utility functions in the case of the optimization problem of the reinsurer. Hence, the utility function U_R and the marginal inverse function I_R are not random.

Proof. The proof is based on the proof of Theorem 4.1 in Korn and Trautmann [1999]. For the dynamics of the put option it holds (4.2). Hence, for the wealth process $V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R,\xi)}$ it follows that:

$$V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R,\xi)}(t) = \varphi_{0R}(t)S_0(t) + \varphi_{1R}(t)S_1(t) + \varphi_{2R}(t)S_2(t) + \xi(t)P(t)$$

$$
\begin{aligned}
&\overset{(*)}{=}\varphi_{0R}(t)S_0(t)+\varphi_{1R}(t)S_1(t)+\varphi_{2R}(t)S_2(t)\\
&\quad+\xi(t)\psi_0(t)S_0(t)+\xi(t)\psi_2(t)S_2(t)\\
&=(\varphi_{0R}(t)+\xi(t)\psi_0(t))S_0(t)+\varphi_{1R}(t)S_1(t)+(\varphi_{2R}(t)+\xi(t)\psi_2(t))S_2(t)\\
&=:\zeta_{0R}(t)S_0(t)+\zeta_{1R}(t)S_1(t)+\zeta_{2R}(t)S_2(t),
\end{aligned}
$$

where $(*)$ follows from the Equation (4.2) and

$$
\begin{aligned}
\zeta_R(t)&=(\zeta_{0R}(t),\zeta_{1R}(t),\zeta_{2R}(t))^\top\\
&:=(\varphi_{0R}(t)+\xi(t)\psi_0(t),\varphi_{1R}(t),\varphi_{2R}(t)+\xi(t)\psi_2(t))^\top
\end{aligned}
\tag{4.16}
$$

is a self-financing trading strategy. Hence, the dynamics are given by

$$
dV_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R,\xi)}(t)=\zeta_{0R}(t)dS_0(t)+\zeta_{1R}(t)dS_1(t)+\zeta_{2R}(t)dS_2(t).
$$

The wealth process $V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R,\xi)}$ equals the wealth process $V^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),\zeta_R}$ defined in (1.11). If the trading strategy φ_R is admissible for the optimization problem $(P_R^{\theta_R,(\varphi_R,\xi)})$ (i.e., $\varphi_R\in\Lambda_R^{\theta_R,(\varphi_R,\xi)}$), then the trading strategy ζ_R is admissible to the portfolio optimization problem (P_R^S):

$$
V^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),\zeta_R}(t)=V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R,\xi)}(t)\geq 0\ \forall t\in[0,T]
$$

and

$$
\mathbb{E}[U_R(V^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),\zeta_R}(T))^-]=\mathbb{E}[U_R(V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R,\xi)}(T))^-]<\infty.
$$

Hence, we are in the case of the portfolio optimization problem (P_R^S). By Theorem 1.37, there exists an optimal trading strategy ζ_R^* to the portfolio optimization problem (P_R^S) and the optimal terminal wealth $V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),\zeta_R}$ is given by

$$
V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),\zeta_R}(T)=I_R(y_R^*(\theta_R)\tilde{Z}(T)),
$$

where $y_R^*\equiv y_R^*(\theta_R)$ is determined by the budget constraint

$$
\mathbb{E}[\tilde{Z}(T)I_R(y_R^*\tilde{Z}(T))]=v_R+\xi_I^*(\theta_R)\theta_R P(0).
$$

Therefore, there exists an optimal trading strategy φ_R^* for the optimization problem $(P_R^{\theta_R,(\varphi_R,\xi)})$ and the optimal wealth process $V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}$ is given by

$$
\begin{aligned}
V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(T)&=V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),\zeta_R}(T)\\
&=I_R(y_R^*(\theta_R)\tilde{Z}(T)).
\end{aligned}
$$

By (4.16), we get for the optimal trading strategy φ_R^* the following representation:

$$
\begin{aligned}
\varphi_{1R}^*(t) =& \zeta_{1R}^*(t) \\
\varphi_{2R}^*(t) =& \zeta_{2R}^*(t) - \xi(t)\psi_2(t) \\
=& \zeta_{2R}^*(t) + \xi_I^*(\theta_R)\psi_2(t) \\
\varphi_{0R}^*(t) =& \zeta_{0R}^*(t) - \xi(t)\psi_0(t) \\
\overset{(*)}{=}& \frac{V^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(t) - \sum_{i=1}^2 \zeta_{iR}^*(t)S_i(t)}{S_0(t)} \\
& - \xi(t)\frac{P(t) - \psi_2(t)S_2(t)}{S_0(t)} \\
=& \frac{V^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(t) - \sum_{i=1}^2 \varphi_{iR}^*(t)S_i(t)}{S_0(t)} \\
& - \frac{\xi(t)\psi_2(t)S_2(t) + \xi(t)(P(t) - \psi_2(t)S_2(t))}{S_0(t)} \\
=& \frac{V^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(t) - \sum_{i=1}^2 \varphi_{iR}^*(t)S_i(t) + \xi_I^*(\theta_R)P(t)}{S_0(t)},
\end{aligned}
$$

where $(*)$ follows from Theorem 1.37. $\square$

By Proposition 4.6 we have an explicit solution for the optimal terminal value $V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(T)$ and the optimal trading strategy φ_R^* in the optimization problem $(P_R^{\varphi_R})$, i.e.,

$$
V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(T) = V_R^{v_{0R}(\xi_I^*(\theta_R),\theta_R),\varphi_R^*}(T) - \xi_I^*(\theta_R)P(T).
$$

In the next step, our aim is to find the optimal safety loading θ_R^* of the reinsurer in the optimization problem (P_R).

Proposition 4.7 (Optimal safety loading)**.** Let $\varphi_R^*(\cdot|\theta_R)$ be the optimal trading strategy in the optimization problem $(P_R^{\varphi_R})$ for $\theta_R \in [0, \theta^{\max}]$. Then, the optimal safety loading θ_R^* of the reinsurer is given by

$$
\theta_R^* = \underset{\theta_R \in [0,\theta^{\max}]}{\arg\max}\, \mathbb{E}[U_R(V_R^{v_{0R}(\xi_I^*(\theta_R),\theta_R),\varphi_R^*}(T) - \xi_I^*(\theta_R)P(T))].
$$

Proof. We define the function $\kappa : [0, \theta^{\max}] \to \mathbb{R}$ by

$$
\begin{aligned}
\kappa(\theta_R) :=& \mathbb{E}[U_R(V_R^{v_{0R}(\xi_I^*(\theta_R),\theta_R),\varphi_R^*}(T) - \xi_I^*(\theta_R)P(T))] \\
=& \mathbb{E}[U_R(I_R(y_R^*(\theta_R)\tilde{Z}(T)))],
\end{aligned}
$$

where $y_R^* \equiv y_R^*(\theta_R)$ is the Lagrange multiplier determined by the budget constraint

$$\mathbb{E}[\tilde{Z}(t) I_R(y_R^* \tilde{Z}(T))] = v_{0R} + \xi_I^*(\theta_R)\theta_R P(0).$$

We show that the map $\theta_R \mapsto \kappa(\theta_R)$ is continuous. By Section 4.4.1, the map $\xi_I^*(\cdot)$ is given by

$$\xi_I^*(\theta) = \underset{\xi \in [0, \xi^{\max}(\theta)]}{\arg\max} \ \nu(\xi, \theta)$$

with $\xi^{\max}(\theta) = \min\{\bar{\xi}, \frac{v_I}{(1+\theta)P(0)}\}$ and

$$\nu(\xi, \theta) := \mathbb{E}[U_I(\max\{I_I(y_I^*(\xi, \theta)\tilde{Z}_{\lambda^*}(T)), \xi P(T)\})],$$

where the Lagrange multiplier $y_I^* \equiv y_I^*(\xi, \theta)$ is given by the budget constraint of the insurer

$$\mathbb{E}[\tilde{Z}_{\lambda^*}(T) \hat{I}_I(y_I^* \tilde{Z}_{\lambda^*}(T))] = v_I - \xi(1 + \theta_R)P(0).$$

The Lagrange multiplier $y_I^*(\xi, \theta)$ is continuous with respect to ξ and θ, since $\hat{I}_I$ is a continuous function and $v_I - \xi(1+\theta_R)P(0)$ is continuous with respect to ξ and θ. Furthermore, we have that the functions I_I, max and U_I are continuous. Therefore, the function ν is continuous with respect to ξ and θ. In addition ν is strictly concave with respect to ξ (by Lemma A.3 in Desmettre and Seifried [2016]) if U_I is strictly concave. Since U_I is a utility function, it is strictly concave. Furthermore, the map $\theta \mapsto \xi^{\max}(\theta)$ is continuous. By Berges Maximum Theorem (Theorem 1.12 in Section 1.1) we have that the map $\theta \mapsto \xi_I^*(\theta)$ is continuous.

Next, we argue that $\theta \mapsto \kappa(\theta)$ is continuous. The Lagrange multiplier $y_R^*(\theta)$ is continuous, since I_R is a continuous function and $v_{0R} + \xi_I^*(\theta)\theta P(0)$ is continuous with respect to θ. Furthermore, we have that the functions I_R and U_R are continuous. Therefore, the function κ is continuous with respect to θ.

Since $[0, \theta^{\max}]$ is compact, it follows from the Weierstrass Theorem (Theorem 1.11 in Section 1.1) that there exists θ_R^* such that

$$\theta_R^* = \underset{\theta_R \in [0, \theta^{\max}]}{\arg\max} \ \kappa(\theta_R).$$

□

4.4.3 Stackelberg Game

Proposition 4.8 (Stackelberg equilibrium)**.** The Stackelberg equilibrium of the Stackelberg game (SG) is given by $(\pi_R^*(\cdot|\theta_R^*), \theta_R^*, \pi_I^*(\cdot|\theta_R^*), \xi_I^*(\theta_R^*))$, where

- $\pi_R^*(\cdot|\theta_R^*)$ is given by

$$\pi_{iR}^*(t|\theta_R^*) = \varphi_{iR}^*(t|\theta_R^*) \frac{S_i(t)}{V_R^{v_{0R}(\xi_I^*(\theta_R^*), \theta_R^*), \varphi_R^*}(t)}$$

  where φ_R^* is given by Proposition 4.6,

- θ_R^* is given by Proposition 4.7, and

- $(\pi_I^*(\cdot|\theta_R^*), \xi_I^*(\theta_R^*))$ are given by Proposition 4.4 and 4.5.

4.5 Example: Power utility function

4.5.1 Optimization Problem of the Insurer (Follower)

We fix $\theta_R \in [0, \theta^{\max}]$. The utility function U_I of the insurer is given by a power utility function, i.e.

$$U_I(x) := \frac{1}{b_I} x^{b_I} \tag{4.17}$$

with $b_I \in (-\infty, 1)\backslash\{0\}$. Then we have

$$U_I'(x) = x^{b_I - 1} \text{ and } I_I(y) = y^{\frac{1}{b_I - 1}},$$

where $b_I - 1 < 0$. For $\xi_I \in [0, \xi^{\max}]$, the random utility function $\hat{U}_I$ is given by

$$\hat{U}_I(x) := U_I(x + \xi_I P(T)) = \frac{1}{b_I}(x + \xi_I P(T))^{b_I}.$$

Therefore, we have for $x \in [0, \infty)$ and $y \in (0, \hat{U}_I'(0)]$

$$\hat{U}_I'(x) = U_I'(x + \xi_I P(T)) = (x + \xi_I P(T))^{b_I - 1} \text{ and } \hat{I}_I(y) = y^{\frac{1}{b_I - 1}} - \xi_I P(T). \tag{4.18}$$

First, we calculate the optimal wealth process V_λ^* to the optimization problem (P_I^λ) for a given $\xi_I \in [0, \xi^{\max}]$ and $\lambda \in \mathcal{D}$.
If $\xi_I = \frac{v_I}{(1+\theta_R)P(0)}$, then the insurer invests all his initial wealth in the reinsurance (i.e.,

$v_{I0}(\xi_I, \theta_R) = 0$) and therefore, the optimal wealth process V_λ^* is given by

$$V_\lambda^*(t) = 0$$

for all $t \in [0, T]$. Hence, the optimal terminal surplus is given by $\xi_I P(T)$.
If $\xi_I \in [0, \xi^{\max}]$ with $\xi_I \neq \frac{v_I}{(1+\theta_R)P(0)}$, then the insurer invests only a part of his wealth in the reinsurance. Therefore, we need to calculate the optimal terminal wealth of the insurer to the optimization problem (P_I^λ).
First, we find the optimal Lagrange multiplier y_I^* from the budget constraint

$$\begin{aligned}
\mathbb{E}[\tilde{Z}_\lambda(T)(y_I^* \tilde{Z}_\lambda(T))^{\frac{1}{b_I-1}} - \xi_I \tilde{Z}_\lambda(T)P(T)] &= v_I - \xi_I(1+\theta_R)P(0) \\
&\Leftrightarrow \\
(y_I^*)^{\frac{1}{b_I-1}} \mathbb{E}[\tilde{Z}_\lambda(T)^{\frac{b_I}{b_I-1}}] &= v_I - \xi_I(1+\theta_R)P_\lambda(0) + \xi_I \mathbb{E}[\tilde{Z}_\lambda(T)P_\lambda(T)] \\
&\Leftrightarrow \\
(y_I^*)^{\frac{1}{b_I-1}} &= \frac{v_I - \xi_I(1+\theta_R)P_\lambda(0) + \xi_I \mathbb{E}[\tilde{Z}_\lambda(T)P_\lambda(T)]}{\mathbb{E}[\tilde{Z}_\lambda(T)^{\frac{b_I}{b_I-1}}]} \\
&\Leftrightarrow \\
y_I^* &= \left(\frac{v_I - \xi_I(1+\theta_R)P_\lambda(0) + \xi_I \mathbb{E}[\tilde{Z}_\lambda(T)P_\lambda(T)]}{\mathbb{E}[\tilde{Z}_\lambda(T)^{\frac{b_I}{b_I-1}}]} \right)^{b_I-1}.
\end{aligned} \tag{4.19}$$

Reminder: $P_\lambda(0) \neq \mathbb{E}[\tilde{Z}_\lambda(T)P_\lambda(T)]$, i.e., the price of the fixed-term security is not equal to the price of a put option with payoff $P_\lambda(T)$ in the auxiliary market.
Since y_I^* depends on ξ_I, we will write $y_I^*(\xi_I)$ instead of y_I^* from now on. By Proposition 4.4, the optimal terminal wealth to the optimization problem (P_I^λ) is given by

$$\begin{aligned}
V_\lambda^*(T) =& \hat{I}(y_I^*(\xi_I)\tilde{Z}_\lambda(T)) \\
\overset{(4.18)}{=}& (y_I^*(\xi_I)\tilde{Z}_\lambda(T))^{\frac{1}{b_I-1}} - \xi_I P_\lambda(T) \\
\overset{(4.19)}{=}& \left(\left(\frac{v_I - \xi_I(1+\theta_R)P_\lambda(0) + \xi_I \mathbb{E}[\tilde{Z}_\lambda(T)P_\lambda(T)]}{\mathbb{E}[\tilde{Z}_\lambda(T)^{\frac{b_I}{b_I-1}}]} \right)^{b_I-1} \tilde{Z}_\lambda(T) \right)^{\frac{1}{b_I-1}} - \xi_I P_\lambda(T) \\
=& \frac{v_I - \xi_I(1+\theta_R)P_\lambda(0) + \xi_I \mathbb{E}[\tilde{Z}_\lambda(T)P_\lambda(T)]}{\mathbb{E}[\tilde{Z}_\lambda(T)^{\frac{b}{b-1}}]} \tilde{Z}_\lambda(T)^{\frac{1}{b-1}} - \xi_I P_\lambda(T).
\end{aligned}$$

Hence, the wealth process is given by

$$V_\lambda^*(t) = \frac{1}{\tilde{Z}_\lambda(t)} \mathbb{E}[\tilde{Z}_\lambda(T)V_\lambda^*(T)|\mathcal{F}_t]$$

$$
\begin{aligned}
=&\frac{1}{\tilde{Z}_\lambda(t)}\mathbb{E}\left[\tilde{Z}_\lambda(T)\left(\frac{v_I-\xi_I(1+\theta_R)P(0)+\xi_I\mathbb{E}[\tilde{Z}_\lambda(T)P(T)]}{\mathbb{E}[\tilde{Z}_\lambda(T)^{\frac{b_I}{b_I-1}}]}\tilde{Z}_\lambda(T)^{\frac{1}{b_I-1}}-\xi_I P(T)\right)\middle|\mathcal{F}_t\right]\\
=&\frac{1}{\tilde{Z}_\lambda(t)}\left(\frac{v_I-\xi_I(1+\theta_R)P(0)+\xi_I\mathbb{E}[\tilde{Z}_\lambda(T)P(T)]}{\mathbb{E}[\tilde{Z}_\lambda(T)^{\frac{b_I}{b_I-1}}]}\mathbb{E}[\tilde{Z}_\lambda(T)^{\frac{b_I}{b_I-1}}|\mathcal{F}_t]-\xi_I\mathbb{E}[\tilde{Z}_\lambda(T)P(T)|\mathcal{F}_t]\right).
\end{aligned}
$$

By Proposition 4.4, the optimal portfolio process π^*_λ is given by

$$
\begin{aligned}
\pi^*_\lambda(t)=&\pi^M_\lambda(t)\frac{V^*_\lambda(t)+\xi_I\tilde{Z}_\lambda(t)^{-1}\mathbb{E}[\tilde{Z}_\lambda(T)P(T)|\mathcal{F}_t]}{V^*_\lambda(t)}\\
=&\pi^M_\lambda(t)\frac{\mathbb{E}[\tilde{Z}_\lambda(T)^{\frac{b_I}{b_I-1}}|\mathcal{F}_t](v_I-\xi_I(1+\theta_R)P(0)+\xi_I\mathbb{E}[\tilde{Z}_\lambda(T)P(T)])}{V^*_\lambda(t)\tilde{Z}_\lambda(t)\mathbb{E}[\tilde{Z}_\lambda(T)^{\frac{b_I}{b_I-1}}]},
\end{aligned}
$$

where π^M_λ is Merton's relative portfolio process given by

$$
\pi^M_\lambda(t)=\frac{1}{1-b_I}(\sigma\sigma^\top)^{-1}(\mu+\lambda(t)-r\mathbb{1}).
$$

Next, we calculate the optimal $\lambda^*\in\mathcal{D}$ such that $\pi^*_{\lambda^*}(t)\in K$ $\mathbb{Q}$-a.s. for all $t\in[0,T]$, as then $\pi^*_{\lambda^*}(t)^\top\lambda^*(t)=0$ $\mathbb{Q}$-a.s. for all $t\in[0,T]$. Since π^*_λ is given by π^M_λ multiplied by a random variable bigger than zero, it is sufficient to show that $\pi^M_{\lambda^*}(t)\in K$ $\mathbb{Q}$-a.s. for all $t\in[0,T]$. Let $\lambda^*\in\mathcal{D}$, i.e., $\lambda^*_1(t)=0$ for a.e. $t\in[0,T]$. Then, for $t\in[0,T]$

$$
\begin{aligned}
&\pi^M_{\lambda^*}(t)\in K\\
&\Leftrightarrow\\
\frac{1}{1-b_I}(\sigma\sigma^\top)^{-1}&\begin{pmatrix}\mu_1-r\\ \mu_2+\lambda^*_2(t)-r\end{pmatrix}\in K. \qquad (4.20)
\end{aligned}
$$

Since

$$
\begin{aligned}
\sigma\sigma^\top&=\begin{pmatrix}\sigma_1 & 0\\ \sigma_2\rho & \sigma_2\sqrt{1-\rho^2}\end{pmatrix}\cdot\begin{pmatrix}\sigma_1 & \sigma_2\rho\\ 0 & \sigma_2\sqrt{1-\rho^2}\end{pmatrix}\\
&=\begin{pmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho\\ \sigma_1\sigma_2\rho & \sigma_2^2\end{pmatrix},
\end{aligned}
$$

we have by Theorem 1.2 (since $\sigma_1^2\sigma_2^2-(\sigma_1\sigma_2\rho)^2=\sigma_1^2\sigma_2^2(1-\rho^2)$)

$$
(\sigma\sigma^\top)^{-1}=\frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\begin{pmatrix}\sigma_2^2 & -\sigma_1\sigma_2\rho\\ -\sigma_1\sigma_2\rho & \sigma_1^2\end{pmatrix},
$$

and Equation (4.20) is equivalent to the linear equation

$$\begin{aligned}
0 &= \frac{1}{(1-b_I)\sigma_1^2\sigma_2^2(1-\rho^2)}(-\sigma_1\sigma_2\rho(\mu_1-r)+\sigma_1^2(\mu_2+\lambda_2^*(t)-r)) \\
&\Leftrightarrow \\
\sigma_1\sigma_2\rho(\mu_1-r) &= \sigma_1^2(\mu_2+\lambda_2^*(t)-r) \\
&\Leftrightarrow \\
\lambda_2^*(t) \equiv \lambda_2^* &= \frac{\sigma_2\rho}{\sigma_1}(\mu_1-r)-\mu_2+r.
\end{aligned}$$

Therefore, the optimal $\lambda^* \in \mathcal{D}$ is given by

$$\lambda^*(t) \equiv \lambda^* = \begin{pmatrix} 0 \\ \frac{\sigma_2\rho}{\sigma_1}(\mu_1-r)-\mu_2+r \end{pmatrix}.$$

Remark. This is the same λ^* as if we use the minimization criterion in Example 15.1 in Cvitanić and Karatzas [1992], i.e.

$$\lambda^* = \arg\min_{x\in\tilde{K}} \|\gamma+\sigma^{-1}x\|^2.$$

But this is a coincidence, since the result in Cvitanić and Karatzas [1992] is not for random utility functions.

It remains to determine the optimal reinsurance strategy ξ_I^*. By Proposition 4.4, we have

$$\xi_I^* = \arg\max_{\xi_I\in[0,\xi^{\max}]} \nu(\xi_I),$$

where the function ν is given by

$$\nu(\xi_I) := \mathbb{E}\left[\frac{1}{b_I}\max\{(y_I^*(\xi_I)\tilde{Z}_{\lambda^*}(T))^{\frac{1}{b_I-1}}, \xi_I P(T)\}^{b_I}\right].$$

We will solve this later numerically.

Remark. If we choose $\xi^{\max} = \bar{\xi}$ with $\bar{\xi} < \frac{v_I}{(1+\theta_R)P(0)}$ for all $\theta_R \in [0,\theta^{\max}]$, then the insurer always invests a part of his wealth in the reinsurance. From Equation (4.7), it follows for the function ν

$$\begin{aligned}
\nu(\xi_I) =& \mathbb{E}\left[\frac{1}{b_I}\max\{(y_I^*(\xi_I)\tilde{Z}_{\lambda^*}(T))^{\frac{1}{b_I-1}}, \xi_I P(T)\}^{b_I}\right] \\
\overset{(4.7)}{=}& \mathbb{E}\left[\frac{1}{b_I}(y_I^*(\xi_I)\tilde{Z}_{\lambda^*}(T))^{\frac{b_I}{b_I-1}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{b_I} y_I^*(\xi_I)^{\frac{b_I}{b_I-1}} \mathbb{E}\Big[\tilde{Z}_{\lambda^*}(T)^{\frac{b_I}{b_I-1}}\Big] \\
&\overset{\text{def. } y_I^*(\xi_I)}{=} \frac{1}{b_I}(v_I - \xi_I(1+\theta_R)P(0) + \xi_I \mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)])^{b_I} \mathbb{E}\Big[\tilde{Z}_{\lambda^*}(T)^{\frac{b_I}{b_I-1}}\Big]^{1-b_I}.
\end{aligned}
$$

It follows for the optimal reinsurance strategy ξ_I^*

$$
\begin{aligned}
\xi_I^* &= \underset{\xi_I \in [0,\bar{\xi}]}{\arg\max} \left(\frac{1}{b_I}(v_I - \xi_I(1+\theta_R)P(0) + \xi_I \mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)])^{b_I} \mathbb{E}\Big[\tilde{Z}_{\lambda^*}(T)^{\frac{b_I}{b_I-1}}\Big]^{1-b_I} \right) \\
&= \begin{cases} \bar{\xi}, & \text{if } -(1+\theta_R)P(0) + \mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)] \geq 0, \\ 0, & \text{if } -(1+\theta_R)P(0) + \mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)] < 0 \end{cases} \\
&= \begin{cases} \bar{\xi}, & \text{if } \frac{\mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)] - P(0)}{P(0)} \geq \theta_R, \\ 0, & \text{if } \frac{\mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)] - P(0)}{P(0)} < \theta_R. \end{cases}
\end{aligned} \tag{4.21}
$$

Summary:

- The optimal $\lambda^* \in \mathcal{D}$ is given by

$$
\lambda^* = (0, \frac{\sigma_2 \rho}{\sigma_1}(\mu_1 - r) - \mu_2 + r)^\top.
$$

- The Lagrange multiplier $y^*(\xi_I)$ is given by

$$
y_I^*(\xi_I) = \left(\frac{v_I - \xi_I(1+\theta_R)P(0) + \xi_I \mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)]}{\mathbb{E}\Big[\tilde{Z}_{\lambda^*}(T)^{\frac{b_I}{b_I-1}}\Big]} \right)^{b_I-1}.
$$

- The optimal reinsurance strategy ξ_I^* is given by

$$
\xi_I^* = \underset{\xi_I \in [0,\xi^{\max}]}{\arg\max}\ \nu(\xi_I),
$$

  where the function ν is given by

$$
\nu(\xi_I) := \mathbb{E}\left[\frac{1}{b_I} \max\{(y_I^*(\xi_I)\tilde{Z}_{\lambda^*}(T))^{\frac{1}{b_I-1}}, \xi_I P(T)\}^{b_I} \right]. \tag{4.22}
$$

- The optimal terminal surplus is given by

$$
\begin{aligned}
V_I^*(T) + \xi_I^* P(T) &= \max\{(y_I^*(\xi_I^*)\tilde{Z}_{\lambda^*}(T))^{\frac{1}{b_I-1}}, \xi_I^* P(T)\} \\
&= \max\left\{ \frac{v_I - \xi_I^*(1+\theta_R)P(0) + \xi_I \mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)]}{\mathbb{E}[\tilde{Z}_{\lambda^*}(T)^{\frac{b_I}{b_I-1}}]} \tilde{Z}_{\lambda^*}(T)^{\frac{1}{b_I-1}}, \xi_I^* P(T) \right\},
\end{aligned}
$$

where V_I^* denotes the optimal wealth process of the insurer (4.3).

- The optimal portfolio process is given by

$$\pi_I^*(t) = \pi_{\lambda^*}^M(b_I)\frac{\mathbb{E}[\tilde{Z}_{\lambda^*}(T)^{\frac{b_I}{b_I-1}}|\mathcal{F}_t](v_I - \xi_I^*(1+\theta_R)P(0) + \xi_I^*\mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)])}{V_I^*(t)\tilde{Z}_{\lambda^*}(t)\mathbb{E}[\tilde{Z}_{\lambda^*}(T)^{\frac{b_I}{b_I-1}}]}, \quad (4.23)$$

where

$$\pi_{\lambda^*}^M(b_I) = \frac{1}{1-b_I}(\sigma\sigma^\top)^{-1}(\mu + \lambda^* - r\mathbb{1}).$$

4.5.2 Optimization Problem of the Reinsurer (Leader)

The optimal reinsurance strategy of the insurer is given by $\xi_I^*(\theta_R)$ and the utility function of the reinsurer is given by a power utility function, i.e., for $x \in (0,\infty)$ and $b_R \in (-\infty, 1)\backslash\{0\}$

$$U_R(x) := \frac{1}{b_R}x^{b_R}.$$

Hence, we have for $x, y \in (0,\infty)$

$$U_R'(x) = x^{b_R-1} \text{ and } I_R(y) = y^{\frac{1}{b_R-1}}. \quad (4.24)$$

First, we calculate the Lagrange multiplier y_R^*, which solves the budget constraint

$$\begin{aligned}
\mathbb{E}[\tilde{Z}(T)I_R(y_R^*\tilde{Z}(T))] &= v_R + \xi_I^*(\theta_R)\theta_R P(0)\\
&\overset{(4.24)}{\Leftrightarrow}\\
\mathbb{E}[\tilde{Z}(T)(y_R^*\tilde{Z}(T))^{\frac{1}{b_R-1}}] &= v_R + \xi_I^*(\theta_R)\theta_R P(0)\\
&\Leftrightarrow\\
(y_R^*)^{\frac{1}{b_R-1}}\mathbb{E}[\tilde{Z}(T)^{\frac{b_R}{b_R-1}}] &= v_R + \xi_I^*(\theta_R)\theta_R P(0)\\
&\Leftrightarrow\\
y_R^* &= \left(\frac{v_R + \xi_I^*(\theta_R)\theta_R P(0)}{\mathbb{E}[\tilde{Z}(T)^{\frac{b_R}{b_R-1}}]}\right)^{b_R-1}.
\end{aligned}$$

Since y_R^* depends on θ_R, we will write $y_R^*(\theta_R)$ instead of y_R^* from now on. By Proposition 4.6, the optimal terminal wealth of the optimization problem $(P_R^{\theta_R,(\varphi_R,\xi)})$ is given by

$$V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(T) = I_R(y_R^*(\theta_R)\tilde{Z}(T))$$

$$
\begin{aligned}
&=(y_R^*(\theta_R)\tilde{Z}(T))^{\frac{1}{b_R-1}} \\
&\overset{\text{Lemma } A.1}{=}(v_R+\xi_I^*(\theta_R)\theta_R P(0))\tilde{Z}(T)^{\frac{1}{b_R-1}} \\
&\quad\times \exp\left(\left[r\frac{b_R}{b_R-1}-\frac{1}{2}\|\gamma\|^2\frac{b_R}{(b_R-1)^2}\right]T\right).
\end{aligned}
$$

Hence, the optimal wealth process is given by

$$
\begin{aligned}
V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(t) =&\tilde{Z}(t)^{-1}\mathbb{E}[\tilde{Z}(T)V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(T)|\mathcal{F}_t] \\
=&(v_R+\xi_I^*(\theta_R)\theta_R P(0)) \\
&\times \exp\left(\left[r\frac{b_R}{b_R-1}-\frac{1}{2}\|\gamma\|^2\frac{b_R}{(b_R-1)^2}\right]T\right) \\
&\times \tilde{Z}(t)^{-1}\mathbb{E}[\tilde{Z}(T)^{\frac{b_R}{b_R-1}}|\mathcal{F}_t] \\
\overset{(*)}{=}&(v_R+\xi_I^*(\theta_R)(1+\theta_R)P(0)) \\
&\times \exp\left(\left[r\frac{b_R}{b_R-1}-\frac{1}{2}\|\gamma\|^2\frac{b_R}{(b_R-1)^2}\right]T\right) \\
&\times \exp\left(-\left(r\frac{b_R}{b_R-1}-\frac{1}{2}\|\gamma\|^2\frac{b_R}{(b_R-1)^2}\right)(T-t)\right) \\
&\times \tilde{Z}(t)^{-1}\tilde{Z}(t)^{\frac{b_R}{b_R-1}} \\
=&(v_R+\xi_I^*(\theta_R)(1+\theta_R)P(0)) \\
&\times \exp\left(\left(r\frac{b_R}{b_R-1}-\frac{1}{2}\|\gamma\|^2\frac{b_R}{(b_R-1)^2}\right)t\right)\tilde{Z}(t)^{\frac{1}{b_R-1}},
\end{aligned}
$$

where $(*)$ follows by Lemma A.1 with $k=\frac{b_R}{b_R-1}$ and $\lambda=0$. Since the optimal wealth process $V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}$ has the form

$$
\begin{aligned}
V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(t) =&(v_R+\xi_I^*(\theta_R)(1+\theta_R)P(0)) \\
&\times \exp\left(\left(r\frac{b_R}{b_R-1}-\frac{1}{2}\|\gamma\|^2\frac{b_R}{(b_R-1)^2}\right)t\right)\tilde{Z}(t)^{\frac{1}{b_R-1}} \\
=&(v_R+\xi_I^*(\theta_R)(1+\theta_R)P(0)) \\
&\times \exp\left(\left(r\frac{b_R}{b_R-1}-\frac{1}{2}\|\gamma\|^2\frac{b_R}{(b_R-1)^2}\right)t\right) \\
&\times \exp\left(-\left(r+\frac{1}{2}\|\gamma\|^2\right)\frac{1}{b_R-1}t-\frac{1}{b_R-1}\gamma^\top W(t)\right) \\
=&(v_R+\xi_I^*(\theta_R)(1+\theta_R)P(0)) \\
&\times \exp\left(\left(r-\frac{1}{2}\|\gamma\|^2\left[\frac{1}{b_R-1}+\frac{b_R}{(b_R-1)^2}\right]\right)t-\frac{1}{b_R-1}\gamma^\top W(t)\right) \\
=:&g(t,W_1(t),W_2(t)),
\end{aligned}
$$

where g has all the properties from Theorem 1.38, we can calculate the optimal trading strategy ζ_R^* to the portfolio optimization problem (P_R^S) directly. The gradient of g with respect to W_1 and W_2 is given by

$$\begin{aligned}\nabla_x g(t, W_1(t), W_2(t)) &= -\frac{V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(t)}{b_R - 1}\gamma \\ &= -\frac{V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(t)}{b_R - 1}\sigma^{-1}(\mu - r\mathbb{1}).\end{aligned}$$

By Theorem 1.38, the optimal trading strategy ζ_R^* to the portfolio optimization problem (P_R^S) is given by (for $i = 1, 2$)

$$\zeta_{iR}^*(t) = -\frac{V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(t)}{S_i(t)}\pi_i^M(b_R),$$

where π^M is the Merton portfolio process given by

$$\pi^M(b_R) = \frac{1}{1 - b_R}(\sigma\sigma^\top)^{-1}(\mu - r\mathbb{1}).$$

Hence,

$$\begin{aligned}\zeta_{1R}^*(t) =& \frac{V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(t)}{S_1(t)}\pi_1^M(b_R) \\ \zeta_{2R}^*(t) =& \frac{V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(t)}{S_2(t)}\pi_2^M(b_R).\end{aligned}$$

Hence, by Proposition 4.6, the optimal trading strategy φ_R^* to the optimization problem of the reinsurer (P_R) is given by

$$\begin{aligned}\varphi_{1R}^*(t) =& \zeta_{1R}^*(t) \\ =& \frac{V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(t)}{S_1(t)}\pi_1^M(b_R) \\ \varphi_{2R}^*(t) =& \zeta_{2R}^*(t) + \psi_2(t)\xi_I^*(\theta_R) \\ =& \frac{V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(t)}{S_2(t)}\pi_2^M(b_R) \\ &+ \frac{\pi^{CM}V^{v_I,\pi_B}(t)(\Phi(d_+) - 1)}{S_2(t)}\xi_I^*(\theta_R).\end{aligned}$$

The optimal safety loading strategy θ_R^* is given by

$$\theta_R^* = \underset{\theta_R\in[0,\theta^{\max}]}{\arg\max}\, \mathbb{E}\Big[\frac{1}{b_R}(V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R),\theta_R),(\varphi_R^*,\xi)}(T))^{b_R}\Big].$$

This will be solved numerically.

Remark. If we choose $\xi^{\max} = \bar{\xi}$ with $\bar{\xi} < \frac{v_I}{(1+\theta_R)P(0)}$ for all $\theta_R \in [0, \theta^{\max}]$, then the optimal reinsurance strategy $\xi_I^*(\theta_R)$ is given by (cf. Equation (4.21))

$$\xi_I^*(\theta_R) = \begin{cases} \bar{\xi}, & \text{if } \frac{\mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)]-P(0)}{P(0)} \geq \theta_R, \\ 0, & \text{if } \frac{\mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)]-P(0)}{P(0)} < \theta_R. \end{cases}$$

It follows for the optimal safety loading θ_R^*

$$\begin{aligned} \theta_R^* =& \underset{\theta_R \in [0,\theta^{\max}]}{\arg\max}\, \mathbb{E}\Big[\frac{1}{b_R}(y_R^*(\theta_R)\tilde{Z}(T))^{\frac{b_R}{b_R-1}}\Big] \\ =& \underset{\theta_R \in [0,\theta^{\max}]}{\arg\max}\, \frac{1}{b_R}(v_R + \xi_I^*(\theta_R)\theta_R P(0))^{b_R}\mathbb{E}[\tilde{Z}(T)^{\frac{b_R}{b_R-1}}]^{1-b_R} \\ =& \underset{\theta_R \in [0,\theta^{\max}]}{\arg\max}\, \xi_I^*(\theta_R)\theta_R. \end{aligned}$$

Hence, the reinsurer chooses the largest $\theta_R \in [0, \theta^{\max}]$ such that $\xi_I^*(\theta_R) = \bar{\xi}$, i.e.,

$$\theta_R^* \approx \min\left\{\frac{\mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)] - P(0)}{P(0)}, \theta^{\max}\right\}. \tag{4.25}$$

Summary:

- The Lagrange multiplier $y_R^*(\theta_R)$ is given by

$$y_R^*(\theta_R) = \left(\frac{v_R + \xi_I^*(\theta_R)\theta_R P(0)}{\mathbb{E}[\tilde{Z}(T)^{\frac{b_R}{b_R-1}}]}\right)^{b_R-1}.$$

- The optimal safety loading strategy θ_R^* is given by

$$\theta_R^* = \underset{\theta_R \in [0,\theta^{\max}]}{\arg\max}\, \kappa(\theta_R),$$

  where the function κ is given by

$$\kappa(\theta_R) := \mathbb{E}\Big[\frac{1}{b_R}(y_R^*(\theta_R)\tilde{Z}(T))^{\frac{b_R}{b_R-1}}\Big]. \tag{4.26}$$

- The optimal terminal surplus is given by

$$\begin{aligned} V_R^*(T) - \xi_I^*(\theta_R^*)P(T) =& V_R^{\bar{v}_{0R}(\xi_I^*(\theta_R^*),\theta_R^*),(\varphi_R^*,\xi)}(T) \\ =& (v_R + \xi_I^*(\theta_R^*)\theta_R^* P(0))\tilde{Z}(T)^{\frac{1}{b_R-1}} \end{aligned}$$

$$\times \exp\left(\left[r\frac{b_R}{b_R-1} - \frac{1}{2}\|\gamma\|^2\frac{b_R}{(b_R-1)^2}\right]T\right).$$

- The optimal trading strategy is given by

$$\begin{aligned}
\varphi^*_{1R}(t) =& \frac{V_R^{\bar{v}_{0R}(\xi^*_I(\theta_R),\theta_R),(\varphi^*_R,\xi)}(t)}{S_1(t)}\pi^M_1(b_R) \\
\varphi^*_{2R}(t) =& \frac{V_R^{\bar{v}_{0R}(\xi^*_I(\theta_R),\theta_R),(\varphi^*_R,\xi)}(t)}{S_2(t)}\pi^M_2(b_R) \\
&+ \frac{\pi^{CM}V^{v_I,\pi_B}(t)(\Phi(d_+)-1)}{S_2(t)}\xi^*_I(\theta_R),
\end{aligned}$$

where π^M is the Merton portfolio process given by

$$\pi^M(b_R) = \frac{1}{1-b_R}(\sigma\sigma^\top)^{-1}(\mu - r\mathbb{1}).$$

Hence, the optimal portfolio process is given by

$$\pi^*_R(t) = \pi^M(b_R) + \begin{pmatrix} 0 \\ \frac{\pi^{CM}V^{v_0,\pi_B}(t)(\Phi(d_+)-1)}{V_R^{\bar{v}_{0R}(\xi^*_I(\theta^*_R),\theta^*_R),(\varphi^*_R,\xi)}(t)}\xi^*_I(\theta^*_R)\end{pmatrix}. \tag{4.27}$$

4.5.3 Numerical Analysis

For the numerical analysis, we use the following parameters:

Parameter	**Symbol**	**Values**
Interest rate	r	1.02%
Drift coefficient for S_1	μ_1	17.52%
Drift coefficient for S_2	μ_2	12.37%
Diffusion coefficient for S_1	σ_1	23.66%
Diffusion coefficient for S_2	σ_2	21.98%
Correlation coefficient	ρ	80.12%
Portfolio process CM strategy	π_B	$(0\%, 29.48\%)^\top$
Initial value of S_1	s_1	1
Initial value of S_2	s_2	1
Guarantee	G_T	100
Initial wealth of insure	v_I	100
Initial wealth of reinsurer	v_R	300
Relative risk aversion of insurer	$1 - b_I$	10
Relative risk aversion of reinsurer	$1 - b_R$	10
Time horizon	T	10
Maximal safety loading of reinsurer	$\theta^{\max}$	50%
Maximal amount of reinsurance	$\xi^{\max} = \bar{\xi}$	1.5

Table 4.1: Parameters for the numerical analysis

Parameter Selection

In this part, we discuss the selection of the parameters, which are summarized in Table 4.1. For the majority of parameters, we choose the same values as in Escobar-Anel et al. [2021]. There, the market parameters are calibrated to the German market in the period from January 1, 2003, till June 8, 2020.

The risk-free rate is modeled by the Euro OverNight Index Average (EONIA) daily data which describes the interest rates on overnight unsecured loans between banks by using a weighted average[4]. For calibrating the parameters of S_1, we use the TecDAX daily data and for calibrating the parameters of S_2, we use the DAX daily data. The DAX (Deutscher Aktienindex) is a performance index which represents the 30 largest companies[5] and the TecDAX represents the 30 largest technology companies on the German Stock Exchange[6].

In this way, we model the following situation. The insurer invests in bonds and prefers

[4]cf. `https://www.ecb.europa.eu/explainers/tell-me-more/html/benchmark\_rates\_qa.en.html`

[5]cf. `https://www.dax-indices.com/index-details?isin=DE0008469008`

[6]cf. `https://www.dax-indices.com/index-details?isin=DE0007203275`

the technological sector instead of the overall portfolio in its stock portfolio. One reason could be that the insurer's asset manager has special knowledge in the technology sector and/or believes that the TecDAX has a better performance than the DAX. In contrast, the reinsurer considers the insurer's technology-focused portfolio too risky to be reinsured. Therefore, the reinsurer offers reinsurance only on a mixed portfolio consisting of bonds and DAX.

The relative proportion π^{CM} of the constant mix portfolio is selected such that it equals the optimal initial proportion of the insurer in the risky asset without reinsurance.

In the German Life Insurance Market, the capital guarantee for the representative client is usually less than or equal to 100% of the representative client's initial endowment. For example, ERGO offers the life insurance product ERGO Rente Garantie where the guarantee lies between 80% and 100%. In contrast, Allianz offers a guarantee between 60% and 90%. Hence, we assume that the representative client has a 100% capital guarantee of the initial capital and investigate in the sensitivity analysis G_T varying from 60% to 110% of v_I.

For convenience, we set the insurer's initial wealth v_I to 100. Since the reinsurer is the leader of the Stackelberg game, it is natural to assume that it is a larger company with more initial capital. Therefore, we set v_R to 300. Chen and Shen [2018] select the initial capital of the reinsurer and the insurer in a similar way in their numerical studies.

In Chen and Shen [2018], the authors assume that the reinsurer and insurer have the same risk aversion. In contrast, Bai et al. [2019] assume that the insurer is more risk averse than the reinsurer. For this, we first consider the situation where the reinsurer and the insurer have the same risk aversion. Afterwards, we explore the situations where the parties have different risk aversion. In the base case, we choose $b_I = b_R = -9$, which is consistent with Escobar-Anel et al. [2021].

For the maximal level of safety loading we choose 50%. It was chosen because Chen and Shen [2018] and Chen and Shen [2019] choose an upper bound on the safety loading of the reinsurer with 45%. But Bai et al. [2019] choose a safety loading of the reinsurer with 200%.

We do not allow that the insurer can speculate with the reinsurance by going short or buying to much of it. Since the underlying of the put option is not the portfolio of the insurer but a correlated portfolio, we allow that $\xi^{\max} = \bar{\xi} = 1.5$.

Algorithm for calculation Stackelberg equilibrium:

1. Choose a sequence of θ_R in the interval $[0, \theta^{\max}]$ with $\Delta\theta_R = 0.0001$.

2. Determine for any θ_R the corresponding optimal reinsurance strategy $\xi_I^*(\theta_R)$:

   2a. choose a sequence of ξ_I in the interval $[0, \xi^{\max}]$ with $\Delta\xi_I = 0.01$.

   2b. apply the function ν to each ξ_I using (4.22).

   2c. then $\xi_I^* = \arg\max \nu(\xi_I)$.

3. Apply the function κ to each pair $(\theta_R, \xi_I^*(\theta_R))$ using (4.26).

4. Then $\theta_R^* = \arg\max \kappa(\theta_R)$.

5. Calculate the optimal investment strategies π_I^* and π_R^* using (4.23) and (4.27), respectively.

Remark.

(a) The functions ν and κ are independent of π_I^* and π_R^*. Therefore, we can calculate the reinsurance strategy ξ_I^* and the safety loading θ_R^* first and afterwards the investment strategies π_I^* and π_R^*.

(b) In our case, it holds

$$\begin{aligned}\bar{\xi} &< \frac{v_{0I}}{(1+\theta^{\max})P(0)} (\approx 17.3064) \\ &\le \frac{v_{0I}}{(1+\theta_R)P(0)}\end{aligned}$$

for all $\theta_R \in [0, \theta^{\max}]$. Hence, by Equations (4.21) and (4.25), we have the concrete formulas

$$\begin{aligned}\theta_R^* &= \min\left\{\frac{\mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)] - P(0)}{P(0)}, \theta^{\max}\right\} \\ \xi_I^*(\theta_R^*) &= 1.5.\end{aligned}$$

Therefore, it is not necessary to use the above algorithm.

For detailed information about the Matlab functions, see Appendix B.1.

Stackelberg Equilibrium

The relative portfolio processes at time 0 of the reinsurer and the insurer without reinsurance (i.e., $\xi_I = 0$) are given by

$$\pi_R^*(0) = (33.48\%, -5.38\%)^\top,$$

$$\pi_I^*(0) = (29.48\%, 0\%)^\top.$$

The reinsurer has a short position in the stock S_2 due to speculation. In comparison, the Stackelberg equilibrium is given by

$$\begin{aligned}\theta_R^* &= 20.86\%,\\ \xi_I^*(\theta_R^*) &= 1.5,\\ \pi_R^*(0) &= (33.48\%, -9.41\%)^\top,\\ \pi_I^*(0) &= (31.69\%, 0\%)^\top.\end{aligned}$$

As we see, the short position of the reinsurer in the stock S_2 increases due to the hedge of the short position in the put option. Figure 4.1 shows the dependence of ξ_I^* on $\theta_R \in [0, \theta^{\max}]$. If θ_R increases then $\xi_I^*(\theta_R)$ decreases from the maximal value $\xi^{\max} = 1.5$ to the minimal value 0. But the optimal reinsurance strategy fo the insurer is to buy the maximal amount of put options (i.e., $\xi_I^*(\theta_R^*) = \xi^{\max}$).

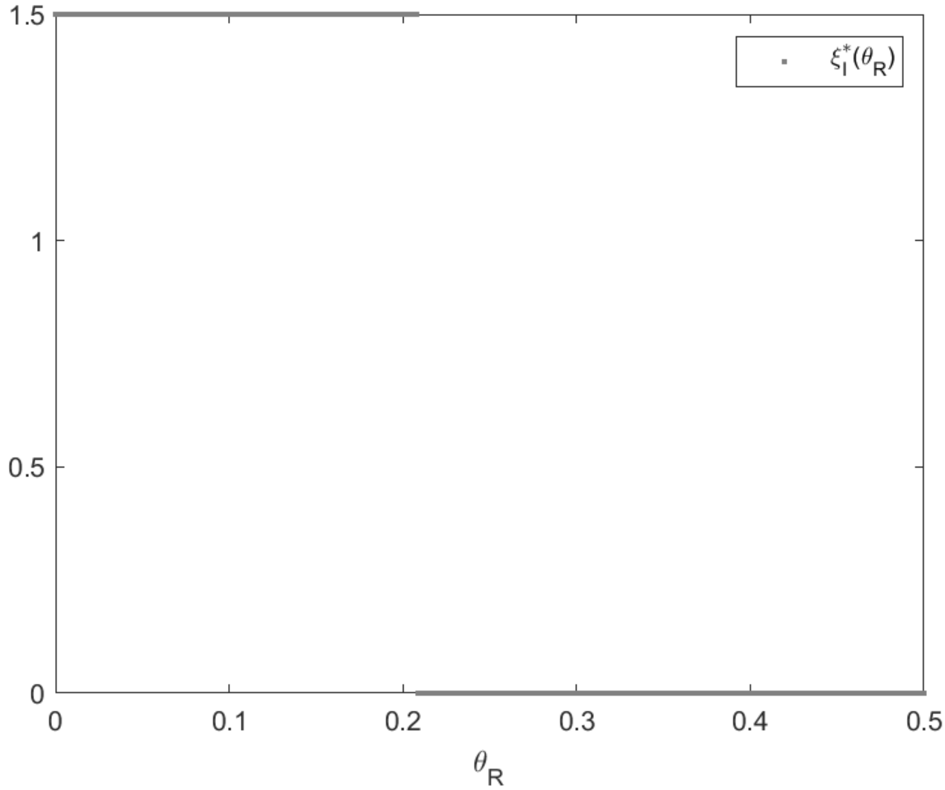

Figure 4.1: Dependence of ξ_I^* on θ_R

This is exactly what we expect due to Equation (4.21) and Equation (4.25), since

$$\frac{\mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)] - P(0)}{P(0)} = 20.86\%.$$

Hence, the reinsurer sets the maximal price for the reinsurance such that the insurer still buys reinsurance.
$\mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)]$ is the fair price of the put option P at time 0 in the optimal axuiliary market $\mathcal{M}_{\lambda^*}$, whereas $P(0)$ is the fair price of the put option P at time 0 in the basic market $\mathcal{M}$. Hence, the optimal safety loading of the reinsurer is the difference between the fair price of the put option P at time 0 in the optimal auxiliary market and in the basic market in relation to the fair price of the put option P at time 0 in the basic market.

Sensitivity Analysis

In the sensitivity analysis, we consider two possible changes in the Stackelberg game. First, we consider a change in the behavior of the leader and the follower, i.e., we vary the relative risk aversion of the reinsurer and the insurer. Next, we take a closer look at the change of the put option price with respect to changes of the interest rate r, the time horizon T, and the capital guarantee G_T.

The relative risk aversion of a decision maker indicates its risk appetite. If the relative risk aversion increases, the decision maker tries to reduce its risk. For the sensitivity analysis we consider $1 - b_R = RRA_R \in \{5, 7.5, 10, 12.5, 15\}$ and $1 - b_I = RRA_I \in \{5, 7.5, 10, 12.5, 15\}$. See Tables B.1 and B.2 in Appendix B.2 for the exact values of the numerical analysis.

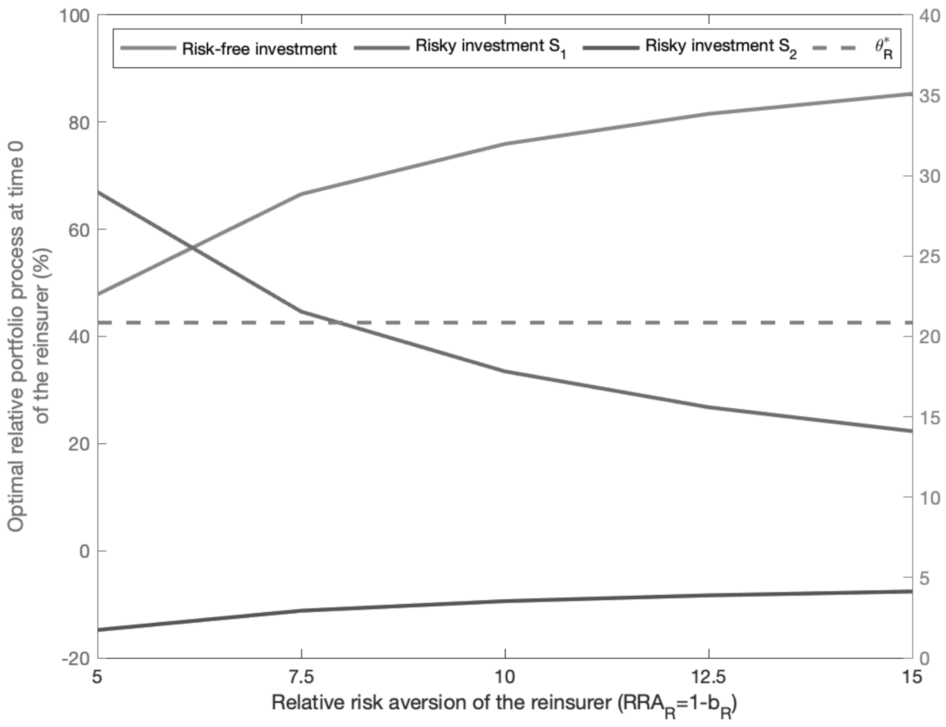

Figure 4.2: Sensitivity of the reinsurer's part of the Stackelberg equilibrium w.r.t. RRA_R

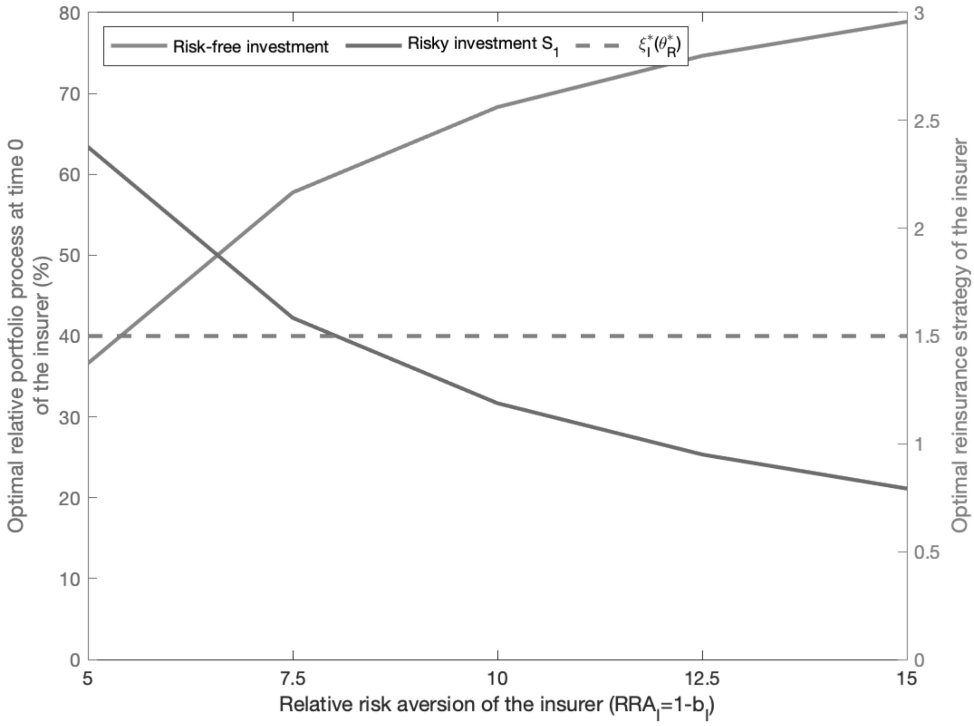

Figure 4.3: Sensitivity of the insurer's part of the Stackelberg equilibrium w.r.t. RRA_I

Surprisingly, a change of the relative risk aversion of the reinsurer and the insurer does not influence the optimal reinsurance strategy $\xi_I^*(\theta_R^*)$ and the optimal safety loading θ_R^*. The insurer always buys the maximal amount of reinsurance, independent of its risk appetite. Since the safety loading of the reinsurer only depends on the put option price, it is independent of the behavior of the reinsurer and the insurer and therefore, it does not change under the relative risk aversion of the reinsurer and the insurer.
In contrast, the optimal relative portfolio processes at time 0 of the reinsurer $\pi_R^*(0)$ and of the insurer $\pi_I^*(0)$ changes under the relative risk aversion, see Figures 4.2 and 4.3. The optimal relative portfolio process of the reinsurer is only influenced by the behavior of the reinsurer and vice versa for the insurer.
If the relative risk aversion of the reinsurer increases, i.e., the reinsurer wants to reduce its risk, then the investment in the risk-free security S_0 increases whereas the investment in the risky asset S_1 decreases and the short position in the risky security S_2 is reduced. Hence, the reinsurer reduces his risk by investing less in the risky asset S_1 and speculates less in the risky asset S_2.
If the relative risk aversion of the insurer increases, the proportion of the investment in the risky asset S_1 decreases. Hence, the insurer reduces its risk by investing less in the risky asset S_1.

In contrast to the parties' relative risk aversion, a change of the interest rate r, of the time horizon T and of the guarantee G_T (being the strike price of the put option) influences the put option price. Hence, we also investigate how the Stackelberg equilibrium changes when the interest rate $r \in \{-2\%, -1\%, 0\%, 1\%, 2\%\}$, the time horizon $T \in \{1, 5, 10, 15, 20\}$, and the guarantee $G_T \in \{0.6\% \cdot v_I, 0.7\% \cdot v_I, 0.8\% \cdot v_I, 0.9\% \cdot v_I, 1\% \cdot v_I, 1.1\% \cdot v_I\}$ change. See Tables B.3, B.4 and B.5 in Appendix B.2 for the values of the analysis.

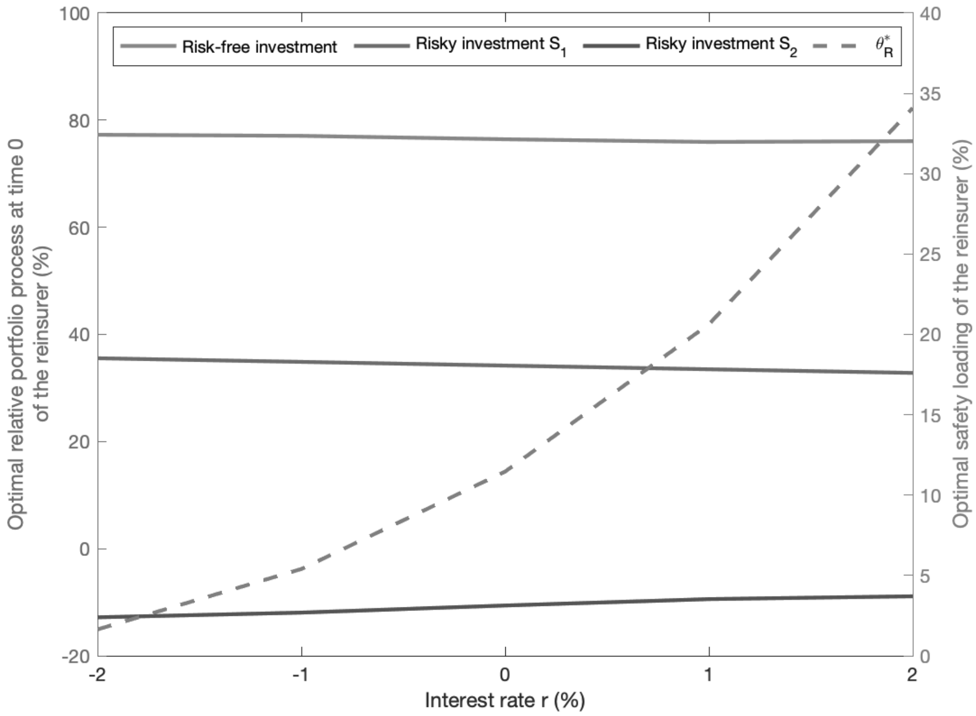

Figure 4.4: Sensitivity of the reinsurer's part of the Stackelberg equilibrium w.r.t. r

In Figures 4.4 and 4.5, we see the influence of the interest rate r on the Stackelberg equilibrium. Again, the optimal reinsurance strategy $\xi_I^*(\theta_R^*)$ of the insurer is $\xi^{\max} = 1.5$, i.e., the insurer chooses the maximum amount of reinsurance independently of the interest rate as long as the safety loading is not unbearably high. Since the optimal safety loading θ_R^* of the reinsurer depends on the put option price, the interest rate influences the optimal safety loading. If the interest rate r increases, then the fair price of reinsurance (without safety loading) decreases, whereas the optimal safety loading θ_R^* increases. In other words, once it is cheaper for the reinsurer to hedge its liabilities toward insurer, the reinsurer tries to increase profit by increasing the safety loading to the maximal value at which the insurer is still willing to purchase reinsurance.
If the interest rate r increases, then the investment in the risky asset S_1 decreases and the short position of the risky asset S_2 is reduced. The decrease of the investment in S_1 follows from the fact that it becomes more attractive to invest in the risk-free asset S_0. Since the put option price decreases with an increasing interest rate, the reinsurer has to hedge the short position in the put less and therefore the investment in S_2 increases (i.e., the short position in S_2 is reduced).

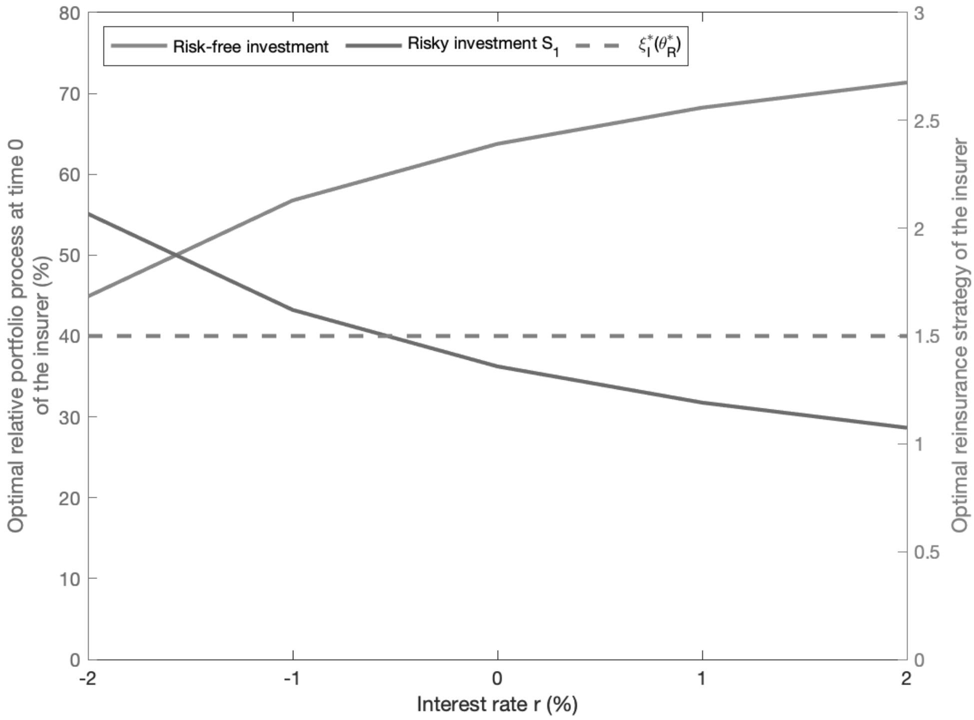

Figure 4.5: Sensitivity of the insurer's part of the Stackelberg equilibrium w.r.t. r

The relative portfolio process of the insurer behaves in the same way: if the interest rate increases, the proportion invested in the risky asset S_1 decreases, since the investment in the risk-free security S_0 becomes more attractive for the insurer.

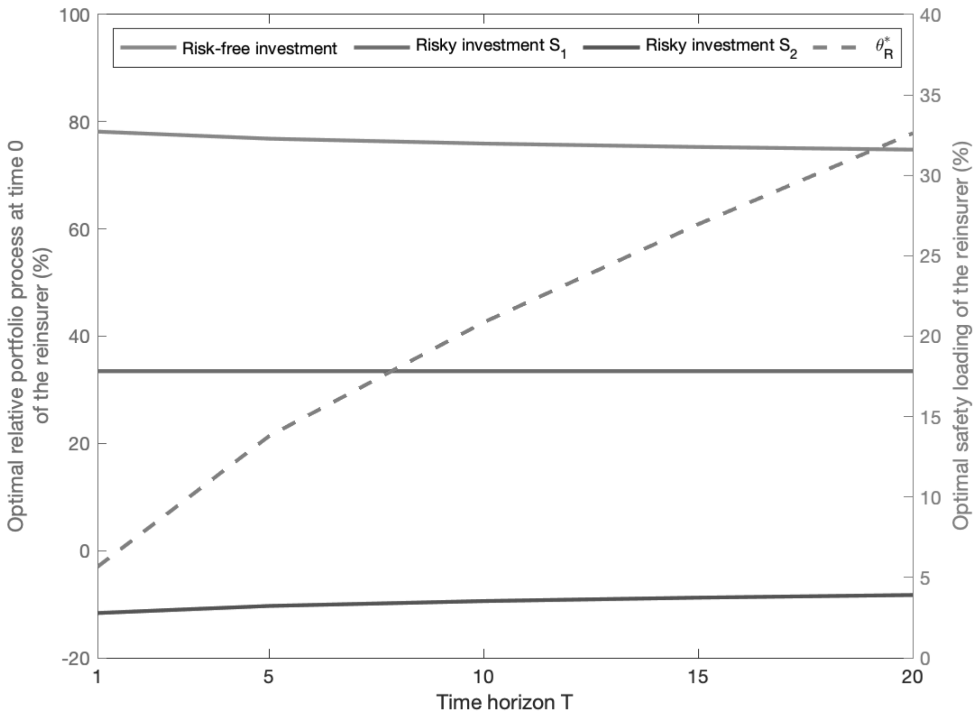

Figure 4.6: Sensitivity of the reinsurer's part of the Stackelberg equilibrium w.r.t. T

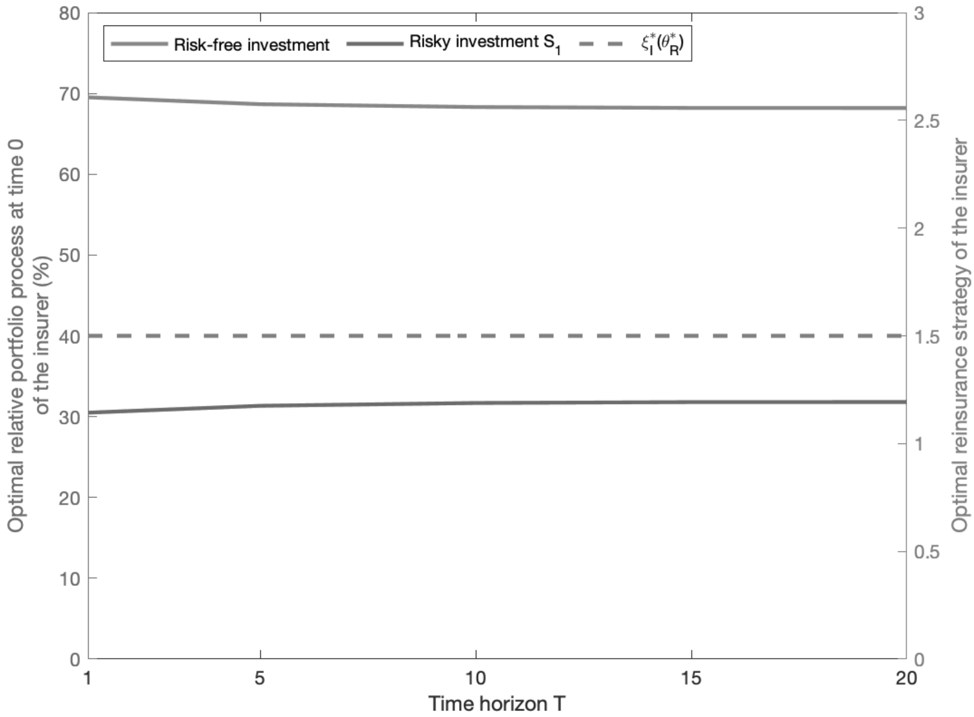

Figure 4.7: Sensitivity of the insurer's part of the Stackelberg equilibrium w.r.t. T

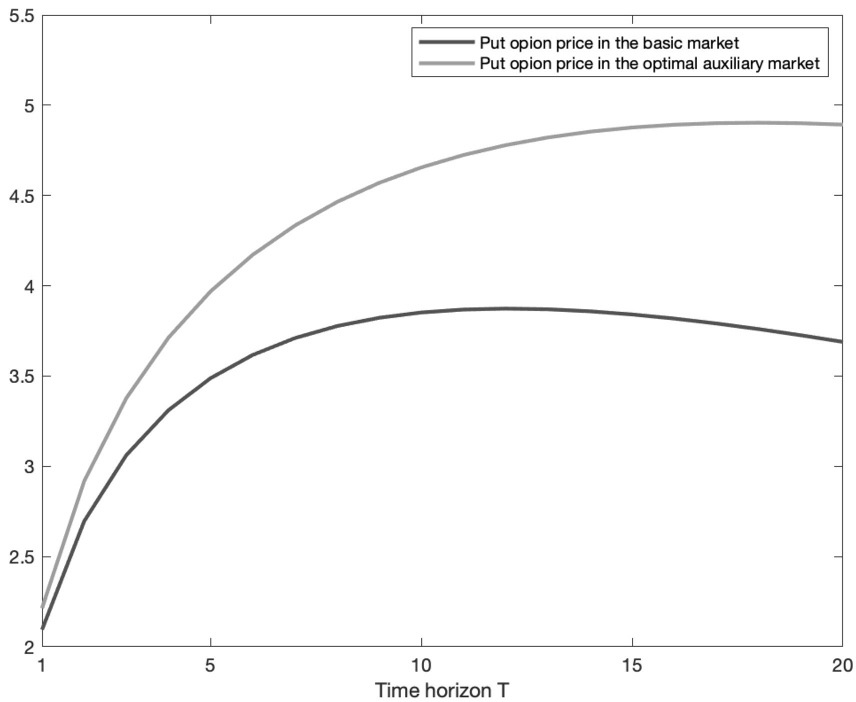

Figure 4.8: Put option price in the basic market and optimal auxiliary market w.r.t. T

The impact of the time horizon T is seen in Figures 4.6 and 4.7. The optimal reinsurance strategy $\xi_I^*(\theta_R^*)$ of the insurer again is the maximum amount $\xi^{\max}$. The fair put option price in the basic market does not change notably if the time horizon T exceeds 5 years (cf.

Figure 4.8). In contrast, the fair price of the put option in the optimal auxiliary market increases significantly and, therefore, the difference between these two prices increases (cf. Figure 4.8). Accordingly, the optimal safety loading θ_R^* of the reinsurer increases with respect to an increasing time horizon.

For the relative portfolio processes of the reinsurer and insurer we only see small changes. The investment of the reinsurer in the risky asset S_1 stays constant when varying the time horizon. The short position in the risky asset S_2 is reduced if the time horizon increases. The relative portfolio process of the insurer increases very slightly if the time horizon increases.

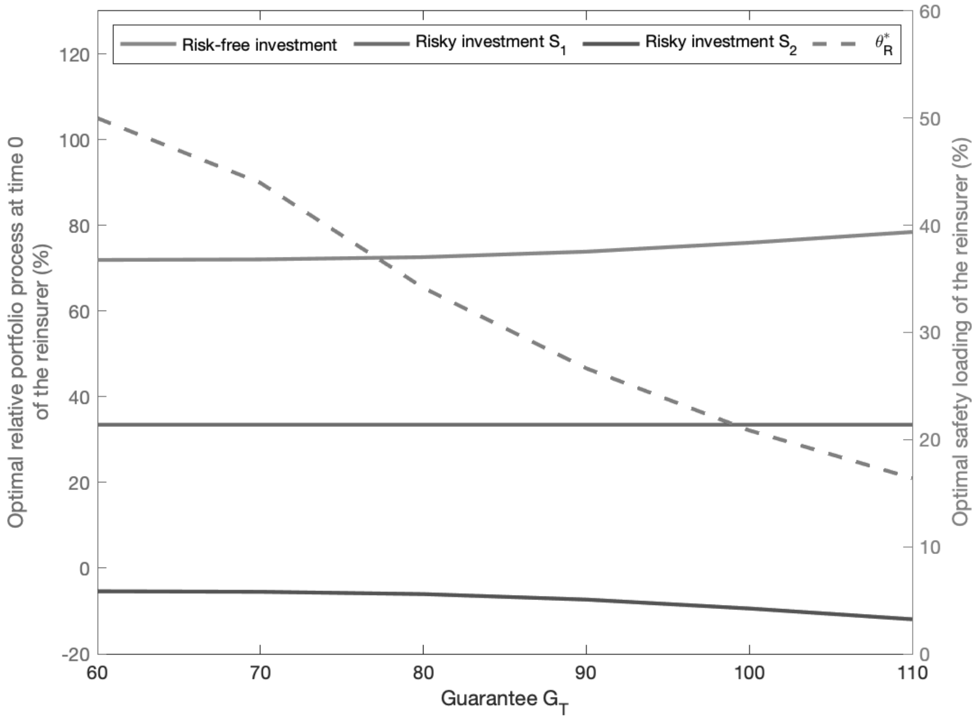

Figure 4.9: Sensitivity of the reinsurer's part of the Stackelberg equilibrium w.r.t. G_T

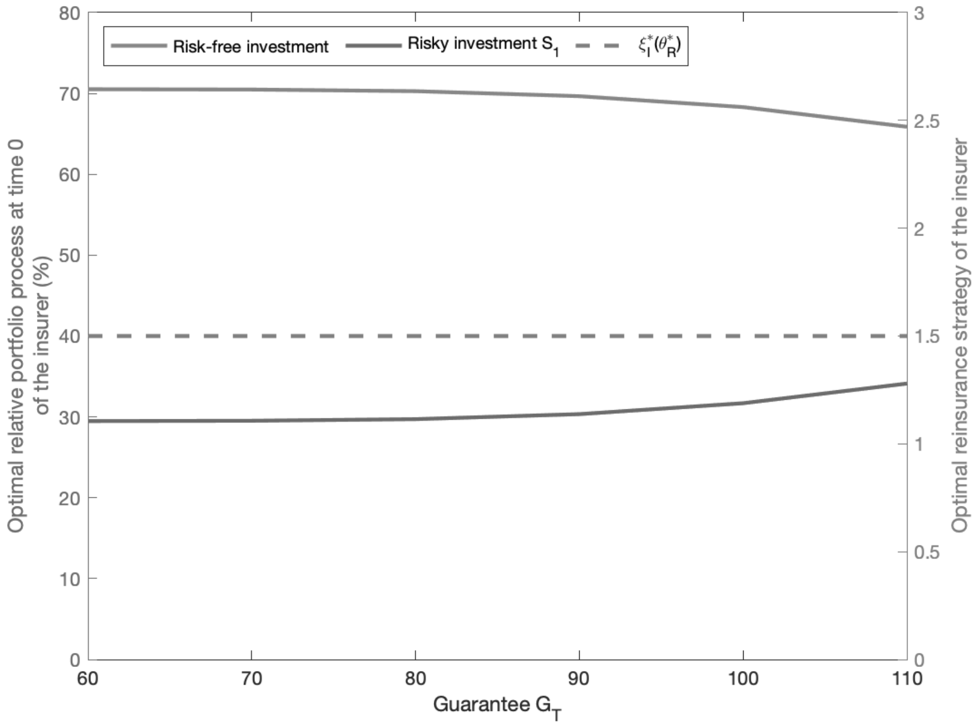

Figure 4.10: Sensitivity of the insurer's part of the Stackelberg equilibrium w.r.t. G_T

In Figures 4.9 and 4.10 we see the influence of the guarantee G_T on the Stackelberg equilibrium. Again, the optimal reinsurance strategy $\xi_I^*(\theta_R^*)$ is constant with respect to the guarantee. The optimal safety loading θ_R^* of the reinsurer decreases with the guarantee, since the put option price increases.

The portfolio process of the reinsurer only changes in the short position of S_2 due to the short put position and the increase of the put option price. The short position in S_2 increase if the guarantee increases, since the reinsurer has to hedge a higher put option price.

The investment of the insurer in the risky asset S_1 increases very slightly when the guarantee increases.

Chapter 5

Conclusion

This master thesis studies Stackelberg games between a reinsurance company and an insurance company. In Chapter 3, we solve a Stackelberg game between a reinsurer and an insurer, where the insurer buys reinsurance on the whole aggregated claims and adjusts its reinsurance strategy dynamically. The game we set up and investigate is a special case from Bai et al. [2019].

In Chapter 4, we solve a more realistic product-specific Stackelberg game between a reinsurer and an insurer. Instead of a reinsurance on the whole company level, we consider reinsurance in the context of a life insurance product with a capital guarantee. The reinsurance is modeled by an excess-of-loss reinsurance, i.e. a put option that protects the insurer against potential losses in the case the investment portfolio of the insurer does not cover the guarantee. The underlying of the put option is a (reinsurable) benchmark portfolio, which is not the same as the insurer's portfolio but highly correlated. The reason for this is that the individual investment strategy of the insurer is too risky for the reinsurer to offer reinsurance on it. Since continuous adjustment of the reinsurance strategy of the insurer is not realistic, we assume that the reinsurance is only purchased at the beginning of the investment horizon.

To solve the Stackelberg game we use backward induction. First, we apply the methods introduced by Cvitanić and Karatzas [1992] and Desmettre and Seifried [2016] to solve the optimization problem of the insurer. This problem has two challenges: fixed-term reinsurance modeled as a put option and market incompleteness due to the inability to hedge the put option within the investment universe of the decision maker. Second, we solve the optimization problem of the reinsurer, with the idea introduced by Korn and Trautmann [1999].

The Stackelberg equilibrium is given by the optimal strategy chosen by the reinsurer and by the insurer. The reinsurer selects its optimal safety loading, whereas the insurer chooses its reinsurance strategy (i.e., the optimal amount of put options) and both choose their own optimal investment strategy.

In the numerical studies, we limit the number of put options to 1.5 to avoid the use of the reinsurance for speculation purposes by the insurer. We find that in the Stackelberg equilibrium the reinsurer selects the largest safety loading such that the insurer is still willing to buy reinsurance and the insurer then buys the maximal amount of reinsurance. The investment strategies of the reinsurer and insurer are mostly influenced by the interest rate and the relative risk aversion of the reinsurer and the insurer, respectively.

Since the Stackelberg equilibrium depends on the underlying model, it may change if we adjust or extend the model. A future research direction could be the extension of the model to allow the reinsurer and the insurer to adjust the reinsurance contract at regular intervals, e.g., annually. Another direction is the usage of more advanced financial market models, e.g., models with jumps, stochastic interest rates, etc. Analogous to Escobar-Anel et al. [2021], the model can be extend by including no-short-selling constraints or Value-at-Risk constraints.

Appendix A

Appendix to Chapter 4

Lemma A.1. For $k \in \mathbb{R}$, $\lambda \in \mathbb{R}^2$ and $t \in [0,T]$ it holds

$$\mathbb{E}[\tilde{Z}_\lambda(T)^k|\mathcal{F}_t] = \exp\bigg(- \Big(rk + \frac{1}{2}\|\gamma_\lambda\|^2 k - \frac{1}{2}\|\gamma_\lambda\|^2 k^2\Big)(T-t)\bigg)\tilde{Z}_\lambda(t)^k.$$

Proof. For $\lambda \in \mathbb{R}^2$ and $t \in [0,T]$ we define

$$\tilde{Z}_\lambda(t) := \exp\bigg(- \Big(r + \frac{1}{2}\|\gamma_\lambda\|^2\Big)t - \gamma_\lambda^\top W(t)\bigg),$$

where $\gamma_\lambda := \gamma + \sigma^{-1}\lambda$. First, we will prove that for any $k \in \mathbb{R}$

$$\bigg(\exp\Big(- \frac{1}{2}\|\gamma_\lambda\|^2 k^2 t - k\gamma_\lambda^\top W(t)\Big)\bigg)_{t\in[0,T]}$$

is a martingale. It holds

$$\mathbb{E}\bigg[\exp\Big(\frac{1}{2}k^2\|\gamma_\lambda\|^2 T\Big)\bigg] < \infty$$

since $k, \|\gamma_\lambda\|, T < \infty$. Therefore, by Novikov's condition (cf. Theorem 1.28) we have that

$$\bigg(\exp\Big(- \frac{1}{2}\|\gamma_\lambda\|^2 k^2 t - k\gamma_\lambda^\top W(t)\Big)\bigg)_{t\in[0,T]}$$

is a martingale. Next, we calculate the expectation $\mathbb{E}[\tilde{Z}_\lambda(T)^k|\mathcal{F}_t]$:

$$\begin{aligned}\mathbb{E}[\tilde{Z}_\lambda(T)^k|\mathcal{F}_t] =& \mathbb{E}\bigg[\exp\bigg(- \Big(r + \frac{1}{2}\|\gamma_\lambda\|^2\Big)kT - k\gamma_\lambda^\top W(T)\bigg)\bigg|\mathcal{F}_t\bigg] \\ =& \exp\bigg(- \Big(r + \frac{1}{2}\|\gamma_\lambda\|^2\Big)kT + \frac{1}{2}\|\gamma_\lambda\|^2 k^2 T\bigg)\end{aligned}$$

$$
\begin{aligned}
&\times \mathbb{E}\Big[\exp\Big(-\frac{1}{2}\|\gamma_\lambda\|^2k^2T - k\gamma_\lambda^\top W(T)\Big)\Big|\mathcal{F}_t\Big] \\
&= \exp\Big(-\Big(r+\frac{1}{2}\|\gamma_\lambda\|^2\Big)kT + \frac{1}{2}\|\gamma_\lambda\|^2k^2T\Big) \\
&\quad \times \exp\Big(-\frac{1}{2}\|\gamma_\lambda\|^2k^2t - k\gamma_\lambda^\top W(t)\Big) \\
&= \exp\Big(-\Big(rk+\frac{1}{2}\|\gamma_\lambda\|^2k - \frac{1}{2}\|\gamma_\lambda\|^2k^2\Big)(T-t)\Big)\tilde{Z}_\lambda(t)^k,
\end{aligned}
$$

where the third equality follows from the martingale property. □

Lemma A.2. Let $\lambda \in \mathbb{R}^2$ and $t \in [0,T]$. Furthermore, let P be a put option with strike price $G_T > 0$ and the constant mix portfolio V^{v_I,π_B} as underlying. Then

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\tilde{Z}_\lambda(T)P(T)|\mathcal{F}_t] = \tilde{Z}_\lambda(t)[&e^{-r(T-t)}G_T\Phi(-d_1(t)) \\
&- V^{v_I,\pi_B}(t)e^{(\tilde{\mu}-\tilde{\sigma}^\top\gamma_\lambda)(T-t)}\Phi(-d_2(t))],
\end{aligned}
\tag{A.1}
$$

where Φ is the cumulative function of the standard normal distribution and

$$
\begin{aligned}
d_1(t) &:= d_1(t, V^{v_I,\pi_B}(t)) := \frac{\ln\big(\frac{V^{v_I,\pi_B}(t)}{G_T}\big) + \big(r+\tilde{\mu}-\frac{1}{2}\|\tilde{\sigma}\|^2 - \tilde{\sigma}^\top\gamma_\lambda\big)(T-t)}{\pi^{CM}\sigma_2\sqrt{T-t}}, \\
d_2(t) &:= d_2(t, V^{v_I,\pi_B}(t)) := d_1(t, V^{v_I,\pi_B}(t)) + \frac{\|\tilde{\sigma}\|^2\sqrt{T-t}}{\pi^{CM}\sigma_2}
\end{aligned}
$$

with $\tilde{\mu} := \pi^{CM}(\mu_2 - r)$ and $\tilde{\sigma} := \pi^{CM}\sigma_2(\rho, \sqrt{1-\rho^2})^\top$. Note that $\mathbb{E}_{\mathbb{Q}}$ denotes the expectation with respect to the probability measure $\mathbb{Q}$.

Proof. Under the probability measure $\mathbb{Q}$ the stochastic process W is a Brownian motion and the underlying V^{v_I,π_B} of the put option is given by

$$
V^{v_I,\pi_B}(t) = V^{v_I,\pi_B}(s)\exp\Big(\Big(r+\tilde{\mu}-\frac{1}{2}\|\tilde{\sigma}\|^2\Big)(t-s) + \tilde{\sigma}^\top(W(t)-W(s))\Big)
$$

with $s \leq t$.
First, we define the probability measure $\tilde{\mathbb{Q}}$ as the equivalent probability measure to $\mathbb{Q}$ with the Radon-Nikodym derivative

$$
\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = Z_\lambda(t) = e^{-\frac{1}{2}\|\gamma_\lambda\|^2t - \gamma_\lambda^\top W(t)}.
$$

By Girsanov's theorem (cf. Theorem 1.29), the stochastic process $\tilde{W}$ defined by

$$
\tilde{W}(t) := W(t) + \gamma_\lambda t
$$

is a Brownian motion and the underlying process V^{v_I,π_B} of the put option has the following dynamics in terms of $\tilde{W}$:

$$V^{v_I,\pi_B}(t) = V^{v_I,\pi_B}(s)\exp\left(\left(r+\tilde{\mu}-\frac{1}{2}\|\tilde{\sigma}\|^2-\tilde{\sigma}^\top\gamma_\lambda\right)(t-s)+\tilde{\sigma}^\top(\tilde{W}(t)-\tilde{W}(s))\right) \tag{A.2}$$

with $s \le t$. We denote by $\mathbb{E}_{\tilde{\mathbb{Q}}}$ the expectation with respect to the probability measure $\tilde{\mathbb{Q}}$. For the payoff $P(T)$ of the put option it holds

$$\begin{aligned} P(T) =&(G_T - V^{v_I,\pi_B}(T))^+ \\ =&(G_T - V^{v_I,\pi_B}(T))\mathbf{1}_{\{G_T > V^{v_I,\pi_B}(T)\}} \end{aligned} \tag{A.3}$$

Then

$$\begin{aligned} \tilde{Z}_\lambda(t)^{-1}\mathbb{E}_{\mathbb{Q}}[\tilde{Z}_\lambda(T)P(T)|\mathcal{F}_t] \underset{\text{Thm. 1.27}}{\overset{\text{Bayes thm.}}{=}}&\mathbb{E}_{\tilde{\mathbb{Q}}}[e^{-r(T-t)}P(T)|\mathcal{F}_t] \\ =&e^{-r(T-t)}\mathbb{E}_{\tilde{\mathbb{Q}}}[P(T)|\mathcal{F}_t] \\ \overset{(A.3)}{=}&e^{-r(T-t)}\mathbb{E}_{\tilde{\mathbb{Q}}}[G_T\mathbf{1}_{\{G_T > V^{v_I,\pi_B}(T)\}}|\mathcal{F}_t] \\ &-e^{-r(T-t)}\mathbb{E}_{\tilde{\mathbb{Q}}}[V^{v_I,\pi_B}(T)\mathbf{1}_{\{G_T > V^{v_I,\pi_B}(T)\}}|\mathcal{F}_t]. \end{aligned}$$

1. We calculate the expectation $\mathbb{E}_{\tilde{\mathbb{Q}}}[G_T\mathbf{1}_{\{G_T > V^{v_I,\pi_B}(T)\}}|\mathcal{F}_t]$. Therefore,

$$\begin{aligned} &\{G_T > V^{v_I,\pi_B}(T)\} \\ \overset{(A.2)}{=}&\left\{G_T > V^{v_I,\pi_B}(t)\exp\left(\left(r+\tilde{\mu}-\frac{1}{2}\|\tilde{\sigma}\|^2-\tilde{\sigma}^\top\gamma_\lambda\right)(T-t)+\tilde{\sigma}^\top(\tilde{W}(T)-\tilde{W}(t))\right)\right\} \\ =&\left\{\ln\left(\frac{G_T}{V^{v_I,\pi_B}(t)}\right)-\left(r+\tilde{\mu}-\frac{1}{2}\|\tilde{\sigma}\|^2-\tilde{\sigma}^\top\gamma_\lambda)\right)(T-t) > \tilde{\sigma}^\top(\tilde{W}(T)-\tilde{W}(t))\right\}. \end{aligned}$$

We have

$$\begin{aligned} \tilde{\sigma}^\top(\tilde{W}(T)-\tilde{W}(t)) =&\pi^{CM}\sigma_2(\rho(\tilde{W}_1(T)-\tilde{W}_1(t))+\sqrt{1-\rho^2}(\tilde{W}_2(T)-\tilde{W}_2(t))) \\ \overset{d}{=}&\pi^{CM}\sigma_2\sqrt{T-t}Z \end{aligned}$$

where $Z \sim \mathcal{N}(0,1)$ under $\tilde{\mathbb{Q}}$. Hence,

$$\begin{aligned} &\left\{\ln\left(\frac{G_T}{V^{v_I,\pi_B}(t)}\right)-\left(r+\tilde{\mu}-\frac{1}{2}\|\tilde{\sigma}\|^2-\tilde{\sigma}^\top\gamma_\lambda)\right)(T-t) > \tilde{\sigma}^\top(\tilde{W}(T)-\tilde{W}(t))\right\} \\ =&\left\{\frac{\ln\left(\frac{G_T}{V^{v_I,\pi_B}(t)}\right)-\left(r+\tilde{\mu}-\frac{1}{2}\|\tilde{\sigma}\|^2-\tilde{\sigma}^\top\gamma_\lambda)\right)(T-t)}{\pi^{CM}\sigma_2\sqrt{T-t}} > Z\right\} \end{aligned}$$

$$=: \{-d_1(t, V^{v_I,\pi_B}(t)) > Z\}.$$

Since $\tilde{W}(T)-\tilde{W}(t)$ is independent of $\mathcal{F}_t$ (due to the independent increment property of the Brownian motion), we get

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{Q}}}[G_T \mathbf{1}_{\{G_T > V^{v_I,\pi_B}(T)\}}|\mathcal{F}_t] =& G_T \mathbb{E}_{\tilde{\mathbb{Q}}}[\mathbf{1}_{\{G_T > V^{v_I,\pi_B}(T)\}}|\mathcal{F}_t] \\ =& G_T \tilde{\mathbb{Q}}(-d_1(t, V^{v_I,\pi_B}(t)) > Z) \\ =& G_T \Phi(-d_1(t, V^{v_I,\pi_B}(t))),\end{aligned}$$

where Φ is the cumulative function of the standard normal distribution.

2. We calculate the expectation $\mathbb{E}_{\tilde{\mathbb{Q}}}[V^{v_I,\pi_B}(T)\mathbf{1}_{\{G_T > V^{v_I,\pi_B}(T)\}}|\mathcal{F}_t]$. First, we define $\hat{\mathbb{Q}}$ as an equivalent probability measure to $\tilde{\mathbb{Q}}$ with the Radon-Nikodym derivative

$$\left.\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = e^{-\frac{1}{2}\|\tilde{\sigma}\|^2 t + \tilde{\sigma}^\top \tilde{W}(t)}.$$

By Girsanov's theorem (cf. Theorem 1.29), the stochastic process $\hat{W}$ defined by

$$\hat{W}(t) := \tilde{W}(t) - \tilde{\sigma} t$$

is a Brownian motion with respect to $\hat{\mathbb{Q}}$ and the underlying process V^{v_I,π_B} of the put option can be rewritten in terms of $\hat{W}$ as follows:

$$\begin{aligned}V^{v_I,\pi_B}(t) =& V^{v_I,\pi_B}(s)\exp\left(\left(r + \tilde{\mu} - \frac{1}{2}\|\tilde{\sigma}\|^2 - \tilde{\sigma}^\top \gamma_\lambda + \|\tilde{\sigma}\|^2\right)(t-s) + \tilde{\sigma}^\top(\hat{W}(t) - \hat{W}(s))\right) \\ =& V^{v_I,\pi_B}(s)\exp\left(\left(r + \tilde{\mu} + \frac{1}{2}\|\tilde{\sigma}\|^2 - \tilde{\sigma}^\top \gamma_\lambda\right)(t-s) + \tilde{\sigma}^\top(\hat{W}(t) - \hat{W}(s))\right)\end{aligned}$$

with $s \leq t$. By $\mathbb{E}_{\hat{\mathbb{Q}}}$ we denote the expectation with respect to the probability measure $\hat{\mathbb{Q}}$. Then

$$\begin{aligned}&\mathbb{E}_{\tilde{\mathbb{Q}}}[V^{v_I,\pi_B}(T)\mathbf{1}_{\{G_T > V^{v_I,\pi_B}(T)\}}|\mathcal{F}_t] \\ =& V^{v_I,\pi_B}(t)\mathbb{E}_{\tilde{\mathbb{Q}}}\Bigg[\exp\left(\left(r + \tilde{\mu} - \frac{1}{2}\|\tilde{\sigma}\|^2 - \tilde{\sigma}^\top \gamma_\lambda\right)(T-t) + \tilde{\sigma}^\top(\tilde{W}(T) - \tilde{W}(t))\right) \\ & \mathbf{1}_{\{G_T > V^{v_I,\pi_B}(T)\}}|\mathcal{F}_t\Bigg] \\ =& V^{v_I,\pi_B}(t)\mathbb{E}_{\hat{Q}}[\exp((r + \tilde{\mu} - \tilde{\sigma}^\top \gamma_\lambda)(T-t))\mathbf{1}_{\{G_T > V^{v_I,\pi_B}(T)\}}|\mathcal{F}_t].\end{aligned}$$

With the same arguments as in point 1, we get

$$
\begin{aligned}
&\{G_T > V^{v_I,\pi_B}(T)\} \\
&= \left\{ G_T > V^{v_I,\pi_B}(t) \exp\left(\Big(r + \tilde{\mu} + \frac{1}{2}\|\tilde{\sigma}\|^2 - \tilde{\sigma}^\top \gamma_\lambda \Big)(T-t) + \tilde{\sigma}^\top (\hat{W}(T) - \hat{W}(t)) \right) \right\} \\
&= \left\{ \frac{\ln\big(\frac{G_T}{V^{v_I,\pi_B}(t)}\big) - \big(r + \tilde{\mu} + \frac{1}{2}\|\tilde{\sigma}\|^2 - \tilde{\sigma}^\top \gamma_\lambda)\big)(T-t)}{\pi^{CM}\sigma_2\sqrt{T-t}} > \hat{Z} \right\} \\
&=: \{-d_2(t, V^{v_I,\pi_B}(t)) > \hat{Z}\},
\end{aligned}
$$

where $\hat{Z} \sim \mathcal{N}(0,1)$ under $\hat{\mathbb{Q}}$. Since $\hat{W}(T) - \hat{W}(t)$ is independent of $\mathcal{F}_t$, we get

$$
\begin{aligned}
&\mathbb{E}_{\tilde{\mathbb{Q}}}[V^{v_I,\pi_B}(T)\mathbf{1}_{\{G_T > V^{v_I,\pi_B}(T)\}}|\mathcal{F}_t] \\
&= V^{v_I,\pi_B}(t)\mathbb{E}_{\hat{Q}}[\exp((r + \tilde{\mu} - \tilde{\sigma}^\top \gamma_\lambda)(T-t))\mathbf{1}_{\{G_T > V^{v_I,\pi_B}(T)\}}|\mathcal{F}_t] \\
&= V^{v_I,\pi_B}(t)\exp((r + \tilde{\mu} - \tilde{\sigma}^\top \gamma_\lambda)(T-t))\hat{\mathbb{Q}}(-d_2(t, V^{v_I,\pi_B}(t)) > \hat{Z}) \\
&= V^{v_I,\pi_B}(t)\exp((r + \tilde{\mu} - \tilde{\sigma}^\top \gamma_\lambda)(T-t))\Phi(-d_2(t, V^{v_I,\pi_B}(t))),
\end{aligned}
$$

where Φ is the cumulative function of the standard normal distribution.

All in all, we have

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\tilde{Z}_\lambda(T)P(T)|\mathcal{F}_t] =& \tilde{Z}_\lambda(t)\tilde{Z}_\lambda(t)^{-1}\mathbb{E}_{\mathbb{Q}}[\tilde{Z}_\lambda(T)P(T)|\mathcal{F}_t] \\
=& \tilde{Z}_\lambda(t)e^{-r(T-t)}\mathbb{E}_{\tilde{\mathbb{Q}}}[G_T\mathbf{1}_{\{G_T > V^{v_I,\pi_B}(T)\}}|\mathcal{F}_t] \\
&- \tilde{Z}_\lambda(t)e^{-r(T-t)}\mathbb{E}_{\tilde{\mathbb{Q}}}[V^{v_I,\pi_B}(T)\mathbf{1}_{\{G_T > V^{v_I,\pi_B}(T)\}}|\mathcal{F}_t] \\
=& \tilde{Z}_\lambda(t)e^{-r(T-t)}G_T\Phi(-d_1(t)) \\
&- \tilde{Z}_\lambda(t)e^{-r(T-t)}V^{v_I,\pi_B}(t)\exp((r + \tilde{\mu} - \tilde{\sigma}^\top \gamma_\lambda)(T-t))\Phi(-d_2(t)) \\
=& \tilde{Z}_\lambda(t)[e^{-r(T-t)}G_T\Phi(-d_1(t)) \\
&- V^{v_I,\pi_B}(t)e^{(\tilde{\mu} - \tilde{\sigma}^\top \gamma_\lambda)(T-t)}\Phi(-d_2(t))],
\end{aligned}
$$

where

$$
\begin{aligned}
d_1(t) &:= d_1(t, V^{v_I,\pi_B}(t)) := \frac{\ln\big(\frac{V^{v_I,\pi_B}(t)}{G_T}\big) + \big(r + \tilde{\mu} - \frac{1}{2}\|\tilde{\sigma}\|^2 - \tilde{\sigma}^\top \gamma_\lambda)\big)(T-t)}{\pi^{CM}\sigma_2\sqrt{T-t}} \\
d_2(t) &:= d_2(t, V^{v_I,\pi_B}(t)) := d_1(t, V^{v_I,\pi_B}(t)) + \frac{\|\tilde{\sigma}\|^2\sqrt{T-t}}{\pi^{CM}\sigma_2}.
\end{aligned}
$$

□

Lemma A.3 (Replicating strategy of the put option)**.** The replicating strategy $\psi(t) :=$

$(\psi_0(t), \psi_1(t), \psi_2(t))^\top$ of the put option P is given by

$$\psi(t) = \left(\frac{P(t) - V(t)\pi^{CM}(\Phi(d_+) - 1)}{S_0(t)}, 0, \frac{\pi^{CM} V^{v_I,\pi_B}(t)(\Phi(d_+) - 1)}{S_2(t)} \right),$$

i.e., for the put option price it holds

$$\begin{aligned} P(t) =& \psi_0(t)S_0(t) + \psi_1(t)S_1(t) + \psi_2(t)S_2(t) \\ =& \frac{P(t) - V(t)\pi^{CM}(\Phi(d_+) - 1)}{S_0(t)} S_0(t) + \frac{\pi^{CM} V^{v_I,\pi_B}(t)(\Phi(d_+) - 1)}{S_2(t)} S_2(t), \end{aligned}$$

where

$$d_+ := d_+(t, V^{v_I,\pi_B}(t)) := \frac{\ln\left(\frac{V^{v_I,\pi_B}(t)}{G_T}\right) + \left(r + \frac{1}{2}(\pi^{CM}\sigma_2)^2\right)(T-t)}{\pi^{CM}\sigma_2\sqrt{T-t}}.$$

The strategy ψ is self-financing, i.e., for the dynamics of the put option it holds

$$dP(t) = \psi_0(t)dS_0(t) + \psi_1(t)dS_1(t) + \psi_2(t)dS_2(t).$$

Proof. By Equation (A.1) (if we set $\lambda = 0$), the price of the put option is given by

$$\begin{aligned} P(t) =& \tilde{Z}(t)^{-1}\mathbb{E}[\tilde{Z}(T)P(T)|\mathcal{F}_t] \\ \overset{(A.1)}{=}& e^{-r(T-t)}G_T\Phi(-d_1(t, V^{v_I,\pi_B}(t))) \\ & - V^{v_I,\pi_B}(t)e^{(\tilde{\mu}-\tilde{\sigma}^\top\gamma)(T-t)}\Phi(-d_2(t, V^{v_I,\pi_B}(t))), \end{aligned} \tag{A.4}$$

where

$$\begin{aligned} d_1(t, V^{v_I,\pi_B}(t)) &:= \frac{\ln\left(\frac{V^{v_I,\pi_B}(t)}{G_T}\right) + \left(r + \tilde{\mu} - \frac{1}{2}\|\tilde{\sigma}\|^2 - \tilde{\sigma}^\top\gamma\right)(T-t)}{\pi^{CM}\sigma_2\sqrt{T-t}}, \\ d_2(t, V^{v_I,\pi_B}(t)) &:= d_1(t, V^{v_I,\pi_B}(t)) + \frac{\|\tilde{\sigma}\|^2\sqrt{T-t}}{\pi^{CM}\sigma_2}. \end{aligned}$$

Reminder: $\tilde{\mu} := \pi^{CM}(\mu_2 - r)$, $\tilde{\sigma} := \pi^{CM}\sigma_2(\rho, \sqrt{1-\rho^2})^\top$ and

$$\begin{aligned} \gamma :=& \sigma^{-1}(\mu - r\mathbb{1}) \\ =& \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \begin{pmatrix} \sigma_2\sqrt{1-\rho^2} & 0 \\ -\sigma_2\rho & \sigma_1 \end{pmatrix} \begin{pmatrix} \mu_1 - r \\ \mu_2 - r \end{pmatrix} \\ =& \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \begin{pmatrix} \sigma_2\sqrt{1-\rho^2}(\mu_1 - r) \\ -\sigma_2\rho(\mu_1 - r) + \sigma_1(\mu_2 - r) \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \frac{\mu_1 - r}{\sigma_1} \\ -\frac{\rho}{\sqrt{1-\rho^2}} \frac{\mu_1 - r}{\sigma_1} + \frac{1}{\sqrt{1-\rho^2}} \frac{\mu_2 - r}{\sigma_2} \end{pmatrix}.$$

Hence,

$$\begin{aligned}
\tilde{\mu} - \tilde{\sigma}^\top \gamma =& \pi^{CM}(\mu_2 - r) - \pi^{CM}\sigma_2 \begin{pmatrix} \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} \frac{\mu_1 - r}{\sigma_1} \\ -\frac{\rho}{\sqrt{1-\rho^2}} \frac{\mu_1 - r}{\sigma_1} + \frac{1}{\sqrt{1-\rho^2}} \frac{\mu_2 - r}{\sigma_2} \end{pmatrix} \\
=& \pi^{CM}(\mu_2 - r) - \pi^{CM}\sigma_2 \rho \frac{\mu_1 - r}{\sigma_1} + \pi^{CM}\sigma_2 \cancel{\sqrt{1-\rho^2}} \frac{\rho}{\cancel{\sqrt{1-\rho^2}}} \frac{\mu_1 - r}{\sigma_1} \\
& - \pi^{CM} \cancel{\sigma_2} \cancel{\sqrt{1-\rho^2}} \frac{1}{\cancel{\sqrt{1-\rho^2}}} \frac{\mu_2 - r}{\cancel{\sigma_2}} \\
=& \pi^{CM}(\mu_2 - r) - \pi^{CM}\sigma_2 \rho \frac{\mu_1 - r}{\sigma_1} + \pi^{CM}\sigma_2 \rho \frac{\mu_1 - r}{\sigma_1} - \pi^{CM}(\mu_2 - r) \\
=& 0.
\end{aligned}$$

For the put option price in (A.4) it follows

$$\begin{aligned}
P(t) =& e^{-r(T-t)} G_T \Phi(-d_1(t, V^{v_I,\pi_B}(t))) - V^{v_I,\pi_B}(t) e^{(\tilde{\mu} - \tilde{\sigma}^\top \gamma)(T-t)} \Phi(-d_2(t, V^{v_I,\pi_B}(t))) \\
=& e^{-r(T-t)} G_T \Phi(-d_1(t, V^{v_I,\pi_B}(t))) - V^{v_I,\pi_B}(t) \Phi(-d_2(t, V^{v_I,\pi_B}(t))),
\end{aligned}$$

where

$$\begin{aligned}
d_1(t, V^{v_I,\pi_B}(t)) &= \frac{\ln\left(\frac{V^{v_I,\pi_B}(t)}{G_T}\right) + \left(r - \frac{1}{2}\|\tilde{\sigma}\|^2\right)\right)(T-t)}{\pi^{CM}\sigma_2\sqrt{T-t}} \\
&= \frac{\ln\left(\frac{V^{v_I,\pi_B}(t)}{G_T}\right) + \left(r - \frac{1}{2}(\pi^{CM}\sigma_2)^2\right)\right)(T-t)}{\pi^{CM}\sigma_2\sqrt{T-t}}, \\
d_2(t, V^{v_I,\pi_B}(t)) &= d_1(t, V^{v_I,\pi_B}(t)) + \frac{\|\tilde{\sigma}\|^2\sqrt{T-t}}{\pi^{CM}\sigma_2} \\
&= d_1(t, V^{v_I,\pi_B}(t)) + \pi^{CM}\sigma_2\sqrt{T-t}.
\end{aligned}$$

The stock price S_2 and the constant mix portfolio value V^{v_I,π_B} are given by

$$\begin{aligned}
S_2(t) =& S_2(0) \exp\left(\left(\mu_2 - \frac{1}{2}\sigma_2^2\right)t + \sigma_2(\rho W_1(t) + \sqrt{1-\rho^2} W_2(t))\right) \\
V^{v_I,\pi_B}(t) =& v_I \exp\left(\left(r + \pi^{CM}(\mu_2 - r) - \frac{1}{2}(\sigma_2\pi^{CM})^2\right)t + \sigma_2\pi^{CM}(\rho W_1(t) + \sqrt{1-\rho^2} W_2(t))\right).
\end{aligned}$$

Hence, the relation between S_2 and V is given by

$$V^{v_I,\pi_B}(t) = \frac{v_I}{S_2(0)} \exp\left(\left(r + \frac{1}{2}\pi^{CM}\sigma_2^2\right)(1 - \pi^{CM})t\right) S_2(t)^{\pi^{CM}}.$$

Therefore,

$$\frac{\partial P(t)}{\partial S_1(t)} = 0$$

and

$$\begin{aligned}\frac{\partial P(t)}{\partial S_2(t)} =& e^{-r(T-t)} G_T \frac{\partial}{\partial S_2(t)} \Phi(-d_1(t, V^{v_I,\pi_B}(t))) - \frac{\partial V^{v_I,\pi_B}(t)}{\partial S_2(t)} \Phi(-d_2(t, V^{v_I,\pi_B}(t))) \\ & - V^{v_I,\pi_B}(t) \frac{\partial}{\partial S_2(t)} \Phi(-d_2(t, V^{v_I,\pi_B}(t))).\end{aligned}$$

We have for $i = 1, 2$

$$\begin{aligned}\frac{\partial V^{v_I,\pi_B}(t)}{\partial S_2(t)} =& \frac{\partial}{\partial S_2(t)} \left(\frac{v_I}{S_2(0)} \exp\left(\left(r + \frac{1}{2}\pi^{CM}\sigma_2^2\right)(1-\pi^{CM})t\right) S_2(t)^{\pi^{CM}} \right) \\ =& \frac{v_I}{S_2(0)} \exp\left(\left(r + \frac{1}{2}\pi^{CM}\sigma_2^2\right)(1-\pi^{CM})t\right) \pi^{CM} S_2(t)^{\pi^{CM}-1} \\ =& V^{v_I,\pi_B}(t) \pi^{CM} S_2(t)^{-1}, \\ \frac{\partial (d_i(t, V^{v_I,\pi_B}(t)))}{\partial S_2(t)} =& \frac{\partial \ln\left(\frac{V^{v_I,\pi_B}(t)}{G_T}\right)}{\partial S_2(t)} \frac{1}{\pi^{CM}\sigma_2\sqrt{T-t}} \\ =& \frac{G_T}{V^{v_I,\pi_B}(t)} \frac{\partial}{\partial S_2(t)} \left(\frac{V^{v_I,\pi_B}(t)}{G_T} \right) \frac{1}{\pi^{CM}\sigma_2\sqrt{T-t}} \\ =& \frac{1}{\cancel{\pi^{CM}}\sigma_2\sqrt{T-t}} \frac{\cancel{G_T}}{\cancel{V^{v_I,\pi_B}(t)}} \frac{\cancel{\pi^{CM}}\,\cancel{V^{v_I,\pi_B}(t)}}{\cancel{G_T} S_2(t)} \\ =& \frac{1}{\sigma_2\sqrt{T-t}} S_2(t)^{-1}, \\ \frac{\partial \Phi(-d_i(t, V^{v_I,\pi_B}(t)))}{\partial S_2(t)} =& -\phi(-d_i(t, V^{v_I,\pi_B}(t))) \frac{\partial (d_i(t, V^{v_I,\pi_B}(t)))}{\partial S_2(t)},\end{aligned}$$

where ϕ is the density of the standard normal distribution, i.e., for $x \in \mathbb{R}$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

Hence,

$$\begin{aligned}\frac{\partial P(t)}{\partial S_2(t)} =& e^{-r(T-t)} G_T \frac{\partial}{\partial S_2(t)} \Phi(-d_1(t, V^{v_I,\pi_B}(t))) - \frac{\partial V^{v_I,\pi_B}(t)}{\partial S_2(t)} \Phi(-d_2(t, V^{v_I,\pi_B}(t))) \\ & - V^{v_I,\pi_B}(t) \frac{\partial}{\partial S_2(t)} \Phi(-d_2(t, V^{v_I,\pi_B}(t))) \\ =& - e^{-r(T-t)} G_T \phi(-d_1(t, V^{v_I,\pi_B}(t))) \frac{1}{\sigma_2\sqrt{T-t}} S_2(t)^{-1} \\ & - \frac{\pi^{CM} V^{v_I,\pi_B}(t) \Phi(-d_2(t, V^{v_I,\pi_B}(t)))}{S_2(t)}\end{aligned}$$

$$
\begin{aligned}
&+ V^{v_I,\pi_B}(t)\phi(-d_2(t,V^{v_I,\pi_B}(t)))\frac{1}{\sigma_2\sqrt{T-t}}S_2(t)^{-1}\\
=&- \frac{\pi^{CM}V^{v_I,\pi_B}(t)\Phi(-d_2(t,V^{v_I,\pi_B}(t)))}{S_2(t)},
\end{aligned}
$$

since

$$
\begin{aligned}
&- e^{-r(T-t)}G_T\phi(-d_1(t,V^{v_I,\pi_B}(t)))\frac{1}{\sigma_2\sqrt{T-t}}S_2(t)^{-1}\\
&+ V^{v_I,\pi_B}(t)\phi(-d_2(t,V^{v_I,\pi_B}(t)))\frac{1}{\sigma_2\sqrt{T-t}}S_2(t)^{-1}\\
=&\frac{1}{\sigma_2\sqrt{T-t}}S_2(t)^{-1}V^{v_I,\pi_B}(t)\Bigg(\phi(-d_2(t,V^{v_I,\pi_B}(t)))\\
&- e^{-r(T-t)}\frac{G_T}{V^{v_I,\pi_B}(t)}\phi(-d_1(t,V^{v_I,\pi_B}(t)))\Bigg)\\
=&\frac{1}{\sigma_2\sqrt{T-t}\sqrt{2\pi}}S_2(t)^{-1}V^{v_I,\pi_B}(t)\Bigg(e^{-\frac{1}{2}d_2(t,V^{v_I,\pi_B}(t))^2}\\
&- e^{-r(T-t)-ln\left(\frac{V^{v_I,\pi_B}(t)}{G_T}\right)-\frac{1}{2}d_1(t,V^{v_I,\pi_B}(t))^2}\Bigg)\\
=&\frac{1}{\sigma_2\sqrt{T-t}\sqrt{2\pi}}S_2(t)^{-1}V^{v_I,\pi_B}(t)\Bigg(e^{-\frac{1}{2}(d_1(t,V^{v_I,\pi_B}(t))+\pi^{CM}\sigma_2\sqrt{T-t})^2}\\
&- e^{-r(T-t)-ln\left(\frac{V^{v_I,\pi_B}(t)}{G_T}\right)-\frac{1}{2}d_1(t,V^{v_I,\pi_B}(t))^2}\Bigg)\\
=&\frac{1}{\sigma_2\sqrt{T-t}\sqrt{2\pi}}S_2(t)^{-1}V^{v_I,\pi_B}(t)\Bigg(e^{-\frac{1}{2}(d_1(t,V^{v_I,\pi_B}(t))^2+2\pi^{CM}\sigma_2\sqrt{T-t}d_1(t,V^{v_I,\pi_B}(t)))}\\
&\times e^{-\frac{1}{2}(\pi^{CM}\sigma_2)^2(T-t)} - e^{-r(T-t)-ln\left(\frac{V^{v_I,\pi_B}(t)}{G_T}\right)-\frac{1}{2}d_1(t,V^{v_I,\pi_B}(t))^2}\Bigg)\\
=&\frac{1}{\sigma_2\sqrt{T-t}\sqrt{2\pi}}S_2(t)^{-1}V^{v_I,\pi_B}(t)\Bigg(e^{-\frac{1}{2}d_1(t,V^{v_I,\pi_B}(t))^2-r(T-t)-ln\left(\frac{V^{v_I,\pi_B}(t)}{G_T}\right)}\\
&\times e^{\frac{1}{2}(\pi^{CM}\sigma_2)^2(T-t)-\frac{1}{2}(\pi^{CM}\sigma_2)^2(T-t)} - e^{-r(T-t)-ln\left(\frac{V^{v_I,\pi_B}(t)}{G_T}\right)-\frac{1}{2}d_1(t,V^{v_I,\pi_B}(t))^2}\Bigg)\\
=&\frac{1}{\sigma_2\sqrt{T-t}\sqrt{2\pi}}S_2(t)^{-1}V^{v_I,\pi_B}(t)\Bigg(e^{-\frac{1}{2}d_1(t,V^{v_I,\pi_B}(t))^2-r(T-t)-ln\left(\frac{V^{v_I,\pi_B}(t)}{G_T}\right)}\\
&- e^{-r(T-t)-ln\left(\frac{V^{v_I,\pi_B}(t)}{G_T}\right)-\frac{1}{2}d_1(t,V^{v_I,\pi_B}(t))^2}\Bigg)\\
=&0.
\end{aligned}
$$

We define $d_2(t,V^{v_I,\pi_B}(t)) =: d_+$. Since it holds $\Phi(-x) = 1-\Phi(x)$, we get

$$
\begin{aligned}
\frac{\partial P(t)}{\partial S_2(t)} =& - \frac{\pi^{CM}V^{v_I,\pi_B}(t)\Phi(-d_2(t,V^{v_I,\pi_B}(t)))}{S_2(t)}\\
=& - \frac{\pi^{CM}V^{v_I,\pi_B}(t)(1-\Phi(d_2(t,V^{v_I,\pi_B}(t))))}{S_2(t)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\pi^{CM} V^{v_I,\pi_B}(t)(\Phi(d_2(t, V^{v_I,\pi_B}(t))) - 1)}{S_2(t)} \\
&= \frac{\pi^{CM} V^{v_I,\pi_B}(t)(\Phi(d_+) - 1)}{S_2(t)}.
\end{aligned}
$$

$\square$

Appendix B

Appendix to Numerical Analysis

B.1 Matlab Functions

In the following, we give some short explanations to the Matlab functions, which we use in the numerical studies. The seed in the analysis is set to 1 (i.e., rng(1)).

- Matlab function "PayoffPutOption": This function calculates a realization of the payoff of a put option with strike price G_T and underlying V^{v_I,π_B} for a realization of the Brownian motion W, i.e.,

$$P(T) = (G_T - V^{v_I,\pi_B}(T))^+.$$

  We need the function for the calculation of the expectation in the function ν.

```
function [payoffPutOption] = PayoffPutOption(WT,vI,GT,r,mu,sigma,
    piCM,T)
% Calculation of the payoff of the put option in the basic market
%   The underlying of the put option is a constant mix portfolio V
%   and the strike price is given by G_T.
%   WT is a realization of the Brownian motion W at time T

tildeMu = piCM*(mu(2)-r);
tildeSigma = piCM*sigma(2,:).';

% Constant mix portfolio at time T
VT = vI*exp((r+tildeMu-0.5*norm(tildeSigma)^2)*T+tildeSigma.'*WT);

% Payoff put option at time T
payoffPutOption = max(GT-VT,0);
end
```

- Matlab function "TerminalDiscFactor": This function calculates a realization of the discounting factor at the terminal time T: $\tilde{Z}_{\lambda^*}(T)$ in the auxiliary market $\mathcal{M}_{\lambda^*}$ and $\tilde{Z}(T)$ in the basic market $\mathcal{M}$. If we want the discounting factor for the basic market, then we set the input argument "lam" to 0 and otherwise to 1.

```
function [ZT] = TerminalDiscFactor(WT,r,mu,sigma,T,lam)
% Discounting Factor Z at time T
%   Function for both dicounting factors Z: for the basic market
%   and the auxiliary market with the optimal lambda.
%   lam == 0: Discounting factor Z in the basic financial market
%   WT is a realization of the Brownian motion W at time T

lambda = [0;(sigma(2,1)*(mu(1)-r))/sigma(1,1)-mu(2)+r];
gammaLambda = inv(sigma)*(mu+lambda-r);
gamma = inv(sigma)*(mu-r);

% Discounting factor at time T
if lam == 0
    ZT = exp(-(r+0.5*norm(gamma)^2)*T-gamma.'*WT);
else
    ZT = exp(-(r+0.5*norm(gammaLambda)^2)*T-gammaLambda.'*WT);
end

end
```

- Matlab function "LagrangeMultiplierI": With this function we can determine the Lagrange multiplier y_I^* of the insurer, i.e.,

$$y_I^* = \left(\frac{v_I - \xi_I(1+\theta_R)P(0) + \xi_I \mathbb{E}[\tilde{Z}_{\lambda^*}(T)P(T)]}{\mathbb{E}[\tilde{Z}_{\lambda^*}(T)^{\frac{b_I}{b_I-1}}]} \right)^{b_I-1}.$$

```
function [y] = LagrangeMultiplierI(xiI,thetaR,vI,bI,GT,r,mu,sigma,
   piCM,P0,T)
% Lagrange multiplier for the insurer
%   Determiniation of the Lagrange multiplier which solves the
%   budget constraint of the insurer

    % Definitions
    tildeMu = piCM*(mu(2)-r);
    tildeSigma = piCM*sigma(2,:).';
    k = bI/(bI-1);
    lambda = [0;(sigma(2,1)*(mu(1)-r))/sigma(1,1)-mu(2)+r];
    gammaLambda = inv(sigma)*(mu+lambda-r);

```

```
    % Expectation of the discounting factor to the power k in the
    auxiliary
    % market
    expectationZAuxiliary = exp(-r*k*T-0.5*norm(gammaLambda)^2*(k-k
    ^2)*T);

    % Price of the put option in the auxiliary market at time 0
    d1 = (log(vI/GT)+(r+tildeMu-0.5*norm(tildeSigma)^2-...
        tildeSigma.'*gammaLambda)*T)/(piCM*norm(sigma(2,:))*sqrt(T)
    );
    d2 = d1+(norm(tildeSigma)^2*sqrt(T))/(piCM*norm(sigma(2,:)));
    pricePutOptionAuxiliary0 = exp(-r*T)*GT*normcdf(-d1)-...
        vI*exp((tildeMu-tildeSigma.'*gammaLambda)*T)*normcdf(-d2);

    % Function of the Lagrange multiplier
    y = ((vI-xiI*(1+thetaR)*P0+xiI*pricePutOptionAuxiliary0)/...
        expectationZAuxiliary).^(bI-1);
end
```

- Matlab function "nu": This function determines the following expectation for a $\xi_I \in [0, \xi^{\max}]$:

$$\nu(\xi_I) = \mathbb{E}\left[\frac{1}{b_I} \max\{(y_I^*(\xi_I)\tilde{Z}_{\lambda^*}(T))^{\frac{1}{b_I - 1}}, \xi_I P(T)\}^{b_I}\right].$$

  $y_I^*(\xi_I)$ is calculated by the Matlab function "LagrangeMultiplierI", $\tilde{Z}_{\lambda^*}(T)$ by the Matlab function "TerminalDiscFactor" and $P(T)$ by the Matlab function "Payoff-PutOption". For the calculation of the expectation, we use the Monte-Carlo Method and the method of antithetic variables (cf. Chapter 4.4 in Korn [2014]).

```
function [nuXiI] = nu(xiI,thetaR,vI,bI,GT,r,mu,sigma,piCM,P0,T,WT)
% Function nu
%   For the calculation of the optimal reinsurance startegy
%   xi_I^ast we maximze the function nu.
%   WT is a realization of the Brownian motion W at time T

% Lagrange multiplier of the insurer
y = LagrangeMultiplierI(xiI,thetaR,vI,bI,GT,r,mu,sigma,piCM,P0,T);

% The arguments of the maximum in the expectation;
% We use the idea of antithetic variates to double the variables
argument1Positive = (y.'*TerminalDiscFactor(WT,r,mu,sigma,T,1))
    .^(1/(bI-1));
argument1Negative = (y.'*TerminalDiscFactor(-WT,r,mu,sigma,T,1))
    .^(1/(bI-1));
```

```
argument2Positive = xiI.'*PayoffPutOption(WT,vI,GT,r,mu,sigma,piCM,
    T);
argument2Negative = xiI.'*PayoffPutOption(-WT,vI,GT,r,mu,sigma,piCM
    ,T);

expectationPositive = (1/bI).*max(argument1Positive,
    argument2Positive).^(bI);
expectationNegative = (1/bI).*max(argument1Negative,
    argument2Negative).^(bI);

nuXiI = mean((expectationPositive+expectationNegative)./2,2);
end
```

- Matlab function "OptimalInvestmentStrategyI": With this function, we can determine the optimal investment strategy of the insurer at time 0. With the input argument "typ" we can choose if we want to calculate the portfolio process π_I^* or the trading strategy φ_I^* (typ="portfolio" or typ="trading").

```
function [piIOptimal] = OptimalInvestmentStrategyI(xiI,thetaR,r,mu,
    sigma,vI,...
                                                              P0,bI,piCM,GT,T,
    typ)
% Optimal Portfolio Process of the Insurer
%   typ can be portfolio process or trading strategy

arguments
    xiI; thetaR; r; mu; sigma; vI; P0; bI; piCM; GT; T;
    typ  char {mustBeMember(typ,{'portfolio','trading'})} = "
    portfolio"
end % typ can only be 'trading' or 'portfolio'
     % 'portfolio': output is optimal portfolio process
     % 'trading': output is optimal trading strategy

lambda = [0;(sigma(2,1)*(mu(1)-r))/sigma(1,1)-mu(2)+r];
gammaLambda = inv(sigma)*(mu+lambda-r);
tildeMu = piCM*(mu(2)-r);
tildeSigma = piCM*sigma(2,:).';

% Merton portfolio in the auxiliary market
piMertonAuxiliary = inv(sigma*sigma.')*(mu+lambda-r)/(1-bI);

% Price of the put option in the optimal auxiliary market
d1 = (log(vI/GT)+(r+tildeMu-0.5*norm(tildeSigma)^2-tildeSigma.'*
    gammaLambda)*T)/...
      (piCM*norm(sigma(2,:))*sqrt(T));
```

```
d2 = d1+(norm(tildeSigma)^2*sqrt(T))/(piCM*norm(sigma(2,:)));
pricePutOptionAuxiliary0 = exp(-r*T)*GT*normcdf(-d1)-...
                     vI*exp((tildeMu-tildeSigma.'*gammaLambda)*T)*
   normcdf(-d2);
if typ == "portfolio"
   piIOptimal = ((vI-xiI*(1+thetaR)*P0+xiI*pricePutOptionAuxiliary0
   )/...
                     (vI-xiI*(1+thetaR)*P0))*piMertonAuxiliary;
else
   piIOptimal = (vI-xiI*(1+thetaR)*P0+xiI*pricePutOptionAuxiliary0)
   ...
                     *piMertonAuxiliary;
end % Gives optimal portfolio process or trading strategy

end
```

- Matlab function ”LagrangeMultiplierR”: With this function we can determine the Lagrange multiplier y_R^* of the reinsurer, i.e.,

$$y_R^* = \left(\frac{v_R + \xi_I^*(\theta_R)\theta_R P(0)}{\mathbb{E}[\tilde{Z}(T)^{\frac{b_R}{b_R-1}}]} \right)^{b_R-1}.$$

```
function [y] = LagrangeMultiplierR(xiI,thetaR,vR,P0,bR,r,mu,sigma,T
   )
% Lagrange multiplier for the reinsurer
%   Determiniation of the Lagrange multiplier which solves the
%   budget constraint of the reinsurer

% Definitions
gamma = inv(sigma)*(mu-r);
k = bR/(bR-1);

% Expectation of the discounting factor to the power k in the basic
% market
expectationZ = exp(-r*k*T-0.5*norm(gamma)^2*(k-k^2)*T);

% Function Lagrange Multiplier
y = ((vR+xiI.*thetaR*P0)/expectationZ).^(bR-1);

end
```

- Matlab function ”kappa”: This function determines the following expectation for a

$\theta_R \in [0, \theta^{\max}]$:

$$\kappa(\theta_R) = \mathbb{E}\left[\frac{1}{b_R}(y_R^*(\theta_R)\tilde{Z}(T))^{\frac{b_R}{b_R-1}}\right].$$

$y_R^*(\theta_R)$ is calculated by the Matlab function "LagrangeMultiplierR" and $\tilde{Z}(T)$ by the Matlab function "TerminalDiscFactor". For the calculation of the expectation, we use the Monte-Carlo Method and the method of antithetic variables (cf. Chapter 4.4 in Korn [2014]).

```
function [kappaThetaR] = kappa(xiI,thetaR,vR,P0,bR,r,mu,sigma,T,WT)
% Function kappa on page 87
%   For the calculation of the optimal safety loading theta_R^ast
%   we maximize the function kappa.
%   WT is a realization of the Brownian motion W at time T

% Lagrange multiplier of the reinsurer
y = LagrangeMultiplierR(xiI,thetaR,vR,P0,bR,r,mu,sigma,T);

% We use the idea of antithetic variates to double the variables
ZTPositive = TerminalDiscFactor(WT,r,mu,sigma,T,0);
ZTNegative = TerminalDiscFactor(-WT,r,mu,sigma,T,0);

expectationPositive = (y.'*ZTPositive).^(bR/(bR-1))/bR;
expectationNegative = (y.'*ZTNegative).^(bR/(bR-1))/bR;

kappaThetaR = mean((expectationPositive+expectationNegative)./2,2);

end
```

- Matlab function "OptimalInvestmentStrategyR": With this function, we determine the optimal investment strategy of the reinsurer at time 0. With the input argument "typ" we can choose if we want to calculate the portfolio process π_R^* or the trading strategy φ_R^* (typ="portfolio" or typ="trading").

```
function [opInvR] = OptimalInvestmentStrategyR(xiI,thetaR,vI,vR,...
                              GT,r,mu,sigma,piCM,T,P0,bR,typ)
% Optimal Trading Strategy of the Reinsurer
arguments
   xiI; thetaR; vI; vR; GT; r; mu; sigma; piCM; T; P0; bR;
   typ  char {mustBeMember(typ,{'portfolio','trading'})} = "
   portfolio"
end % typ can only be 'trading' or 'portfolio'
    % 'portfolio': output is optimal portfolio process
    % 'trading': output is optimal trading strategy
```

```
% Merton portfolio in the basic market
piM = inv(sigma*sigma.')*(mu-r)./(1-bR);

% Replicating strategy of the put option at time 0
d = (log(vI/GT)+(r+0.5*(piCM*norm(sigma(2,:)))^2)*T)/...
    (piCM*norm(sigma(2,:))*sqrt(T));
psi0 = piCM*vI*(normcdf(d)-1);

% Output is a 2-dimensional Vector
opInvR = zeros(2,1);

if typ == "portfolio"
    opInvR(1,1) = piM(1,1);
    opInvR(2,1) = piM(2,1)+xiI*psi0/(vR+xiI*thetaR*P0);
else
    opInvR(1,1) = piM(1,1)*(vR+xiI*thetaR*P0);
    opInvR(2,1) = piM(2,1)*(vR+xiI*thetaR*P0)-xiI*psi0;
end % Gives optimal portfolio process or trading strategy

end
```

- Matlab function "stackelbergEquilibrium": This function is the algorithm from Section 4.5.3. We use it to determine the Stackelberg equilibrium at time 0, i.e.,

$$(\pi_R^*(0), \theta_R^*, \pi_I^*(0|\theta_R^*), \xi_I^*(\theta_R^*)).$$

```
function [xiIOptimal, xiMaxOptimal, investmentIOptimal,
   thetaROptimal,investmentROptimal] ...
    = stackelbergEquilibrium(xiMax,stepSizeXi,thetaMax,
   stepSizeTheta,vI,vR,...
                                   bI,bR,GT,r,mu1,mu2,sigma1,sigma2,rho,
   piCM,s1,s2,T,N,typ)
% Stackelberg equilibrium to the Stackelberg game in Chapter 4
%   Compute the Stackleberg equilibrium which consists of the
%   optimal reinsurance strategy of the insurer
%   xi_I^ast(theta_R^ast), optimal investment strategy of the
%   insurer pi_I^ast(theta_R^ast) or phi_I^ast(theta_R^ast),
%   optimal safety loading of the reinsurer theta_R^ast, and the
%   optimal investment strategy of the reinsurer pi_R^ast or
%   phi_R^ast. The second output is the maximal amount of xi_I,
%   i.e., we can compare it with the optimal reinsurance strategy
%   if we get the maximum value or less than the maximum.
%   With the input-argument "typ" we can choose if we want the
```

```
%   optimal portfolio process of optimal trading strategy of
%   the insurer and reinsurer.
%   typ == "portfolio" or "trading"

% Additional parameters
thetaR = 0:stepSizeTheta:thetaMax;
tildeMu = piCM*(mu2-r);
tildeSigma = piCM*sigma2*[rho;sqrt(1-rho^2)];
mu = [mu1;mu2];
sigma = [sigma1,0;sigma2*rho,sigma2*sqrt(1-rho^2)];
gamma = inv(sigma)*(mu-r);
lambda = [0;(sigma2*rho*(mu1-r))/sigma1-mu2+r];
gammaLambda = inv(sigma)*(mu+lambda-r);
k = bI/(bI-1);

% Price of the put option in the basic market
d1 = (log(vI/GT)+(r+tildeMu-0.5*norm(tildeSigma)^2-...
     tildeSigma.'*gamma)*T)/(piCM*sigma2*sqrt(T));
d2 = d1+(norm(tildeSigma)^2*sqrt(T))/(piCM*sigma2);
P0 = exp(-r*T)*GT*normcdf(-d1)-...
     vI*exp((tildeMu-tildeSigma.'*gamma)*T)*normcdf(-d2);

% N independent normal distributed random variables W1 and W2
WT = normrnd(0,1,2,N)*sqrt(T);

if xiMax == 'No upper bound'
    xiIOptimalGivenTheta = [];
    xiMaxThetaR = []; % If we have no upper bound on xiMax,
    % we will use the sequence in line 45
    parfor nTheta=1:size(thetaR,2)
        xiI = 0:stepSizeXi:(vI/((1+thetaR(nTheta))*P0));
        ExpectationInsurer = nu(xiI,thetaR(nTheta),vI,bI,GT,...
                                r,mu,sigma,piCM,P0,T,WT);
        [maximalValueInsurer,maximalIndexInsurer] = ...
                    max(ExpectationInsurer);
        xiMaxThetaR(nTheta) = vI/((1+thetaR(nTheta))*P0);
        xiIOptimalGivenTheta(nTheta) = xiI(maximalIndexInsurer);
    end

    ExpectationReinsurer = kappa(xiIOptimalGivenTheta,thetaR,...
                                 vR,P0,bR,r,mu,sigma,T,WT);

    [maximalValueReinsurer,maximalIndexReinsurer] = ...
                    max(ExpectationReinsurer);
    thetaROptimal = thetaR(maximalIndexReinsurer);
```

```
    xiIOptimal = xiIOptimalGivenTheta(maximalIndexReinsurer);
    xiMaxOptimal = xiMaxThetaR(maximalIndexReinsurer);
else
    xiIOptimalGivenTheta = [];
    % We will use this sequence, if we have an upper bound of xiMax
    xiI = 0:stepSizeXi:xiMax;
    parfor nTheta=1:size(thetaR,2)
        ExpectationInsurer = nu(xiI,thetaR(nTheta),vI,bI,GT,...
                                r,mu,sigma,piCM,P0,T,WT);
        [maximalValueInsurer,maximalIndexInsurer] = ...
                        max(ExpectationInsurer);
        xiIOptimalGivenTheta(nTheta) = xiI(maximalIndexInsurer);
    end

    ExpectationReinsurer = kappa(xiIOptimalGivenTheta,thetaR,...
                                 vR,P0,bR,r,mu,sigma,T,WT);

    [maximalValueReinsurer,maximalIndexReinsurer] = ...
                        max(ExpectationReinsurer);
    thetaROptimal = thetaR(maximalIndexReinsurer);
    xiIOptimal = xiIOptimalGivenTheta(maximalIndexReinsurer);
    xiMaxOptimal = xiMax;
end % Do we assume that xi^max has an upper bound?

% Optimal investment strategy of the insurer
investmentIOptimal = OptimalInvestmentStrategyI(xiIOptimal,...
    thetaROptimal,r,mu,sigma,vI,P0,bI,piCM,GT,T,typ);

% Optimal trading strategy reinsurer
investmentROptimal = OptimalInvestmentStrategyR(xiIOptimal,...
    thetaROptimal,vI,vR,GT,r,mu,sigma,piCM,T,P0,bR,typ);


end
```

For the sensitivity analysis, we calculate the Stackelberg equilibrium with the Monte Carlo Method for different values of b_I, b_R, r, T and G_T. We check the outcome of the method with the explicit formula for θ_R^* (cf. (4.25)).

```
%% Choosen paremeters
r = 0.0102;
mu1 = 0.1752;
mu2 = 0.1237;
sigma1 = 0.2366;
sigma2 = 0.2198;
```

```
rho = 0.8012;
piCM = 0.2947;
s1 = 1;
s2 = 1;
GT = 100;
T = 10;
vI = 100;
vR = 300;
thetaMax = 0.5;
bI = -9;
bR = -9;
N = 10000;
stepSizeXi = 0.01;
stepSizeTheta = 0.0001;
xiMax = 1.5;
typ = "portfolio";

%% Stackelberg equilibrium
rng(1);
[xiIOptimal, xiMaxOptimal, investmentIOptimal, thetaROptimal,...
    investmentROptimal]  = stackelbergEquilibrium(xiMax,...
    stepSizeXi,thetaMax,stepSizeTheta,vI,vR,bI,bR,GT,r,mu1,...
    mu2,sigma1,sigma2,rho,piCM,s1,s2,T,N,typ);

% To check that the calculation of the Monte Carlo Simulation
% is correct, we calculate the explicit formula of the
% optimal safety loading of the reinsurer

% Additional parameters
tildeMu = piCM*(mu2-r);
tildeSigma = piCM*sigma2*[rho;sqrt(1-rho^2)];
mu = [mu1;mu2];
sigma = [sigma1,0;sigma2*rho,sigma2*sqrt(1-rho^2)];
gamma = inv(sigma)*(mu-r);
lambda = [0;(sigma2*rho*(mu1-r))/sigma1-mu2+r];
gammaLambda = inv(sigma)*(mu+lambda-r);

% Put price basic market
d1 = (log(vI/GT)+(r+tildeMu-0.5*norm(tildeSigma)^2-...
     tildeSigma.'*gamma).*T)./(piCM.*sigma2.*sqrt(T));
d2 = d1+(norm(tildeSigma)^2.*sqrt(T))./(piCM*sigma2);
P0 = exp(-r.*T).*GT.*normcdf(-d1)-...
     vI.*exp((tildeMu-tildeSigma.'*gamma).*T).*normcdf(-d2);

 % Put price auxiliary market
```

```
d1 = (log(vI/GT)+(r+tildeMu-0.5*norm(tildeSigma)^2-...
     tildeSigma.'*gammaLambda).*T)./(piCM*norm(sigma(2,:)).*sqrt(T));
d2 = d1+(norm(tildeSigma)^2.*sqrt(T))./(piCM*norm(sigma(2,:)));
PA0 = exp(-r.*T).*GT.*normcdf(-d1)-...
      vI*exp((tildeMu-tildeSigma.'*gammaLambda).*T).*normcdf(-d2);

thetaROptimalCheck = min((PA0-P0)/P0,thetaMax);

%% Plot: Dependence of xi_I^ast on theta_R
rng(1)
% Additional parameters
thetaR = 0:stepSizeTheta:thetaMax;
tildeMu = piCM*(mu2-r);
tildeSigma = piCM*sigma2*[rho;sqrt(1-rho^2)];
mu = [mu1;mu2];
sigma = [sigma1,0;sigma2*rho,sigma2*sqrt(1-rho^2)];
gamma = inv(sigma)*(mu-r);
lambda = [0;(sigma2*rho*(mu1-r))/sigma1-mu2+r];
gammaLambda = inv(sigma)*(mu+lambda-r);
k = bI/(bI-1);

% Price of the put option in the basic market
d1 = (log(vI/GT)+(r+tildeMu-0.5*norm(tildeSigma)^2-...
     tildeSigma.'*gamma)*T)/(piCM*sigma2*sqrt(T));
d2 = d1+(norm(tildeSigma)^2*sqrt(T))/(piCM*sigma2);
P0 = exp(-r*T)*GT*normcdf(-d1)-...
     vI*exp((tildeMu-tildeSigma.'*gamma)*T)*normcdf(-d2);

% N independent normal distributed random variables W1 and W2
WT = normrnd(0,1,2,N)*sqrt(T);

xiIOptimalGivenTheta = [];
xiI = 0:stepSizeXi:xiMax;
parfor nTheta=1:size(thetaR,2)
    ExpectationInsurer = nu(xiI,thetaR(nTheta),vI,bI,GT,...
                            r,mu,sigma,piCM,P0,T,WT);
    [maximalValueInsurer,maximalIndexInsurer] = ...
                    max(ExpectationInsurer);
    xiIOptimalGivenTheta(nTheta) = xiI(maximalIndexInsurer);
end

plot(thetaR,xiIOptimalGivenTheta,'.r');
legend('\xi_I^\ast(\theta_R)');
xlabel('\theta_R');

```

```

%% Analysis of put option price in auxiliary and basic market
% The analysis is w.r.t. the time horizon T
T = 1:1:20;

% Additional parameters
tildeMu = piCM*(mu2-r);
tildeSigma = piCM*sigma2*[rho;sqrt(1-rho^2)];
mu = [mu1;mu2];
sigma = [sigma1,0;sigma2*rho,sigma2*sqrt(1-rho^2)];
gamma = inv(sigma)*(mu-r);
lambda = [0;(sigma2*rho*(mu1-r))/sigma1-mu2+r];
gammaLambda = inv(sigma)*(mu+lambda-r);

% Put price basic market
d1 = (log(vI/GT)+(r+tildeMu-0.5*norm(tildeSigma)^2-...
     tildeSigma.'*gamma).*T)./(piCM.*sigma2.*sqrt(T));
d2 = d1+(norm(tildeSigma)^2.*sqrt(T))./(piCM*sigma2);
P0 = exp(-r.*T).*GT.*normcdf(-d1)-...
     vI.*exp((tildeMu-tildeSigma.'*gamma).*T).*normcdf(-d2);

 % Put price auxiliary market
d1 = (log(vI/GT)+(r+tildeMu-0.5*norm(tildeSigma)^2-...
     tildeSigma.'*gammaLambda).*T)./(piCM*norm(sigma(2,:)).*sqrt(T));
d2 = d1+(norm(tildeSigma)^2.*sqrt(T))./(piCM*norm(sigma(2,:)));
PA0 = exp(-r.*T).*GT.*normcdf(-d1)-...
      vI*exp((tildeMu-tildeSigma.'*gammaLambda).*T).*normcdf(-d2);

% plot of the put prices w.r.t. T
p = plot(T,P0,T,PA0,'LineWidth',2);
p(1).Color = [0.4940 0.1840 0.5560];
p(2).Color = [0.4660 0.6740 0.1880];
legend('Put opion price in the basic market',...
       'Put opion price in the optimal auxiliary market');
ylim([2,5.5]);
xlabel('Time horizon T');
xlim([1,20]);
xticks([1;5;10;15;20]);
```

For the plots in the sensitivity analysis, we used the following script:

```
%% Relative portfolio process reinsurer w.r.t. risk aversion
piR1 = [66.96;44.64;33.48;26.78;22.32];
piR2 = [-14.79;-11.2;-9.41;-8.33;-7.61];
RRA=[5;7.5;10;12.5;15];
thetaROptimal = [20.86;20.86;20.86;20.86;20.86];
```

```
% Plot
yyaxis left
p = plot(RRA,100-piR1-piR2,'-',RRA,piR1,'-',RRA,piR2,'-','LineWidth',2);
p(1).Color = [.2 .6 .5];
p(2).Color = [0 0.4470 0.7410];
p(3).Color = [0.6350 0.0780 0.1840];
ylim([-20,100]);
ylabel({'Optimal relative portfolio process at time 0';'of the reinsurer
    (%)'});
yyaxis right
q = plot(RRA,thetaROptimal,'--','LineWidth',2);
ylim([0,40]);
ylabel({'Optimal safety loading of the reinsurer (%)'});

legend('Risk-free investment','Risky investment S_1','Risky investment
   S_2',...
       '\theta_R^\ast','Location','northwest','NumColumns',4);
xlabel('Relative risk aversion of the reinsurer (RRA_R=1-b_R)');
xticks(RRA);

%% Relative portfolio process insurer w.r.t. risk aversion
piI1 = [63.38;42.25;31.69;25.35;21.13];
RRA=[5;7.5;10;12.5;15];
xiIOptimal = [1.5;1.5;1.5;1.5;1.5];

% Plot
yyaxis left
p = plot(RRA,100-piI1,'-',RRA,piI1,'-','LineWidth',2);
ylabel({'Optimal relative portfolio process at time 0';'of the insurer
   (%)'});
ylim([0,80]);
p(1).Color = [.2 .6 .5];
p(2).Color = [0 0.4470 0.7410];
yyaxis right
q = plot(RRA,xiIOptimal,'--','LineWidth',2);
ylim([0,3]);
ylabel('Optimal reinsurance strategy of the insurer');

legend('Risk-free investment','Risky investment S_1','\xi_I^\ast(\
   theta_R^\ast)',...
       'Location','northwest','NumColumns',3);
xlabel('Relative risk aversion of the insurer (RRA_I=1-b_I)');
xticks(RRA);

```

```

%% Relative portfolio process reinsurer w.r.t. interest rate
piR1 = [35.55;34.87;34.18;33.49;32.81];
piR2 = [-12.82;-11.94;-10.61;-9.42;-8.89];
r = [-2,-1,0,1,2];
thetaROptimal = [1.64;5.4;11.46;20.63;34.09];

% Plot
yyaxis left
p = plot(r,100-piR1-piR2,'-',r,piR1,'-',r,piR2,'-','LineWidth',2);
p(1).Color = [.2 .6 .5];
p(2).Color = [0 0.4470 0.7410];
p(3).Color = [0.6350 0.0780 0.1840];
ylim([-20,100]);
ylabel({'Optimal relative portfolio process at time 0';'of the reinsurer (%)'});
yyaxis right
q = plot(r,thetaROptimal,'--','LineWidth',2);
ylim([0,40]);
ylabel({'Optimal safety loading of the reinsurer (%)'});

legend('Risk-free investment','Risky investment S_1','Risky investment S_2',...
       '\theta_R^\ast','Location','northwest','NumColumns',4);
xlabel('Interest rate r (%)');
xticks(r);


%% Relative portfolio process insurer w.r.t. interest rate
piI1 = [55.11;43.23;36.24;31.76;28.65];
r = [-2,-1,0,1,2];
xiIOptimal = [1.5;1.5;1.5;1.5;1.5];

% Plot
yyaxis left
p = plot(r,100-piI1,'-',r,piI1,'-','LineWidth',2);
ylabel({'Optimal relative portfolio process at time 0';'of the insurer (%)'});
ylim([0,80]);
p(1).Color = [.2 .6 .5];
p(2).Color = [0 0.4470 0.7410];
yyaxis right
q = plot(r,xiIOptimal,'--','LineWidth',2);
ylim([0,3]);
ylabel('Optimal reinsurance strategy of the insurer');
```

```
legend('Risk-free investment','Risky investment S_1','\xi_I^\ast(\theta_R^\ast)',...
       'Location','northwest','NumColumns',3);
xlabel('Interest rate r (%)');
xticks(r);


%% Relative portfolio process reinsurer w.r.t. time horizon
piR1 = [33.48;33.48;33.48;33.48;33.48];
piR2 = [-11.64;-10.32;-9.41;-8.77;-8.28];
T = [1,5,10,15,20];
thetaROptimal = [5.66;13.79;20.86;26.96;32.62];

% Plot
yyaxis left
p = plot(T,100-piR1-piR2,'-',T,piR1,'-',T,piR2,'-','LineWidth',2);
p(1).Color = [.2 .6 .5];
p(2).Color = [0 0.4470 0.7410];
p(3).Color = [0.6350 0.0780 0.1840];
ylim([-20,100]);
ylabel({'Optimal relative portfolio process at time 0';'of the reinsurer (%)'});
yyaxis right
q = plot(T,thetaROptimal,'--','LineWidth',2);
ylim([0,40]);
ylabel({'Optimal safety loading of the reinsurer (%)'});

legend('Risk-free investment','Risky investment S_1','Risky investment S_2',...
       '\theta_R^\ast','Location','northwest','NumColumns',4);
xlabel('Time horizon T');
xticks(T);
xlim([1,20]);


%% Relative portfolio process insurer w.r.t. time horizon
piI1 = [30.49;31.34;31.69;31.8;31.81];
T = [1,5,10,15,20];
xiIOptimal = [1.5;1.5;1.5;1.5;1.5];

% Plot
yyaxis left
p = plot(T,100-piI1,'-',T,piI1,'-','LineWidth',2);
ylabel({'Optimal relative portfolio process at time 0';'of the insurer
```

```
    (%)'});
ylim([0,80]);
p(1).Color = [.2 .6 .5];
p(2).Color = [0 0.4470 0.7410];
yyaxis right
q = plot(T,xiIOptimal,'--','LineWidth',2);
ylim([0,3]);
ylabel('Optimal reinsurance strategy of the insurer');

legend('Risk-free investment','Risky investment S_1','\xi_I^\ast(\
    theta_R^\ast)',...
       'Location','northwest','NumColumns',3);
xlabel('Time horizon T');
xticks(T);
xlim([1,20]);



%% Alternative Plots: Guarantee Reinsurer
piR1 = [33.48;33.48;33.48;33.48;33.48;33.48];
piR2 = [-5.4;-5.52;-6.05;-7.33;-9.41;-11.92];
piR = [100-piR1-piR2,piR1,piR2];
GT = [60;70;80;90;100;110];
thetaROptimal = [50;43.96;34.16;26.65;20.86;16.4];

yyaxis left
p = plot(GT,100-piR1-piR2,'-',GT,piR1,'-',GT,piR2,'-',...
    'LineWidth',2);
p(1).Color = [.2 .6 .5];
p(2).Color = [0 0.4470 0.7410];
p(3).Color = [0.6350 0.0780 0.1840];
yyaxis right
q = plot(GT,thetaROptimal,'--','LineWidth',2);
xlabel('Guarantee G_T');
yyaxis left
ylabel({'Optimal relative portfolio process at time 0';'of the reinsurer
    (%)'});
ylim([-20,130]);
yyaxis right
ylabel({'Optimal safety loading of the reinsurer (%)'});
ylim([0,60]);
xticks(GT);
legend('Risk-free investment','Risky investment S_1','Risky investment
    S_2',...
       '\theta_R^\ast','Location','northwest','NumColumns',4);
```

```


%% Alternative Plots: Guarantee Insurer
piI1 = [29.48;29.52;29.72;30.34;31.69;34.12];
piI = [100-piI1,piI1];
GT = [60,70,80,90,100,110];
xiIOptimal = [1.5;1.5;1.5;1.5;1.5;1.5];

% Plot
yyaxis left
p = plot(GT,100-piI1,'-',GT,piI1,'-','LineWidth',2);
ylim([0,80]);
p(1).Color = [.2 .6 .5];
p(2).Color = [0 0.4470 0.7410];
ylabel({'Optimal relative portfolio process at time 0';'of the insurer
    (%)'});
yyaxis right
q = plot(GT,xiIOptimal,'--','LineWidth',2);
ylim([0,3]);
ylabel({'Optimal reinsurance strategy of the insurer'});

legend('Risk-free investment','Risky investment S_1','\xi_I^\ast(\
    theta_R^\ast)',...
        'Location','northwest','NumColumns',4);
xlim([60,110]);
xlabel('Guarantee G_T');
xticks(GT);
```

B.2 Sensitivity Analysis

This section contains the values to the numerical study in Section 4.5.3.

RRA_R \ RRA_I		5	7.5	10	12.5	15
5	θ_R^*	20.86%	20.86%	20.86%	20.86%	20.86%
	$\xi_I^*(\theta_R^*)$	1.5	1.5	1.5	1.5	1.5
7.5	θ_R^*	20.86%	20.86%	20.86%	20.86%	20.86%
	$\xi_I^*(\theta_R^*)$	1.5	1.5	1.5	1.5	1.5
10	θ_R^*	20.86%	20.86%	20.86%	20.86%	20.86%
	$\xi_I^*(\theta_R^*)$	1.5	1.5	1.5	1.5	1.5
12.5	θ_R^*	20.86%	20.86%	20.86%	20.86%	20.86%
	$\xi_I^*(\theta_R^*)$	1.5	1.5	1.5	1.5	1.5
15	θ_R^*	20.86%	20.86%	20.86%	20.86%	20.86%
	$\xi_I^*(\theta_R^*)$	1.5	1.5	1.5	1.5	1.5

Table B.1: Sensitivity of θ_R^* and ξ_I^* w.r.t. RRA_R and RRA_I

RRA_R \ RRA_I		5	7.5	10	12.5	15
5	$\pi_R^*(0)$	$\begin{pmatrix}66.96\%\\-14.79\%\end{pmatrix}$	$\begin{pmatrix}66.96\%\\-14.79\%\end{pmatrix}$	$\begin{pmatrix}66.96\%\\-14.79\%\end{pmatrix}$	$\begin{pmatrix}66.96\%\\-14.79\%\end{pmatrix}$	$\begin{pmatrix}66.96\%\\-14.79\%\end{pmatrix}$
	$\pi_I^*(0)$	$\begin{pmatrix}63.38\%\\0\%\end{pmatrix}$	$\begin{pmatrix}42.25\%\\0\%\end{pmatrix}$	$\begin{pmatrix}31.69\%\\0\%\end{pmatrix}$	$\begin{pmatrix}25.35\%\\0\%\end{pmatrix}$	$\begin{pmatrix}21.13\%\\0\%\end{pmatrix}$
7.5	$\pi_R^*(0)$	$\begin{pmatrix}44.64\%\\-11.20\%\end{pmatrix}$	$\begin{pmatrix}44.64\%\\-11.20\%\end{pmatrix}$	$\begin{pmatrix}44.64\%\\-11.20\%\end{pmatrix}$	$\begin{pmatrix}44.64\%\\-11.20\%\end{pmatrix}$	$\begin{pmatrix}44.64\%\\-11.20\%\end{pmatrix}$
	$\pi_I^*(0)$	$\begin{pmatrix}63.38\%\\0\%\end{pmatrix}$	$\begin{pmatrix}42.25\%\\0\%\end{pmatrix}$	$\begin{pmatrix}31.69\%\\0\%\end{pmatrix}$	$\begin{pmatrix}25.35\%\\0\%\end{pmatrix}$	$\begin{pmatrix}21.13\%\\0\%\end{pmatrix}$
10	$\pi_R^*(0)$	$\begin{pmatrix}33.48\%\\-9.41\%\end{pmatrix}$	$\begin{pmatrix}33.48\%\\-9.41\%\end{pmatrix}$	$\begin{pmatrix}33.48\%\\-9.41\%\end{pmatrix}$	$\begin{pmatrix}33.48\%\\-9.41\%\end{pmatrix}$	$\begin{pmatrix}33.48\%\\-9.41\%\end{pmatrix}$
	$\pi_I^*(0)$	$\begin{pmatrix}63.38\%\\0\%\end{pmatrix}$	$\begin{pmatrix}42.25\%\\0\%\end{pmatrix}$	$\begin{pmatrix}31.69\%\\0\%\end{pmatrix}$	$\begin{pmatrix}25.35\%\\0\%\end{pmatrix}$	$\begin{pmatrix}21.13\%\\0\%\end{pmatrix}$
12.5	$\pi_R^*(0)$	$\begin{pmatrix}26.78\%\\-8.33\%\end{pmatrix}$	$\begin{pmatrix}26.78\%\\-8.33\%\end{pmatrix}$	$\begin{pmatrix}26.78\%\\-8.33\%\end{pmatrix}$	$\begin{pmatrix}26.78\%\\-8.33\%\end{pmatrix}$	$\begin{pmatrix}26.78\%\\-8.33\%\end{pmatrix}$
	$\pi_I^*(0)$	$\begin{pmatrix}63.38\%\\0\%\end{pmatrix}$	$\begin{pmatrix}42.25\%\\0\%\end{pmatrix}$	$\begin{pmatrix}31.69\%\\0\%\end{pmatrix}$	$\begin{pmatrix}25.35\%\\0\%\end{pmatrix}$	$\begin{pmatrix}21.13\%\\0\%\end{pmatrix}$
15	$\pi_R^*(0)$	$\begin{pmatrix}22.32\%\\-7.61\%\end{pmatrix}$	$\begin{pmatrix}22.32\%\\-7.61\%\end{pmatrix}$	$\begin{pmatrix}22.32\%\\-7.61\%\end{pmatrix}$	$\begin{pmatrix}22.32\%\\-7.61\%\end{pmatrix}$	$\begin{pmatrix}22.32\%\\-7.61\%\end{pmatrix}$
	$\pi_I^*(0)$	$\begin{pmatrix}63.38\%\\0\%\end{pmatrix}$	$\begin{pmatrix}42.25\%\\0\%\end{pmatrix}$	$\begin{pmatrix}31.69\%\\0\%\end{pmatrix}$	$\begin{pmatrix}25.35\%\\0\%\end{pmatrix}$	$\begin{pmatrix}21.13\%\\0\%\end{pmatrix}$

Table B.2: Sensitivity of $\pi_R^*(0)$ and $\pi_I^*(0)$ w.r.t. RRA_R and RRA_I

r	-2%	-1%	0%	1%	2%
θ_R^*	1.64%	5.40%	11.46%	20.63%	34.09%
$\xi_I^*(\theta_R^*)$	1.5	1.5	1.5	1.5	1.5
$\pi_R^*(0)$	$\begin{pmatrix} 35.55\% \\ -12.82\% \end{pmatrix}$	$\begin{pmatrix} 34.87\% \\ -11.94\% \end{pmatrix}$	$\begin{pmatrix} 34.18\% \\ -10.61\% \end{pmatrix}$	$\begin{pmatrix} 33.49\% \\ -9.42\% \end{pmatrix}$	$\begin{pmatrix} 32.81\% \\ -8.89\% \end{pmatrix}$
$\pi_I^*(0)$	$\begin{pmatrix} 55.11\% \\ 0\% \end{pmatrix}$	$\begin{pmatrix} 43.23\% \\ 0\% \end{pmatrix}$	$\begin{pmatrix} 36.24\% \\ 0\% \end{pmatrix}$	$\begin{pmatrix} 31.76\% \\ 0\% \end{pmatrix}$	$\begin{pmatrix} 28.65\% \\ 0\% \end{pmatrix}$

Table B.3: Sensitivity of the Stackelberg equilibrium w.r.t. r

T	1%	5%	10%	15%	20%
θ_R^*	5.66%	13.79%	20.86%	26.96%	32.62%
$\xi_I^*(\theta_R^*)$	1.5	1.5	1.5	1.5	1.5
$\pi_R^*(0)$	$\begin{pmatrix} 33.48\% \\ -11.64\% \end{pmatrix}$	$\begin{pmatrix} 33.48\% \\ -10.32\% \end{pmatrix}$	$\begin{pmatrix} 33.48\% \\ -9.41\% \end{pmatrix}$	$\begin{pmatrix} 33.48\% \\ -8.77\% \end{pmatrix}$	$\begin{pmatrix} 33.48\% \\ -8.28\% \end{pmatrix}$
$\pi_I^*(0)$	$\begin{pmatrix} 30.49\% \\ 0\% \end{pmatrix}$	$\begin{pmatrix} 31.34\% \\ 0\% \end{pmatrix}$	$\begin{pmatrix} 31.69\% \\ 0\% \end{pmatrix}$	$\begin{pmatrix} 31.80\% \\ 0\% \end{pmatrix}$	$\begin{pmatrix} 31.81\% \\ 0\% \end{pmatrix}$

Table B.4: Sensitivity of the Stackelberg equilibrium w.r.t. T

G_T	60%	70%	80%	90%	100%	110%
θ_R^*	50%	43.96%	34.18%	26.65%	20.86%	16.40%
$\xi_I^*(\theta_R^*)$	1.5	1.5	1.5	1.5	1.5	1.5
$\pi_R^*(0)$	$\begin{pmatrix} 33.48\% \\ -5.40\% \end{pmatrix}$	$\begin{pmatrix} 33.48\% \\ -5.52\% \end{pmatrix}$	$\begin{pmatrix} 33.48\% \\ -6.05\% \end{pmatrix}$	$\begin{pmatrix} 33.48\% \\ -7.33\% \end{pmatrix}$	$\begin{pmatrix} 33.48\% \\ -9.41\% \end{pmatrix}$	$\begin{pmatrix} 33.48\% \\ -11.92\% \end{pmatrix}$
$\pi_I^*(0)$	$\begin{pmatrix} 29.48\% \\ 0\% \end{pmatrix}$	$\begin{pmatrix} 29.52\% \\ 0\% \end{pmatrix}$	$\begin{pmatrix} 29.72\% \\ 0\% \end{pmatrix}$	$\begin{pmatrix} 30.34\% \\ 0\% \end{pmatrix}$	$\begin{pmatrix} 31.69\% \\ 0\% \end{pmatrix}$	$\begin{pmatrix} 34.12\% \\ 0\% \end{pmatrix}$

Table B.5: Sensitivity of the Stackelberg equilibrium w.r.t. G_T

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