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Utility Maximization in an Investment Pool

Master Thesis

in Mathematics and Management

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1. Introduction

1.1. Motivation

In developed countries, defined benefit (DB) and defined contribution (DC) pension schemes are still the most popular types of occupational retirement plans. In a DB scheme, the employee's pension benefit is determined by a formula that takes into account years of service for the employer and wages or salary. So in a DB scheme the sponsoring companies basically promise their employees a guaranteed pension payment. In DC schemes, on the other hand, the sponsoring company (and often also its employees) pay fixed contributions, usually as a percentage of the salary, to an external pension fund. These contributions are then invested in financial assets and the benefit at retirement depends on the performance of investment returns experienced during the membership.
In a DB scheme the market risk is borne completely by the employer, while in a DC scheme the customer carries all of the market risk. Another form of occupational pension schemes are hybrid pension plans which are a combination of DB and DC schemes where both parties carry some risk. For example, in Germany, "pure" DC plans without any guarantee had not existed at all until the "Betriebsrentenstärkungsgesetz" came into effect in 2018 and the new pension scheme "Zielrente" started being used. This is a retirement plan without guarantee, so basically a pure DC scheme. Before this new law was enacted, in all the pension schemes in Germany some sort of guarantee had been prescribed by law. Such a shift from DB towards DC schemes could not only be observed in Germany in the last years, but actually in most industrialized countries of today's world. The main reasons for this are the constantly increasing life expectancy and the current low interest rate environment.
In such a case of forced joint investments, where the pension beneficiaries carry the investment risk, the question, how an optimal investment strategy along with an adequate rule how to share the aggregate terminal wealth at retirement can be obtained, arises. Of course, this question is of high relevance for both pension schemes with and without

guarantees.

In order to fully understand the optimization problem faced by a collective of investors it seems inevitable to first take a closer look at the corresponding individual optimization problem. The question about optimal investments for individuals, unrestricted or under portfolio insurance, is not really new to the academic world. The individual unrestricted optimization problem is analyzed in Merton (1969, 1971) or, for instance, also in Karatzas et al. (1987) and Cox and Huang (1989). The investment under portfolio insurance is considered, for example, in Basak (1995), Grossman and Zhou (1996) and Jensen and Sørensen (2001).
The collective optimization where a group of agents is tied together in their investment decision has also already been studied heavily, for example in Dumas (1989), Weinbaum (2009) or, more recently, also in Jensen and Nielsen (2016) and Branger et al. (2018).

1.2. Objectives

In the thesis, we will consider various possibilities to address the problems of finding an optimal investment strategy and redistributing the obtained terminal payoff among the participants in the pool. All the approaches will be considered with and without an interest rate guarantee.
We will use a Black-Scholes market and assume that each individual is modeled by a constant relative risk aversion (CRRA) utility function with her own specific risk aversion parameter. To be able to present approaches that yield a terminal payoff to a pool of investors with different risk aversions along with an adequate sharing rule, we first need to take a closer look at the optimal investment problem for a single investor in the market. The individual unrestricted terminal payoff can then be used to assess the losses or gains incurred by the joint investment for each individual.
Whenever dealing with a collective of investors, we would like to use a Pareto efficient payoff for a given sharing rule. A terminal payoff is Pareto efficient if it is not possible to increase one investor's expected utility without simultaneously decreasing another individual's expected utility by using a different payoff. Pareto efficiency is a frequently used criterion in economics when dealing with a group of agents facing a joint decision in a financial environment, see for example Wilson (1968), Amershi and Stoeckenius (1983), Huang and Litzenberger (1985) or Kreps (2012).

In the setting considered here, where the collective consists of investors with different CRRA utility functions, such Pareto efficient payoffs can be obtained by maximizing a weighted sum of the individual utility levels. For any choice of the weights the resulting payoff is going to be Pareto efficient which means that there are basically infinitely many such Pareto efficient payoffs. Therefore, the focus of the thesis lies on analyzing these payoffs and finding a neat choice of these weights that take into account all the members of the pool in a fair way.

1.3. Structure

There are three chapters in the remainder of the thesis. We will start with reviewing the unrestricted individual solution for each investor who maximizes her expected utility of terminal wealth in Chapter 2. This is the classic Merton problem which results in a constant mix strategy. After that, we include an interest rate guarantee in the individual optimization problem. In this case, the investment strategy is no longer constant but state dependent. The chapter is concluded by a comparison between the setting with and without guarantee. This chapter is mainly based on Jensen and Sørensen (2001).
In Chapter 3 we then consider a collective of individuals who are forced to jointly invest their total initial wealth. The first best payoff is equal to the sum over all individually optimal payoffs, and the corresponding sharing rule is state-dependent. Realistic contracts, however, will have a linear sharing rule where the percentage of the terminal payoff obtained at the end is known before the initial wealth is invested. Having introduced linear sharing rules, we first consider strategies where the total capital of the pool is invested according to some risk aversion parameter that usually lies between the smallest and the largest risk aversion parameter of the pool. So the problem is basically reduced to the individual investment problem from Chapter 2 and all the relevant characteristics can be determined explicitly. For the case without a guarantee this results in a constant mix strategy. If guarantees are included, the investment strategy is no longer constant but can still be determined explicitly. The main reason for using this approach is, among further advantages which will be presented later, its simplicity which makes the procedure very easy to communicate. Chapter 3 is then concluded with an analysis of Pareto efficient payoffs which are obtained from maximizing a sum of the weighted individual utility functions. These payoffs will also be analyzed with and without an interest rate guarantee. Since it is not clear from the beginning how the weights should be chosen when dealing

with Pareto efficient payoffs the chapter will be concluded by a numerical analysis with respect to the weights and the derivation of an appropriate choice of them. Chapter 3 is mainly based on the ideas presented in Branger et al. (2018), Jensen and Nielsen (2016) and Jensen and Sørensen (2001).

Chapter 4 concludes the thesis and briefly summarizes the most important results presented in Chapters 2 and 3. It is followed by the appendix, in which the basic methods of Monte Carlo simulation applied throughout the thesis are explained and the R codes used for the numerical analyses are listed.

2. Individual Optimization Problem

In this chapter we will consider one individual in a financial market starting with an initial wealth v. This individual is modeled by a CRRA utility function $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ where $\gamma > 0$, $\gamma \neq 1$. This investor aims to maximize her expected utility of the final accumulated wealth at a finite time horizon $T < \infty$.
In this chapter the most important, mostly well known results in this field of study will be presented. First the individual investment problem without any constraints will be reviewed. After that, we will assume that the terminal payoff needs to meet a specific interest rate guarantee. But before that, we will first have a closer look at the financial market that will be used throughout all the following chapters.

2.1. Financial Market

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P\right)$ be a stochastic basis. The filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the standard filtration of $\{W_t\}_{t\in[0,T]}$ which is a standard Brownian motion under P. Throughout the thesis we will assume that the assets traded in the financial market follow the Black-Scholes model, that is, there are the following two assets traded in the market:

- a risk-free asset B that earns a constant interest rate r, that is,

$$\begin{aligned} &dB_t = rB_t dt, \quad B_0 = 1, \quad \text{or, equivalently,} \\ &B_t = e^{rt}. \end{aligned}$$

- a risky asset following a geometric Brownian motion with instantaneous rate of return $\mu > r$ and volatility $\sigma > 0$, that is,

$$\begin{aligned} &dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 = s, \quad \text{or, equivalently,} \\ &S_t = s \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right). \end{aligned} \tag{2.1}$$

Under these assumptions the state price density process $\{M_t\}_{t\in[0,T]}$ is uniquely determined and has the following properties (see for example Rogers (2013) or Karatzas and Shreve (1998)):

- The state price density process is determined by the following stochastic differential equation:

$$dM_t = -M_t \left(r dt + \lambda dW_t\right), \quad M_0 = 1, \quad \lambda = \frac{\mu - r}{\sigma}.$$

  The solution is given by

$$M_t = \exp\left(-rt - \frac{1}{2}\lambda^2 t - \lambda W_t\right). \tag{2.2}$$

  In particular, M_t follows a log-normal distribution for all $t \in (0, T]$.

- For any contingent T-claim V_T the corresponding price at time t can be determined as

$$V_t = \frac{1}{M_t}\mathbb{E}[M_T V_T \mid \mathcal{F}_t] \quad \text{for all } t \leq T. \tag{2.3}$$

- Rearranging equation (2.1), we obtain

$$W_t = \frac{1}{\sigma}\left(\log\left(\frac{S_t}{S_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right) t\right).$$

  Now we can combine this equation with (2.2) and obtain the following representation of the state price density:

$$\begin{aligned} M_t &= \exp\left(-rt - \frac{1}{2}\lambda^2 t - \frac{\lambda}{\sigma}\left(\log\left(\frac{S_t}{S_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right) t\right)\right) \\ &= \exp\left(-rt - \frac{1}{2}\lambda^2 t + \frac{\lambda}{\sigma}\left(\mu - \frac{1}{2}\sigma^2\right) t\right)\left(\frac{S_t}{S_0}\right)^{-\frac{\lambda}{\sigma}}. \end{aligned} \tag{2.4}$$

From (2.4) we make an important observation. Since λ are σ are always greater than 0, we see that high stock prices, or a good performance of the financial market, lead to a low state price density. On the other hand, low stock prices imply an increase in the state price density. In other words, the market performs well (bad) if and only if the state price density is low (high). This observation is important for some of the following sections, where we will plot a specific terminal payoff against the state price density to see how this payoff behaves depending on the performance of the financial market.

2.2. Unrestricted Optimization Problem

In the first setting the investor is allowed to invest her initial capital without any restrictions. The only assumption made is that her trading strategy is self-financing. So if π_t denotes the fraction of wealth allocated to the risky asset and V_t is the wealth at time t, then the wealth process is going to evolve according to the following stochastic differential equation:

$$dV_t = \left(r + \pi_t(\mu - r)\right) V_t dt + \sigma \pi_t V_t dW_t \,, \quad V_0 = v. \tag{2.5}$$

The optimization problem of this investor will then have the following form:

$$\max_{\{\pi_t\}_{t\in[0,T]}} \mathbb{E}\left[\frac{V_T^{1-\gamma}}{1-\gamma}\right] \quad \text{subject to} \quad (2.5). \tag{2.6}$$

This is a classic Merton problem which can be solved through dynamic programming (see Merton (1969, 1971)). We will, however, approach this problem in a different way (see also Pliska (1986), Karatzas et al. (1987) or Cox and Huang (1989)): Let us denote by $\mathcal{C}_T = \mathcal{C}_T(v)$ the set of contingent claims payable at time T for which a self-financing, replicating investment strategy exists, given that the initial capital available is v. Then we can rewrite our dynamic problem (2.6) as a static one, in which the focus does not lie on the optimal investment strategy but instead on the terminal wealth maximizing the expected utility:

$$\max_{V_T} \mathbb{E}\left[\frac{V_T^{1-\gamma}}{1-\gamma}\right] \quad \text{subject to} \quad V_T \in \mathcal{C}_T \,.$$

This problem can be reformulated one more time by noting that all contingent claims are replicable in a complete market. As the Black-Scholes market is complete, we observe that for any random variable $V_T \in \mathcal{F}_T$ the following two conditions are equivalent:

$$V_T \in \mathcal{C}_T \quad \Leftrightarrow \quad \mathbb{E}\left[M_T V_T\right] = v.$$

Using this observation, we can now proceed in the same way as Jensen and Sørensen (2001) instead of solving the dynamic optimization problem (2.6):

- First we solve the static problem

$$\max_{V_T} \mathbb{E}\left[\frac{V_T^{1-\gamma}}{1-\gamma}\right] \quad \text{subject to} \quad \mathbb{E}\left[M_T V_T\right] = v. \tag{2.7}$$

- Having determined the optimal terminal payoff V_T^*, it is possible to determine the corresponding self-financing investment strategy. For this we will first determine the optimal wealth process $\{V_t^*\}_{t\in[0,T]}$ explicitly, and then use Itô's formula (see for example Korn (2014) or Brigo and Mercurio (2007)) in combination with (2.5). In section 2.2.2 this approach will be explained in detail.

In the following three subsections we will focus on optimization problem (2.7). First we will determine the optimal terminal wealth, followed by the corresponding investment strategy. In the last subsection we will compute the optimal level of expected utility and introduce the certainty equivalent which will be used to compare the well-being of different investors throughout the thesis.

2.2.1. Optimal Terminal Wealth

Denoting by η the Lagrangian multiplier, the first order conditions of (2.7) are given by:

$$V_T^{-\gamma} - \eta M_T = 0, \tag{2.8}$$

$$\mathbb{E}[M_T V_T] = v. \tag{2.9}$$

As a first step, we can rewrite (2.8) to obtain

$$V_T = \left(\eta M_T\right)^{-\frac{1}{\gamma}}. \tag{2.10}$$

Now we need to compute the Lagrangian multiplier η by combining this equation with (2.9):

$$\eta^{-\frac{1}{\gamma}} = \frac{v}{\mathbb{E}\left[M_T^{1-\frac{1}{\gamma}}\right]} = v \exp\left(\left(r + \frac{1}{2}\lambda^2\right)\left(1 - \frac{1}{\gamma}\right)T - \frac{1}{2}\lambda^2\left(1 - \frac{1}{\gamma}\right)^2 T\right),$$

where we have used (2.2) and that M_T follows a log-normal distribution under P. Therefore, the Lagrangian multiplier is given by

$$\eta = v^{-\gamma} \exp\left(\left(r + \frac{1}{2}\lambda^2\right)(1-\gamma)T + \frac{1}{2}\lambda^2\gamma\left(1 - \frac{1}{\gamma}\right)^2 T\right). \tag{2.11}$$

Combining this result with (2.10), we obtain the well-known solution of (2.7) in the same form as Branger et al. (2018):

$$V_T^* = (\eta M_T)^{-\frac{1}{\gamma}} = v \exp\left(\left(r + \frac{1}{2}\lambda^2\right)\left(1 - \frac{1}{\gamma}\right)T - \frac{1}{2}\lambda^2\left(1 - \frac{1}{\gamma}\right)^2 T\right) M_T^{-\frac{1}{\gamma}}. \tag{2.12}$$

Here we clearly see that the optimal terminal wealth is decreasing in the state price density M_T. Recalling the observation made from (2.4), this makes perfect sense, as low (high) values of M_T imply a good (bad) performance of the financial market and thus, a high (low) terminal wealth.

It is also possible to express V_T^* in terms of S_T (see also Branger et al. (2018)). Combining (2.4) with (2.12), we get

$$V_T^* = v \exp\left(\left(r + \frac{1}{2}\lambda^2 - \frac{1}{2}\lambda^2\left(1 - \frac{1}{\gamma}\right)^2 - \frac{\lambda}{\sigma\gamma}\left(\mu - \frac{1}{2}\sigma^2\right)\right)T\right)\left(\frac{S_T}{S_0}\right)^{\frac{\lambda}{\sigma\gamma}}.$$

Using the notation

$$m(\gamma) := \frac{\mu - r}{\gamma\sigma^2} = \frac{\lambda}{\gamma\sigma},$$

we end up with

$$V_T^* = v \exp\left((1 - m(\gamma))\left(r + \frac{1}{2}m(\gamma)\sigma^2\right)T\right)\left(\frac{S_T}{S_0}\right)^{m(\gamma)}, \tag{2.13}$$

so V_T^* is a power function of the stock price S_T. As $m(\gamma)$ is strictly positive for all $\gamma > 0$, the terminal wealth is clearly increasing in the stock price S_T. This is basically the same observation as the one we already made from (2.12). In the next section we will also see that $m(\gamma)$ is exactly the constant fraction held in the risky asset at all times. We can now distinguish two cases (the case $S_T = S_0$ can only occur with probability zero and is therefore ignored):

- $S_T/S_0 > 1$: In this case a higher proportion $m(\gamma)$ yields a higher payoff.
- $S_T/S_0 < 1$: In such a scenario a higher fraction $m(\gamma)$ leads to a lower terminal wealth.

So people allocating a larger fraction of their wealth to the risky asset face higher gains and losses than people who act more conservatively.

2.2.2. Investment Strategy

Having determined the optimal terminal payoff V_T^* explicitly, we can also determine the corresponding self-financing investment strategy. For this we first need to determine the optimal wealth process $\{V_t^*\}_{t\in[0,T]}$. This can be done using (2.3):

$$\begin{aligned}
V_t^* &= \mathbb{E}\left[\frac{M_T}{M_t} V_T^* \;\middle|\; \mathcal{F}_t\right] \\
&= (\eta M_t)^{-\frac{1}{\gamma}}\, \mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{1-\frac{1}{\gamma}} \;\middle|\; \mathcal{F}_t\right] \\
&= (\eta M_t)^{-\frac{1}{\gamma}}\, \mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{1-\frac{1}{\gamma}}\right],
\end{aligned}$$

where the last equation results from the independent increments of the Brownian motion. Using equation (2.2), we first observe that

$$\left(\frac{M_T}{M_t}\right)^{1-\frac{1}{\gamma}} = \exp\left(-r(T-t)\left(1-\frac{1}{\gamma}\right) - \frac{1}{2}\lambda^2(T-t)\left(1-\frac{1}{\gamma}\right) - \lambda\left(1-\frac{1}{\gamma}\right)(W_T - W_t)\right).$$

As $W_T - W_t$ is normally distributed, we get

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{1-\frac{1}{\gamma}}\right] \\
&= \exp\left(-r(T-t)\left(1-\frac{1}{\gamma}\right) - \frac{1}{2}\lambda^2(T-t)\left(1-\frac{1}{\gamma}\right) + \frac{1}{2}\lambda^2\left(1-\frac{1}{\gamma}\right)^2(T-t)\right). \qquad (2.14)
\end{aligned}$$

Now we can use (2.11) to get

$$\eta^{-\frac{1}{\gamma}}\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{1-\frac{1}{\gamma}}\right] = v\exp\left(rt\left(1-\frac{1}{\gamma}\right) + \frac{1}{2}\lambda^2 t\left(1-\frac{1}{\gamma}\right) - \frac{1}{2}\lambda^2\left(1-\frac{1}{\gamma}\right)^2 t\right). \qquad (2.15)$$

Using again (2.2), we obtain

$$M_t^{-\frac{1}{\gamma}} = \exp\left(\frac{r}{\gamma}t + \frac{1}{2}\frac{\lambda^2}{\gamma}t + \frac{\lambda}{\gamma}W_t\right). \qquad (2.16)$$

Finally, we obtain V_t^* as the product of (2.15) and (2.16) (see also Jensen and Sørensen (2001)):

$$\begin{aligned}
V_t^* &= \eta^{-\frac{1}{\gamma}}\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{1-\frac{1}{\gamma}}\right]M_t^{-\frac{1}{\gamma}} \\
&= v\exp\left(rt + \lambda^2 t\left(\frac{1}{\gamma} - \frac{1}{2\gamma^2}\right) + \frac{\lambda}{\gamma}W_t\right). \qquad (2.17)
\end{aligned}$$

Having determined V_t^*, we can now use Itô's lemma to determine the dynamics of the wealth process. Define

$$f(t,z) = v\exp\left(rt + \lambda^2 t\left(\frac{1}{\gamma} - \frac{1}{2\gamma^2}\right) + \frac{\lambda}{\gamma}z\right),$$

clearly a twice continuously differentiable function. Then we have

$$\begin{aligned}
\frac{\partial f}{\partial t}(t,V_t^*) &= \left(r + \lambda^2\left(\frac{1}{\gamma} - \frac{1}{2\gamma^2}\right)\right)V_t^*, \\
\frac{\partial f}{\partial z}(t,V_t^*) &= \frac{\lambda}{\gamma}V_t^*, \text{ and} \\
\frac{\partial^2 f}{\partial z^2}(t,V_t^*) &= \frac{\lambda^2}{\gamma^2}V_t^*.
\end{aligned}$$

Therefore, the wealth dynamics can be described as

$$\begin{aligned} dV_t^* &= \left(r + \lambda^2\left(\frac{1}{\gamma} - \frac{1}{2\gamma^2}\right)\right) V_t^* dt + \frac{\lambda}{\gamma} V_t^* dW_t + \frac{\lambda^2}{2\gamma^2} V_t^* dt \\ &= \left(r + \frac{\lambda^2}{\gamma}\right) V_t^* dt + \frac{\lambda}{\gamma} V_t^* dW_t \\ &= \left(r + (\mu - r)\frac{\mu - r}{\sigma^2\gamma}\right) V_t^* dt + \frac{\mu - r}{\sigma^2\gamma}\sigma V_t^* dW_t\,. \end{aligned}$$

Comparing this stochastic differential equation with the process given in (2.5), we see that the investment strategy is given by

$$\pi_t = \pi = m(\gamma) = \frac{\mu - r}{\gamma\sigma^2} \tag{2.18}$$

which is called the Merton portfolio (Merton (1971)). So the optimal investment strategy is to hold a constant fraction in the risky asset over time. This fraction is smaller for more risk averse investors and larger for individuals with a low risk aversion.

2.2.3. Expected Utility and Certainty Equivalent

In this section we first want to determine the indirect utility process for each point in time t which is defined by

$$J_t(v, T, \gamma) = \mathbb{E}\left[u(V_T^*) \mid \mathcal{F}_t\right] = \mathbb{E}\left[\frac{(V_T^*)^{1-\gamma}}{1-\gamma} \;\middle|\; \mathcal{F}_t\right].$$

For each t this is the optimal level of expected utility based on the information available at time t. In particular, for $t = 0$ we obtain the optimal level of expected utility. To compute this process we first need to observe from (2.12) that

$$(V_T^*)^{1-\gamma} = (\eta M_T)^{-\frac{1-\gamma}{\gamma}}\,.$$

As M_t is $\mathcal{F}_t$-measurable and using the independent increments of the Brownian motion, we can rewrite the indirect utility process in the following way:

$$\begin{aligned} J_t(v,T,\gamma) &= \frac{1}{1-\gamma}\left(\eta M_t\right)^{-\frac{1-\gamma}{\gamma}} \mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{-\frac{1-\gamma}{\gamma}} \middle| \mathcal{F}_t\right] \\ &= \frac{1}{1-\gamma}\left(\eta M_t\right)^{1-\frac{1}{\gamma}} \mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{1-\frac{1}{\gamma}}\right]. \end{aligned} \tag{2.19}$$

Using equations (2.11) and (2.14), we get

$$\begin{aligned} &\eta^{-\frac{1-\gamma}{\gamma}} \mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{-\frac{1-\gamma}{\gamma}}\right] \\ &= v^{1-\gamma} \exp\left(\left(r+\frac{1}{2\gamma}\lambda^2\right)T - \frac{1}{\gamma}\left(r+\frac{1}{2}\lambda^2\right)t - \frac{1}{2\gamma^2}(1-\gamma)\lambda^2 t\right)^{1-\gamma}. \end{aligned} \tag{2.20}$$

Combining (2.19) and (2.20), we obtain the indirect utility process as

$$J_t(v,T,\gamma) = \frac{v^{1-\gamma}}{1-\gamma}\exp\left(\left(r+\frac{\lambda^2}{2\gamma}\right)T - \frac{1}{\gamma}rt - \frac{\lambda^2}{2\gamma^2}t\right)^{1-\gamma} M_t^{-\frac{1-\gamma}{\gamma}}. \tag{2.21}$$

Therefore, the optimal level of expected utility has the following form (see also Jensen and Sørensen (2001)):

$$J_0(v,T,\gamma) = \frac{\left(v\exp\left(\left(r+\frac{\lambda^2}{2\gamma}\right)T\right)\right)^{1-\gamma}}{1-\gamma}. \tag{2.22}$$

Throughout the thesis we will use the certainty equivalent return to compare the well-being of different investors. The certainty equivalent return can be computed from an investor's expected utility and will allow us to compare the well-being of individuals with different levels of risk aversion and different utility functions. We will follow the definition given in Branger et al. (2018): For a terminal wealth V_T the certainty equivalent wealth is given by

$$\begin{aligned} u\left(\mathrm{CE}(V_T)\right) &= \mathbb{E}[u(V_T)] \quad \text{or, equivalently,} \\ \mathrm{CE}(V_T) &= \mathbb{E}\left[V_T^{1-\gamma}\right]^{\frac{1}{1-\gamma}}. \end{aligned} \tag{2.23}$$

This is the deterministic level of wealth that would lead to the same level of expected utility as the random wealth V_T. The certainty equivalent return is then defined as the deterministic rate of return $y(V_T)$ that yields the same expected utility as the random wealth V_T would:

$$u\left(ve^{y(V_T)T}\right) = \mathbb{E}[u(V_T)] \quad \text{or, equivalently,}$$
$$y(V_T) = \frac{1}{T}\log\left(\frac{\mathrm{CE}(V_T)}{v}\right). \tag{2.24}$$

Using (2.22), we see that the certainty equivalent return for $V_T = V_T^*$ is given by

$$\begin{aligned} y^* = y\left(V_T^*\right) &= \frac{1}{T}\log\left(\frac{\mathrm{CE}\left(V_T^*\right)}{v}\right) \\ &= \frac{1}{T}\log\left(\frac{\left((1-\gamma)J_0(v,T,\gamma)\right)^{\frac{1}{1-\gamma}}}{v}\right) \\ &= r + \frac{1}{2}\frac{(\mu-r)^2}{\gamma\sigma^2}. \end{aligned} \tag{2.25}$$

2.3. Optimization Problem with Interest Rate Guarantee

In practical applications a lot of investors also like to meet a certain, predetermined interest rate guarantee to make sure that their invested wealth is protected from bad scenarios of the financial market. This situation is typical for insurance companies, as insurers often include guarantees in their products to ensure that a minimum benefit is paid to the customer in the end. For example, in unit-linked life insurance products guarantees are often included to protect the customers from a bad performance of the stock market. The purpose of this section is, therefore, to include an interest rate guarantee in the optimization problem (2.7). As a consequence, the investor considered is now no longer able to choose her investment strategy without any restrictions. Let $g < r$ denote the guaranteed rate of return that needs to be achieved by the investor. Then the new optimization problem can be written as

$$\max_{\widetilde{V}_T}\mathbb{E}\left[\frac{\widetilde{V}_T^{1-\gamma}}{1-\gamma}\right] \quad \text{subject to} \quad \mathbb{E}\left[M_T\widetilde{V}_T\right] = v, \text{ and}$$
$$\widetilde{V}_T \geq ve^{gT}.$$

This problem is also known in the literature as the optimal investment problem under portfolio insurance and has been studied extensively already: Grossman and Vila (1989) consider this problem in a discrete-time model. In Jensen and Sørensen (2001) this exact problem is analyzed in a more general continuous-time financial market with more than one risky asset. The approach in this article is basically the same as here since the price processes of the assets are considered to be a given input on a person's investment decision. Grossman and Zhou (1996) and Basak (1995), on the other hand, focus on the effects of portfolio insurance on market equilibrium and asset price processes. A more recent article related to this field of research would also be Gabih et al. (2009). Here no lower bound is put on the terminal wealth but instead only on the expected loss.
In the following we will further investigate this problem. We will determine the optimal terminal wealth, the corresponding investment strategy and the resulting expected utility along with the certainty equivalent return. We will start with the terminal wealth in the next section.

2.3.1. Optimal Terminal Wealth

We will follow Jensen and Sørensen (2001) to determine the optimal terminal payoff of the portfolio insurance problem. Denoting by η_1 and η_2 the Lagrangian multipliers, the first order conditions of the problem can be obtained as:

$$\begin{aligned} \widetilde{V}_T^{-\gamma} - \eta_1 M_T - \eta_2 &= 0, \\ \eta_2 \left(\widetilde{V}_T - ve^{gT} \right) &= 0. \end{aligned}$$

Here we have to distinguish two cases:

- $\eta_2 = 0$: In this case the guarantee is not effective and we end up with $\widetilde{V}_T^* = (\eta_1 M_T)^{-\frac{1}{\gamma}}$.
- $\eta_2 > 0$: If this is the case, the guarantee is effective and we will get $\widetilde{V}_T^* = ve^{gT}$.

So the solution can be written as

$$\widetilde{V}_T^* = \max \left\{ (\eta_1 M_T)^{-\frac{1}{\gamma}}, ve^{gT} \right\}.$$

For the case where $(\eta_1 M_T)^{-\frac{1}{\gamma}} > ve^{gT}$, that is, when the guarantee is not effective, the optimal payoff is proportional to $M_T^{-\frac{1}{\gamma}}$. As a consequence, it is also proportional to V_T^*, as can be seen in (2.12). Therefore, it can be written as

$$\widetilde{V}_T^* = \max\left\{ve^{gT}, xV_T^*\right\}, \tag{2.26}$$

where $x = \left(\frac{\eta_1}{\eta}\right)^{-\frac{1}{\gamma_i}}$ has to lie in the interval $(0,1)$, as $\eta_1 > \eta$ needs to hold necessarily. This can be seen as follows: If we assumed that $\eta_1 \leq \eta$, we would obtain $V_T^* \leq \widetilde{V}_T^*$ a.s. where the strict inequality holds with positive probability. However, this would clearly be a contradiction to the budget constraint. The case $x = 0$ would also lead to contradiction in the budget constraint since we assume $g < r$.

The question is now how this x can be determined. In order to answer this question we rewrite (2.26) as follows:

$$\widetilde{V}_T^* = xV_T^* + \max\left\{ve^{gT} - xV_T^*, 0\right\}. \tag{2.27}$$

The second term in this equation is a put option with underlying xV_T^* and strike price ve^{gT}. Let $P_t = P_t(v, g, T, x)$ denote the price of this put option at time t, then for $t = 0$ we obtain the condition

$$v = xv + P_0\,. \tag{2.28}$$

Our goal is now to determine the price of the put option P_0. We will do this by computing the wealth process $\{\widetilde{V}_t^*\}_{t\in[0,T]}$. From this we can then easily deduce the value of the put option P_t and, in particular, P_0. We start with the following observation resulting from (2.3):

$$\begin{aligned} M_t\widetilde{V}_t^* &= \mathbb{E}\left[M_T\widetilde{V}_T^* \mid \mathcal{F}_t\right] \\ &= \mathbb{E}\left[xM_TV_T^* + M_T\max\{0, ve^{gT} - xV_T^*\} \mid \mathcal{F}_t\right]. \end{aligned}$$

It follows that

$$\begin{aligned}\widetilde{V}_t^* &= xV_t^* + \mathbb{E}\left[\frac{M_T}{M_t}\max\{0, ve^{gT} - xV_T^*\} \;\Big|\; \mathcal{F}_t\right] \\ &= xV_t^* + x\mathbb{E}\left[\frac{M_T}{M_t}\left(\frac{ve^{gT}}{x} - V_T^*\right)\mathbb{1}_{\left\{\frac{ve^{gT}}{x} > V_T^*\right\}} \;\Bigg|\; \mathcal{F}_t\right] \\ &= xV_t^* + x\,\underbrace{\mathbb{E}\left[\frac{M_T}{M_t}\frac{ve^{gT}}{x}\mathbb{1}_{\left\{\frac{ve^{gT}}{x} > V_T^*\right\}} \;\Bigg|\; \mathcal{F}_t\right]}_{\text{(I)}} - x\,\underbrace{\mathbb{E}\left[\frac{M_T}{M_t}V_T^*\mathbb{1}_{\left\{\frac{ve^{gT}}{x} > V_T^*\right\}} \;\Bigg|\; \mathcal{F}_t\right]}_{\text{(II)}}. \end{aligned} \tag{2.29}$$

Now we can use (2.12) to determine the two conditional expectations. For the first one this can be done as follows:

$$\begin{aligned}\text{(I)} &= \frac{ve^{gT}}{x}\mathbb{E}\left[\frac{M_T}{M_t}\mathbb{1}_{\left\{\frac{ve^{gT}}{x} > (\eta M_T)^{-\frac{1}{\gamma}}\right\}} \;\Bigg|\; \mathcal{F}_t\right] \\ &= \frac{ve^{gT}}{x}\mathbb{E}\left[\frac{M_T}{M_t}\mathbb{1}_{\left\{\frac{1}{\eta}\left(\frac{ve^{gT}}{xM_t^{-1/\gamma}}\right)^{-\gamma} < \frac{M_T}{M_t}\right\}} \;\Bigg|\; \mathcal{F}_t\right]. \end{aligned} \tag{2.30}$$

We define

$$X := \frac{W_T - W_t}{\sqrt{T-t}}\,.$$

This is a standard normally distributed random variable under P that is independent of $\mathcal{F}_t$. Now we know from (2.2) that

$$\begin{aligned}\frac{M_T}{M_t} &= \exp\left(-r(T-t) - \frac{1}{2}\lambda^2(T-t) - \lambda(W_T - W_t)\right) \\ &= \exp\left(-r(T-t) - \frac{1}{2}\lambda^2(T-t) - \lambda\sqrt{T-t}X\right),\end{aligned}$$

and thus we observe that

$$\frac{1}{\eta}\left(\frac{ve^{gT}}{xM_t^{-\frac{1}{\gamma}}}\right)^{-\gamma} < \frac{M_T}{M_t}$$

holds if and only if

$$-r(T-t) - \frac{1}{2}\lambda^2(T-t) - \lambda\sqrt{T-t}X > -\gamma \log\left(\frac{ve^{gT}}{xM_t^{-\frac{1}{\gamma}}}\right) - \log\eta\,.$$

This inequality is equivalent to

$$\begin{aligned}
X &< \frac{\gamma\log\left(\frac{ve^{gT}}{xM_t^{-\frac{1}{\gamma}}}\right) + \log\eta - r(T-t) - \frac{1}{2}\lambda^2(T-t)}{\lambda\sqrt{T-t}} \\
&= \frac{\gamma\left(\log v + gT - \log x\right) + \log M_t + \log\eta - r(T-t) - \frac{1}{2}\lambda^2(T-t)}{\lambda\sqrt{T-t}} \\
&=: D_1(t)\,.
\end{aligned}$$

This expression can be further simplified using (2.2) and (2.11):

$$D_1(t) = \frac{\gamma((g-r)T - \log x) + \left(\frac{1}{2\gamma} - 1\right)\lambda^2 T - \lambda W_t}{\lambda\sqrt{T-t}}\,.$$

Since M_t is $\mathcal{F}_t$-measurable and using the independent increments of the Brownian motion, we can rewrite (2.30) as follows:

$$\begin{aligned}
\text{(I)} &= ve^{gT}\int_{-\infty}^{D_1(t)} \exp\left(-r(T-t) - \frac{1}{2}\lambda^2(T-t) - z\lambda\sqrt{T-t}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= ve^{gT}e^{-r(T-t)}\Phi\left(D_1(t) + \lambda\sqrt{T-t}\right).
\end{aligned}$$

Here $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. Now that the first conditional expectation in (2.29) is computed, we can move on to the second one:

$$\begin{aligned}
\text{(II)} &= x\eta^{-\frac{1}{\gamma}}\mathbb{E}\left[\frac{M_T}{M_t}M_t^{-\frac{1}{\gamma}}\left(\frac{M_T}{M_t}\right)^{-\frac{1}{\gamma}}\mathbb{1}_{\{X<D_1(t)\}}\,\middle|\,\mathcal{F}_t\right] \\
&= x\left(\eta M_t\right)^{-\frac{1}{\gamma}}\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{1-\frac{1}{\gamma}}\mathbb{1}_{\{X<D_1(t)\}}\,\middle|\,\mathcal{F}_t\right].
\end{aligned}$$

Since

$$\left(\frac{M_T}{M_t}\right)^{1-\frac{1}{\gamma}} = \exp\left(-r(T-t)\left(1-\frac{1}{\gamma}\right) - \frac{1}{2}\lambda^2(T-t)\left(1-\frac{1}{\gamma}\right) - \lambda\sqrt{T-t}\left(1-\frac{1}{\gamma}\right)X\right)$$

we obtain the following representation of (II):

$$\begin{aligned}(\mathrm{II}) &= x\left(\eta M_t\right)^{-\frac{1}{\gamma}} \int_{-\infty}^{D_1(t)} e^{\left(1-\frac{1}{\gamma}\right)\left(-r(T-t)-\frac{1}{2}\lambda^2(T-t)-z\lambda\sqrt{T-t}\right)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= x\left(\eta M_t\right)^{-\frac{1}{\gamma}} e^{-r(T-t)\left(1-\frac{1}{\gamma}\right)} e^{-\frac{1}{2}\lambda^2(T-t)\left(1-\frac{1}{\gamma}\right)\frac{1}{\gamma}} \Phi\left(D_2(t)\right), \end{aligned} \tag{2.31}$$

where

$$D_2(t) = D_1(t) + \lambda\sqrt{T-t}\left(1-\frac{1}{\gamma}\right).$$

Now we can use (2.11) again to simplify (2.31) by observing

$$\eta^{-\frac{1}{\gamma}} e^{-r(T-t)\left(1-\frac{1}{\gamma}\right)} e^{-\frac{1}{2}\lambda^2(T-t)\left(1-\frac{1}{\gamma}\right)\frac{1}{\gamma}} = v\exp\left(\left(1-\frac{1}{\gamma}\right)\left(rt+\frac{1}{2}\lambda^2 t\frac{1}{\gamma}\right)\right).$$

As a consequence, (2.31) can be further simplified:

$$(\mathrm{II}) = xv\exp\left(\left(1-\frac{1}{\gamma}\right)\left(rt+\frac{1}{2}\lambda^2 t\frac{1}{\gamma}\right)\right)\Phi\left(D_2(t)\right) M_t^{-\frac{1}{\gamma}}.$$

Here we can use (2.17) to simplify (II) even further:

$$\begin{aligned}(\mathrm{II}) &= xv\Phi\left(D_2(t)\right)\exp\left(\left(1-\frac{1}{\gamma}\right)\left(rt+\frac{1}{2}\lambda^2 t\frac{1}{\gamma}\right)+\frac{1}{\gamma}\left(rt+\frac{1}{2}\lambda^2 t+\lambda W_t\right)\right) \\ &= xV_t^*\Phi\left(D_2(t)\right), \end{aligned}$$

where the last equality is obtained from (2.17). Now we can put everything together and obtain the complete process as

$$\widetilde{V}_t^* = xV_t^* + ve^{gT}e^{-r(T-t)}\Phi\left(D_1(t)+\lambda\sqrt{T-t}\right) - xV_t^*\Phi\left(D_2(t)\right). \tag{2.32}$$

In particular, the price of the put option at any time t is given by

$$P_t = ve^{gT}e^{-r(T-t)}\Phi\left(D_1(t)+\lambda\sqrt{T-t}\right) - xV_t^*\Phi\left(D_2(t)\right).$$

For $t=0$ we obtain the same result as Jensen and Sørensen (2001):

$$P_0 = ve^{gT}e^{-rT}\Phi\left(d_1+\lambda\sqrt{T}\right) - xv\Phi\left(d_2\right),$$

where

$$d_1 = D_1(0) = \frac{\gamma\left((g-r)T - \log(x)\right)}{\lambda\sqrt{T}} + \left(\frac{1}{2\gamma} - 1\right)\lambda\sqrt{T}\,,$$
$$d_2 = D_2(0) = d_1 + \left(1 - \frac{1}{\gamma}\right)\lambda\sqrt{T}\,.$$

Since we know from (2.28) that

$$(1-x)v = P_0\,,$$

we obtain

$$1 - x = e^{gT}e^{-rT}\Phi\left(d_1 + \lambda\sqrt{T}\right) - x\Phi\left(d_2\right)$$

which is equivalent to

$$1 = x\Phi\left(-d_2\right) + e^{(g-r)T}\Phi\left(d_1 + \lambda\sqrt{T}\right)\,. \tag{2.33}$$

We need to determine x numerically from this equation. Since the right-hand-side of (2.33) is increasing in x (for a proof see Jensen and Sørensen (2001)), we could for example use the bisection method to obtain x numerically. The corresponding R code can be found in Appendix B.

2.3.2. Investment Strategy

Having determined the wealth process $\widetilde{V}_t^*$ in (2.32), we can now determine the self-financing investment strategy $\{\widetilde{\pi}_t\}_{t\in[0,T]}$ used for obtaining the terminal payoff (2.26). We will do this in a similar way as in Section 2.2.2 using Itô's formula. We define the following twice continuously differentiable function:

$$f(t, W_t) = xV_t^* + ve^{gT}e^{-r(T-t)}\Phi\left(D_1(t) + \lambda\sqrt{T-t}\right) - xV_t^*\Phi\left(D_2(t)\right).$$

Then we have

$$\frac{\partial f}{\partial W_t}(t, W_t) = x\frac{\lambda}{\gamma}V_t^* - ve^{gT}e^{-r(T-t)}\varphi\left(D_1(t) + \lambda\sqrt{T-t}\right)\frac{1}{\sqrt{T-t}} - x\frac{\lambda}{\gamma}V_t^*\Phi\left(D_2(t)\right) + xV_t^*\varphi\left(D_2(t)\right)\frac{1}{\sqrt{T-t}},$$

where

$$\varphi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$$

is the density of the standard normal distribution. Taking a look at the stochastic differential equation in (2.5) and Itô's formula, we observe that

$$\frac{\partial f}{\partial W_t}(t, W_t)dW_t = \sigma\widetilde{\pi}_t\widetilde{V}_t^* dW_t$$

which leads to

$$\widetilde{\pi}_t = \frac{\frac{\partial f}{\partial W_t}(t, W_t)}{\sigma\widetilde{V}_t^*}. \tag{2.34}$$

2.3.3. Expected Utility and Certainty Equivalent

Having computed x and $\widetilde{V}_T^*$, it is also possible to determine the indirect utility process explicitly. Using (2.26), it can be obtained as follows:

$$\begin{aligned}\widetilde{J}_t(v, T, \gamma) &= \mathbb{E}\left[\frac{\left(\widetilde{V}_T^*\right)^{1-\gamma}}{1-\gamma}\,\middle|\,\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\frac{\left(ve^{gT}\right)^{1-\gamma}}{1-\gamma}\mathbb{1}_{\left\{ve^{gT} > xV_T^*\right\}}\,\middle|\,\mathcal{F}_t\right] + \mathbb{E}\left[\frac{\left(xV_T^*\right)^{1-\gamma}}{1-\gamma}\mathbb{1}_{\left\{ve^{gT} \le xV_T^*\right\}}\,\middle|\,\mathcal{F}_t\right].\end{aligned}$$

The first expectation can be obtained in a similar way as in Section 2.3.1 and is given by

$$\mathbb{E}\left[\frac{\left(ve^{gT}\right)^{1-\gamma}}{1-\gamma}\mathbb{1}_{\left\{ve^{gT} > xV_T^*\right\}}\,\middle|\,\mathcal{F}_t\right] = u\left(ve^{gT}\right)\Phi\left(D_1(t)\right). \tag{2.35}$$

The second conditional expectation can be determined as follows:

$$\begin{aligned}
&\mathbb{E}\left[\frac{(xV_T^*)^{1-\gamma}}{1-\gamma}\mathbb{1}_{\left\{ve^{gT}\le xV_T^*\right\}}\;\middle|\;\mathcal{F}_t\right] \\
&= \frac{x^{1-\gamma}}{1-\gamma}\left(\eta M_t\right)^{-\frac{1-\gamma}{\gamma}}\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{-\frac{1-\gamma}{\gamma}}\mathbb{1}_{\left\{ve^{gT}\le xV_T^*\right\}}\;\middle|\;\mathcal{F}_t\right],
\end{aligned} \tag{2.36}$$

where

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{-\frac{1-\gamma}{\gamma}}\mathbb{1}_{\left\{ve^{gT}\le xV_T^*\right\}}\;\middle|\;\mathcal{F}_t\right] &= \mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{1-\frac{1}{\gamma}}\mathbb{1}_{\left\{ve^{gT}\le xV_T^*\right\}}\;\middle|\;\mathcal{F}_t\right] \\
&= e^{-r(T-t)\left(1-\frac{1}{\gamma}\right)}e^{-\frac{1}{2}\lambda^2(T-t)\left(1-\frac{1}{\gamma}\right)\frac{1}{\gamma}}\left(1-\Phi\left(D_2(t)\right)\right)
\end{aligned}$$

has already been computed in Section 2.3.1. Now we see that

$$\eta^{-\frac{1-\gamma}{\gamma}} = v^{1-\gamma}\exp\left(\left(r+\frac{1}{2}\lambda^2\right)(1-\gamma)T+\frac{1}{2}\lambda^2\left(1-\frac{1}{\gamma}\right)^2\gamma T\right)^{1-\frac{1}{\gamma}}.$$

Multiplying the previous two expressions, we obtain

$$\begin{aligned}
&\eta^{-\frac{1-\gamma}{\gamma}}e^{-r(T-t)\left(1-\frac{1}{\gamma}\right)}e^{-\frac{1}{2}\lambda^2(T-t)\left(1-\frac{1}{\gamma}\right)\frac{1}{\gamma}}\left(1-\Phi\left(D_2(t)\right)\right) \\
&= v^{1-\gamma}\exp\left(\left(r+\frac{1}{2\gamma}\lambda^2\right)T-\frac{1}{\gamma}rt-\frac{\lambda^2}{2\gamma^2}t\right)^{1-\gamma}\left(1-\Phi\left(D_2(t)\right)\right).
\end{aligned} \tag{2.37}$$

Using (2.37) and (2.21), we can simplify (2.36) in the following way:

$$\begin{aligned}
&\frac{x^{1-\gamma}}{1-\gamma}\left(\eta M_t\right)^{-\frac{1-\gamma}{\gamma}}\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{-\frac{1-\gamma}{\gamma}}\mathbb{1}_{\left\{ve^{gT}\le xV_T^*\right\}}\;\middle|\;\mathcal{F}_t\right] \\
&= \frac{(xv)^{1-\gamma}}{1-\gamma}M_t^{-\frac{1-\gamma}{\gamma}}\exp\left(\left(r+\frac{1}{2\gamma}\lambda^2\right)T-\frac{1}{\gamma}rt-\frac{\lambda^2}{2\gamma^2}t\right)^{1-\gamma}\left(1-\Phi\left(D_2(t)\right)\right) \\
&= J_t(xv,T,\gamma)(1-\Phi\left(D_2(t)\right)).
\end{aligned} \tag{2.38}$$

Finally, adding (2.35) and (2.38), we can determine the complete indirect utility process as

$$\widetilde{J}_t(v,T,\gamma) = u\left(ve^{gT}\right)\Phi(D_1(t)) + J_t(xv,T,\gamma)(1-\Phi\left(D_2(t)\right)).$$

In particular, the optimal level of expected utility is given by (see also Jensen and Sørensen (2001)):

$$\widetilde{J}_0(v,T,\gamma) = u\left(ve^{gT}\right)\Phi(d_1) + J_0(xv,T,\gamma)(1-\Phi\left(d_2\right)).$$

From this we can easily compute the certainty equivalent wealth. It will be denoted by $\widetilde{\mathrm{CE}}$ and is given by

$$\begin{aligned}\widetilde{\mathrm{CE}} &= \left((1-\gamma)\widetilde{J}_0(v,T,\gamma)\right)^{\frac{1}{1-\gamma}} \\ &= v\left(\left(e^{gT}\right)^{1-\gamma}\Phi\left(d_1\right) + x^{1-\gamma}\exp\left((1-\gamma)\left(r+\frac{\lambda^2}{2\gamma}\right)T\right)\left(1-\Phi\left(d_2\right)\right)\right)^{\frac{1}{1-\gamma}}.\end{aligned}$$

Similarly, we will denote the certainty equivalent return by $\widetilde{y}$. This quantity can be computed as follows:

$$\begin{aligned}\widetilde{y} &= \frac{1}{T}\log\left(\frac{\widetilde{\mathrm{CE}}}{v}\right) \\ &= \frac{1}{T(1-\gamma)}\log\left(\left(e^{gT}\right)^{1-\gamma}\Phi\left(d_1\right) + x^{1-\gamma}\exp\left(\left(r+\frac{\lambda^2}{2\gamma}\right)T\right)^{1-\gamma}\left(1-\Phi\left(d_2\right)\right)\right).\end{aligned} \quad (2.39)$$

Now that we have analyzed the individual optimization problems with and without guarantee, we can compare them to each other and the effects of the guarantee can be determined. This will be done in the following section.

2.4. Comparison

In this section we will compare different characteristics of the individual optimization problem with and without interest rate guarantee. We will consider the following example which will be used throughout the thesis from now on:

- The constant risk-free interest rate is $r = 1.5\%$,
- the instantaneous rate of return of the risky asset is $\mu = 6\%$,
- the volatility of this asset is $\sigma = 11\%$, and
- the maturity is $T = 1$.

All the remaining parameters will be specified separately for each example. In this section we will consider an investor with a relative risk aversion $\gamma = 4$ and an initial wealth $v = 1$. The guarantee is given by $g = 0.5\%$.
We will compare the three main characteristics from the previous two sections, the terminal payoff, the self-financing investment strategy and the certainty equivalent return. The R code used for all these computations can be found in Appendix B.
First we have a look at the terminal payoff obtained by this investor depending on the performance of the stock market in Figure 2.1. We plot the terminal payoff against the state price density M_T. We clearly see the effect of the guarantee in this plot. The

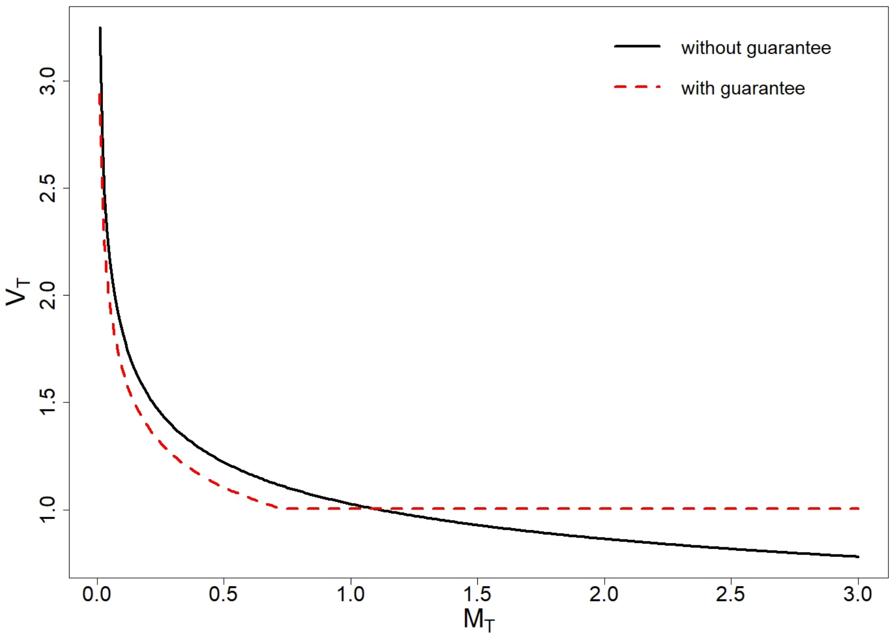


Figure 2.1.: Comparison between individual terminal payoff with and without guarantee, where $\gamma = 4$, $v = 1$, $g = 0.005$.

guarantee prevents the investor from bad performances of the stock market which occur if and only if M_T gets large. The price for this guarantee is, however, that more initial wealth needs to be allocated to the risk-free asset earning a lower rate of return. This prevents the individual from obtaining higher returns when the stock market performs well, that is, when M_T is small.
In Figure 2.2 we compare the Merton portfolio from (2.18) to one path of the investment strategy given in (2.34). For the portfolio of the optimization problem with guarantee we first need to simulate one path of the Brownian motion $\{W_t\}_{t\in[0,T]}$. Having chosen a discretization of the time axis $t_1 = \Delta t, \dots, t_i = t_{i-1} + \Delta t, \dots, t_m = T$, we can use that

a Brownian motion has independent and normally distributed increments to do this (see also Kroese et al. (2013) or Korn (2014)):

1. Initialize T, m, $\Delta t = T/m$.
2. Simulate m independent standard normally distributed random variables $X^i, i = 1, \ldots, m$.
3. Set $W_{t_i} = \sum_{j=1}^{i} \sqrt{\Delta t} X^j$.

The proportion of wealth allocated to the risky asset then varies over time and depends on the performance of the financial market at any point in time t. We clearly see the

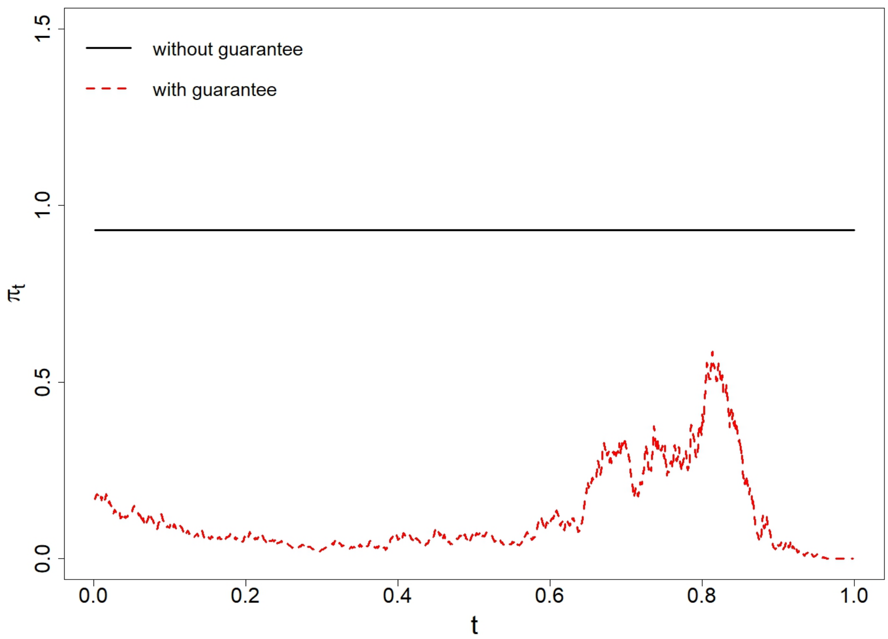


Figure 2.2.: Comparison between the fractions allocated to the risky asset with and without guarantee, where $\gamma = 4$, $v = 1$, $g = 0.005$.

difference in the investment strategies in this plot. Although $\tilde{\pi}_t$ is not constant, it is always below π_t. The reason for this has already been explained: In order to meet the prescribed guarantee more wealth needs to be invested in the risk-free asset than without the restriction where the investor can decide for herself how to allocate the wealth. As a consequence, the fraction held in the risky asset is always below the Merton portfolio.

However, from these two plots it is not clear yet whether the investor suffers a loss in utility from the guarantee and how large this loss is. Therefore, we will now have a look at the certainty equivalent returns. We can choose a discretization $\gamma_1, \ldots, \gamma_n$ for the possible values of γ and then plot the certainty equivalent return y^* from (2.25) and $\tilde{y}$ from (2.39) against these discretized values of γ. Here we use $n = 100$ values for γ that are spread equally in the interval $[1/2, 10]$, that is $\gamma_i = 1/2 + \frac{9.5(i-1)}{99}$. All the remaining values between γ_{i-1} and γ_i are then obtained by linear interpolation for all $i = 2, \ldots, n$. The results of this procedure are given in Figure 2.3 where we compare the certainty equivalent returns with and without guarantee. In this plot we observe the already well-known result that the

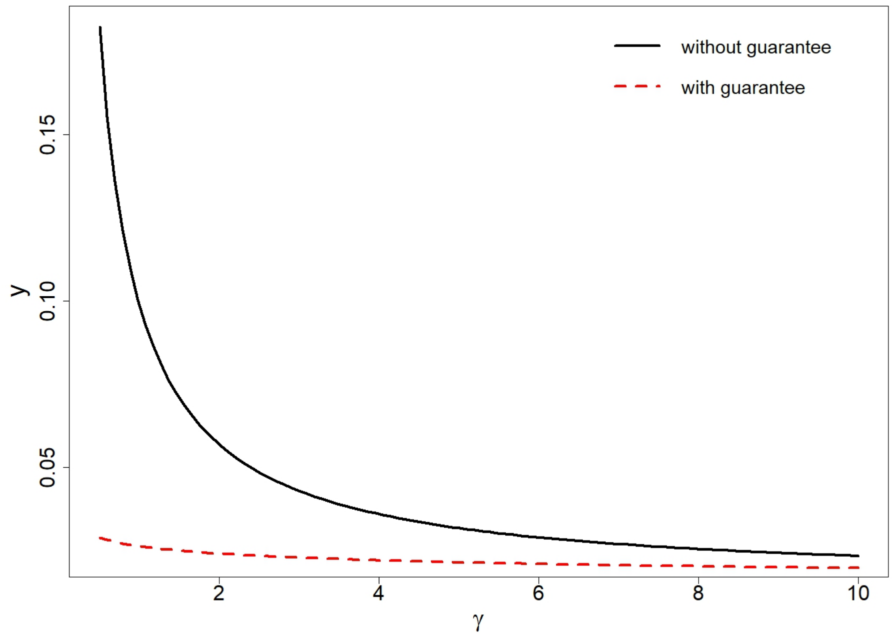


Figure 2.3.: Comparison of the certainty equivalent returns with and without guarantee where $g = 0.005$.

guarantee leads to a loss, especially for the least risk averse investors (see also Jensen and Sørensen (2001)). They could achieve a much higher rate of return if they were allowed to invest in any way they wanted. However, the guarantee forces them to invest in the risk-free asset with a low rate of return which creates a huge loss in the certainty equivalent return. For the more risk averse investors, however, the situation is not that drastic. Since highly risk averse people tend to invest most of their capital in risk-free assets, even if no guarantee is imposed, their losses in the certainty equivalent return are much smaller than the losses of the less risk averse people.

3. Investment Pools

From now on we will consider a collective of n investors, each starting with an initial wealth v_i, $i = 1, \ldots, n$. The ith individual will be modeled by a CRRA utility function $u_i(x) = \frac{x^{1-\gamma_i}}{1-\gamma_i}$ as in the previous chapter. All the investors considered will be tied together in their investment decisions, as it is the case, for example, when dealing with defined contribution pension schemes, as mentioned in the introduction.
There are two questions that need to be addressed when such a collective of investors is considered:

- How should the total initial wealth $v = \sum_{i=1}^{n} v_i$ be invested?
- How should the total terminal wealth V_T obtained from this investment strategy be split among the participants?

We want to find solutions to these questions such that all the participants obtain a certainty equivalent return as close to the individual optimum $y_i^* = r + \frac{\mu - r}{2\gamma_i \sigma^2}$ from (2.25) as possible. However, we would also like all the procedures to be easy to communicate to the members of the collective.
Let us first consider the joint investment problem without guarantees. Clearly, there exists an approach that manages to return to each individual her optimal unrestricted terminal wealth: This approach would be to hold the Merton portfolio given in (2.18) for each investor i, as described in Branger et al. (2018). So the total terminal wealth would be given by $V_T = \sum_{i=1}^{n} V_T^{(i,*)}$, where $V_T^{(i,*)}$ is the optimal terminal wealth of investor i as given in (2.12) with γ replaced by γ_i and v replaced by v_i. Then we can return to each investor her individual optimal payoff $V_T^{(i,*)}$. The corresponding proportion of terminal wealth distributed to investor i is thus given by

$$\alpha_i = \frac{V_T^{(i,*)}}{V_T} = \frac{V_T^{(i,*)}}{\sum_{i=1}^{n} V_T^{(i,*)}} \, .$$

In Branger et al. (2018) this specific way of distributing the accumulated terminal payoff is called the first best sharing rule. A sharing rule is basically a rule after which the total accumulated wealth V_T is distributed among the participants in the pool.
This approach works similarly if all the investors are required to meet an individual interest rate guarantee $g_i < r$ as described in Section 2.3. Then we simply need to replace $V_T^{(i,*)}$ by $\widetilde{V}_T^{(i,*)}$ from (2.26) in the above equation.
However, this sharing rule is very hard to communicate, as it depends on the stock price at maturity S_T (see (2.13)). Many individuals might not like the fact that the fraction of terminal wealth received is not fixed in advance. Therefore, this sharing rule is not used in most practical applications.
We would like to use a simpler sharing rule than the first best. There exists a variety of literature dealing with a collective of investors facing a joint decision under uncertainty and the choice of an adequate sharing rule: Wilson (1968) and Huang and Litzenberger (1985) analyze necessary and sufficient conditions for sharing rules to be Pareto optimal in a rather general setting. Dumas (1989) considers two individuals with different risk preferences interacting dynamically in a complete market. Weinbaum (2009) analyzes a social planner who maximizes a weighted sum of two individual utility functions, and then characterizes the optimal sharing rule by a comparison to a portfolio of options, as the terminal payoff, and therefore also the optimal sharing rule, generally cannot be determined explicitly in the considered setting (see also Wang (1996)). Jensen and Nielsen (2016) consider a similar social planner maximizing a weighted sum of two different utility functions. However, in this paper the sharing rule is fixed to be linear and the focus lies more on the suboptimality of this linear sharing rule. Branger et al. (2018) extend the approach presented in Jensen and Nielsen (2016) to n investors instead of two.
In the following sections of this chapter we are going to follow Jensen and Nielsen (2016) and, particularly, Branger et al. (2018) and fix the sharing rule to be linear. A formal introduction of this specific type of sharing rules will be given in the next section. After that, we will consider different strategies how to obtain a terminal payoff for the collective of investors and combine them with the linear sharing rule also used in Jensen and Nielsen (2016) and Branger et al. (2018).

3.1. Linear Sharing Rules

We will follow the definition given in Chen and Nguyen (2018): A sharing rule is linear (or affine) if there exist two real valued vectors $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, where $a_i \geq 0$, $\sum_{i=1}^n a_i = 1$, and $\sum_{i=1}^n b_i = 0$, such that investor i obtains

$$V_T^{(i)} = a_i V_T + b_i$$

for all $i = 1, \dots, n$. Whenever possible, we would like to use such a linear sharing rule because it is easy to understand and to communicate. In particular, the proportion of terminal wealth obtained by each investor is known from the beginning and not state dependent.

It seems natural to assign to each individual the fraction of terminal wealth being equal to the proportion of initial wealth invested, that is,

$$a_i = \frac{v_i}{v}, \quad b_i = 0 \tag{3.1}$$

for all $i = 1, \dots, n$. This sharing rule is easily understandable and known before the investment is done. Therefore, it is well suited for practical applications. However, as we will see in Section 3.3.1, the terminal payoff obtained by adding the individual optimal wealths $\sum_{i=1}^n V_T^{(i,*)}$ is not the best choice in combination with this sharing rule. Additionally, when interest rate guarantees are imposed, we can only use this sharing rule on the total payoff $\sum_{i=1}^n \widetilde{V}_T^{(i,*)}$ if all the guarantees g_i are the same. Otherwise, it is not certain whether each individual guarantee will be met.

However, these problems are mainly created by the investment strategy and not by the linear sharing rule. We will rather focus on maximizing some function of the total terminal payoff instead of combining existing optimal terminal wealths in the following sections. When doing this, another advantage of this sharing rule becomes clear: Whenever we include individual guarantees in an optimization problem that aims to maximize some function of the total terminal wealth of the pool, the problem reduces to a problem with just one common guarantee. This can be seen as follows. Suppose that $U(\cdot)$ is some function and we want to solve the optimization problem

$$\max_{\widetilde{V}_T} \mathbb{E}\left[U(\widetilde{V}_T)\right] \quad \text{subject to} \quad \mathbb{E}[M_T \widetilde{V}_T] = v, \text{ and}$$
$$\frac{v_i}{v}\widetilde{V}_T \geq v_i e^{g_i T} \text{ for all } i = 1, \dots, n.$$

Then the second condition is equivalent to

$$\widetilde{V}_T \geq ve^{g_i T} \text{ for all } i = 1, \dots, n$$

which is fulfilled if and only if

$$\widetilde{V}_T \geq ve^{\max_i(g_i)T}.$$

Due to the linear sharing rule we only need to consider the maximal guarantee in the optimization problem. Since we were able to derive this result without using the structure of $\widetilde{V}_T$ or $U(\cdot)$, this result holds for all joint investment problems where the linear sharing rule (3.1) is used, especially for the strategies in the following two sections. As a consequence, when considering guarantees in the following, we will only focus on the case where one joint guarantee is imposed.

Now that we have an idea how to distribute the total terminal payoff to each individual, we need to come up with an idea how to invest the aggregate amount of capital v. Based on Chapter 2 we could assume that the fund manager responsible for investing v invests this amount in a similar way as an individual with a relative risk aversion $\bar{\gamma}$ would do. This proceeding has several advantages. First, we can get the terminal wealths explicitly from Chapter 2 for both the case with and without guarantee. Secondly, the investment strategy π_t is constant over time, at least for the case without guarantees, and, for both cases, it is known explicitly. The purpose of the next section is therefore, to consider the joint investment with such a joint risk aversion parameter $\bar{\gamma}$. This approach has already been analyzed in Jensen and Sørensen (2001), Jensen and Nielsen (2016) and Branger et al. (2018). The following section is mainly based on Jensen and Sørensen (2001), where this approach is studied with and without guarantee, and on Branger et al. (2018), where only the case without guarantee is considered, but in more detail than in Jensen and Sørensen (2001).

3.2. Investment with a Joint Risk Aversion Parameter

In this section we assume that the fund manager who is responsible for investing the total initial wealth v invests this amount in a similar way as an individual with a relative risk aversion $\bar{\gamma}$ would do. This new risk aversion parameter $\bar{\gamma}$ is usually chosen between

$\min_i(\gamma_i)$ and $\max_i(\gamma_i)$ to make sure that the utility losses are not spread too widely between the participants. The terminal wealth V_T is then shared linearly as in (3.1), that is, $V_T^{(i)} = \frac{v_i}{v} V_T$. This is obviously the same result as the one that would be obtained if all the individuals invested v_i according to the relative risk aversion $\bar{\gamma}$.
As usual, we will first consider the unrestricted case without guarantee and then move on to the case where an interest rate guarantee is imposed.

3.2.1. Case without Guarantee

For the case without guarantee the terminal wealth V_T and the investment strategy can be obtained from equations (2.12) and (2.18) with $\bar{\gamma}$ instead of γ. In this section, we will denote by V_T^* the optimal terminal payoff with respect to $\bar{\gamma}$, that is,

$$V_T^* = (\eta M_T)^{-\frac{1}{\bar{\gamma}}} ,$$

where

$$\eta = v^{-\bar{\gamma}} \exp\left(\left(r + \frac{1}{2}\lambda^2\right)(1-\bar{\gamma})\,T + \frac{1}{2}\lambda^2\bar{\gamma}\left(1 - \frac{1}{\bar{\gamma}}\right)^2 T\right).$$

Now the only thing remaining for us to compute is the indirect utility process. It is defined as

$$J_t(v_i, T, \gamma_i, \bar{\gamma}) = \mathbb{E}\left[u\left(\frac{v_i}{v}V_T^*\right) \;\middle|\; \mathcal{F}_t\right] = \mathbb{E}\left[\frac{(\frac{v_i}{v}V_T^*)^{1-\gamma_i}}{1-\gamma_i} \;\middle|\; \mathcal{F}_t\right].$$

First we observe that

$$(V_T^*)^{1-\gamma_i} = (\eta M_T)^{-\frac{1-\gamma_i}{\bar{\gamma}}} .$$

Therefore, the indirect utility process can be written as

$$J_t(v_i, T, \gamma_i, \bar{\gamma}) = \frac{1}{1-\gamma_i}\,(\eta M_t)^{-\frac{1-\gamma_i}{\bar{\gamma}}}\left(\frac{v_i}{v}\right)^{1-\gamma_i}\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{-\frac{1-\gamma_i}{\bar{\gamma}}} \;\middle|\; \mathcal{F}_t\right]. \tag{3.2}$$

We can now proceed analogously to Section 2.2.3 to compute this expectation. Using (2.2), (2.11) and the fact that a Brownian motion has independent increments, we obtain

$$\eta^{-\frac{1-\gamma_i}{\bar{\gamma}}}\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{-\frac{1-\gamma_i}{\bar{\gamma}}}\;\middle|\;\mathcal{F}_t\right] = \eta^{-\frac{1-\gamma_i}{\bar{\gamma}}}\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{-\frac{1-\gamma_i}{\bar{\gamma}}}\right]$$
$$= v^{1-\gamma_i}\exp\left(\left(r+\frac{\lambda^2}{\bar{\gamma}}\right)T - \frac{1}{\bar{\gamma}}\left(r+\frac{\lambda^2}{2}\right)t - (1-\gamma_i)\frac{\lambda^2}{2\bar{\gamma}^2}t - \frac{\lambda^2}{2\bar{\gamma}^2}\gamma_i T\right)^{1-\gamma_i}. \tag{3.3}$$

Using (3.3), we can simplify (3.2) and obtain the indirect utility process as

$$J_t(v_i,T,\gamma_i,\bar{\gamma}) = \frac{1}{1-\gamma_i}\left(\eta M_t\right)^{-\frac{1-\gamma_i}{\bar{\gamma}}}\left(\frac{v_i}{v}\right)^{1-\gamma_i}\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{-\frac{1-\gamma_i}{\bar{\gamma}}}\right]$$
$$= \frac{v_i^{1-\gamma_i}}{1-\gamma_i}\exp\left(\left(r+\frac{\lambda^2}{\bar{\gamma}}-\frac{\lambda^2}{2\bar{\gamma}^2}\gamma_i\right)T - \frac{1}{\bar{\gamma}}\left(r+\frac{\lambda^2}{2}+(1-\gamma_i)\frac{\lambda^2}{2\bar{\gamma}}\right)t\right)^{1-\gamma_i} M_t^{-\frac{1-\gamma_i}{\bar{\gamma}}}. \tag{3.4}$$

The expected utility of an investor with relative risk aversion γ_i who invests according to the relative risk aversion $\bar{\gamma}$ is therefore given by (see also Jensen and Sørensen (2001)):

$$J_0(v_i,T,\gamma_i,\bar{\gamma}) = \frac{\left(v_i\exp\left(\left(r+\frac{\lambda^2}{\bar{\gamma}}\right)T - \frac{\lambda^2}{2\bar{\gamma}^2}\gamma_i T\right)\right)^{1-\gamma_i}}{1-\gamma_i}. \tag{3.5}$$

The certainty equivalent wealth for investor i can then be obtained as follows:

$$\overline{\mathrm{CE}}_i = \left((1-\gamma_i)J_0(v_i,T,\gamma_i,\bar{\gamma})\right)^{\frac{1}{1-\gamma_i}}$$
$$= v_i\exp\left(\left(r+\frac{\lambda^2}{\bar{\gamma}}\right)T - \frac{\lambda^2}{2\bar{\gamma}^2}\gamma_i T\right).$$

Hence, the certainty equivalent return is given by

$$\bar{y}_i = r + \frac{\lambda^2}{\bar{\gamma}} - \frac{\lambda^2}{2\bar{\gamma}^2}\gamma_i = r + \left(\frac{1}{\bar{\gamma}} - \frac{\gamma_i}{2\bar{\gamma}^2}\right)\lambda^2. \tag{3.6}$$

Some important observations about this new certainty equivalent are:

- If $\gamma_i = \bar{\gamma}$, we get $y_i^* = \bar{y}_i$, so in this case the investment strategy coincides with that of investor i and thus, this person receives her optimal terminal wealth.

- If $\gamma_i \neq \bar{\gamma}$, then we always have $\bar{y}_i < y_i^*$. This can be seen as follows:

$$\begin{aligned}\bar{y}_i &= y_i^* + \left(\frac{1}{\bar{\gamma}} - \frac{\gamma_i}{2\bar{\gamma}^2}\right)\lambda^2 - \frac{\lambda^2}{2\gamma_i} \\ &= y_i^* - \frac{\lambda^2}{2}\left(\frac{1}{\gamma_i} - \frac{2}{\bar{\gamma}} + \frac{\gamma_i}{\bar{\gamma}^2}\right) \\ &= y_i^* - \frac{\lambda^2}{2\gamma_i}\left(1 - \frac{\gamma_i}{\bar{\gamma}}\right)^2 \\ &< y_i^*.\end{aligned}$$

  So clearly, each investor with a risk aversion differing from $\bar{\gamma}$ is going to suffer a loss in utility from the joint investment which is not a surprising result. Furthermore, we make the following observation:

  - If $\gamma_i < \bar{\gamma}$, the term $\frac{\lambda^2}{2\gamma_i}\left(1 - \frac{\gamma_i}{\bar{\gamma}}\right)^2$ is decreasing in γ_i.
  - If, on the other hand, $\gamma_i > \bar{\gamma}$, the term $\frac{\lambda^2}{2\gamma_i}\left(1 - \frac{\gamma_i}{\bar{\gamma}}\right)^2$ is increasing in γ_i.

  So the largest losses are always suffered by the investors whose risk aversion deviates the most from the joint parameter $\bar{\gamma}$. These results have also been shown in Branger et al. (2018) already.

3.2.2. Case with Guarantee

Now we assume that the accumulated wealth v needs to earn a prescribed interest rate $g < r$, that is, we postulate $V_T \geq ve^{gT}$. The total terminal payoff $\widetilde{V}_T^*$ and the fraction invested in the risky asset can then be obtained from the equations (2.26) and (2.34).
So again, we only need to determine the indirect utility process. We can proceed as follows:

$$\begin{aligned}\widetilde{J}_t(v_i, T, \gamma_i, \bar{\gamma}) &= \mathbb{E}\left[\frac{\left(\frac{v_i}{v}\widetilde{V}_T^*\right)^{1-\gamma_i}}{1-\gamma_i} \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\frac{\left(v_i e^{gT}\right)^{1-\gamma_i}}{1-\gamma_i}\mathbb{1}_{\left\{ve^{gT} > xV_T^*\right\}} \,\middle|\, \mathcal{F}_t\right] + \mathbb{E}\left[\frac{(xV_T^*)^{1-\gamma_i}}{1-\gamma_i}\mathbb{1}_{\left\{ve^{gT} \leq xV_T^*\right\}} \,\middle|\, \mathcal{F}_t\right],\end{aligned}$$

where x is determined numerically from the equation

$$1 = x\Phi\left(-\bar{d}_2\right) + e^{(g-r)T}\Phi\left(\bar{d}_1 + \lambda\sqrt{T}\right)$$

with

$$\begin{aligned}
\bar{d}_1 &= \frac{\bar{\gamma}((g-r)T - \log(x))}{\lambda\sqrt{T}} + \left(\frac{1}{2\bar{\gamma}} - 1\right)\lambda\sqrt{T}, \\
\bar{d}_2 &= \bar{d}_1 + \left(1 - \frac{1}{\bar{\gamma}}\right)\lambda\sqrt{T}\,.
\end{aligned}$$

The first expectation can be obtained in a similar way as in Section 2.3.1 and is given by

$$\mathbb{E}\left[\frac{\left(v_i e^{gT}\right)^{1-\gamma_i}}{1-\gamma_i}\mathbb{1}_{\left\{ve^{gT} > xV_T^*\right\}}\,\middle|\,\mathcal{F}_t\right] = u_i\left(v_i e^{gT}\right)\Phi\left(\overline{D}_1(t)\right), \tag{3.7}$$

where

$$\overline{D}_1(t) = \frac{\bar{\gamma}((g-r)T - \log x) + \left(\frac{1}{2\bar{\gamma}} - 1\right)\lambda^2 T - \lambda W_t}{\lambda\sqrt{T-t}}\,.$$

The second conditional expectation is a straightforward extension of the calculations performed in Section 2.3.3 and can be computed as follows:

$$\begin{aligned}
&\mathbb{E}\left[\frac{\left(x\frac{v_i}{v}V_T^*\right)^{1-\gamma_i}}{1-\gamma_i}\mathbb{1}_{\left\{ve^{gT} \le xV_T^*\right\}}\,\middle|\,\mathcal{F}_t\right] \\
&= \frac{x^{1-\gamma_i}}{1-\gamma_i}\left(\frac{v_i}{v}\right)^{1-\gamma_i}(\eta M_t)^{-\frac{1-\gamma_i}{\bar{\gamma}}}\,\mathbb{E}\left[\left(\frac{M_T}{M_t}\right)^{-\frac{1-\gamma_i}{\bar{\gamma}}}\mathbb{1}_{\left\{ve^{gT} \le xV_T^*\right\}}\,\middle|\,\mathcal{F}_t\right] \\
&= \frac{x^{1-\gamma_i}}{1-\gamma_i}\left(\frac{v_i}{v}\right)^{1-\gamma_i}(\eta M_t)^{-\frac{1-\gamma_i}{\bar{\gamma}}}\,e^{\frac{1-\gamma_i}{\bar{\gamma}}r(T-t) + \frac{1-\gamma_i}{2\bar{\gamma}}\left(1 + \frac{1-\gamma_i}{\bar{\gamma}}\right)\lambda^2(T-t)}\left(1 - \Phi\left(\overline{D}_2^{(i)}(t)\right)\right)
\end{aligned} \tag{3.8}$$

with

$$\overline{D}_2^{(i)}(t) = \overline{D}_1(t) - \frac{1-\gamma_i}{\bar{\gamma}}\lambda\sqrt{T-t}\,.$$

Using (2.11), we obtain

$$\eta^{-\frac{1-\gamma_i}{\bar{\gamma}}} e^{\frac{1-\gamma_i}{\bar{\gamma}} r(T-t)+\frac{1-\gamma_i}{2\bar{\gamma}}\left(1+\frac{1-\gamma_i}{\bar{\gamma}}\right)\lambda^2(T-t)} \left(1-\Phi\left(\overline{D}_2^{(i)}(t)\right)\right)$$
$$= v^{1-\gamma_i} \exp\left(\left(r+\frac{\lambda^2}{\bar{\gamma}}-\frac{\gamma_i}{2\bar{\gamma}^2}\lambda^2\right)T-\left(\frac{r}{\bar{\gamma}}+\frac{\lambda^2}{2\bar{\gamma}}+\frac{\lambda^2}{2\bar{\gamma}^2}-\frac{\gamma_i}{2\bar{\gamma}^2}\lambda^2\right)t\right)^{1-\gamma_i}\left(1-\Phi\left(\overline{D}_2^{(i)}(t)\right)\right).$$

Combining this result with (3.8) and using (3.4), we get

$$\mathbb{E}\left[\frac{\left(x\frac{v_i}{v}\widetilde{V}_T^*\right)^{1-\gamma_i}}{1-\gamma_i}\mathbb{1}_{\left\{ve^{gT}\leq xV_T^*\right\}}\,\middle|\,\mathcal{F}_t\right] = J_t(xv_i,T,\gamma_i,\bar{\gamma})\left(1-\Phi\left(\overline{D}_2^{(i)}(t)\right)\right). \tag{3.9}$$

Finally, adding (3.7) and (3.9), we obtain the complete indirect utility process as

$$\widetilde{J}_t(v_i,T,\gamma_i,\bar{\gamma}) = u_i\left(v_ie^{gT}\right)\Phi(\overline{D}_1(t)) + J_t(xv_i,T,\gamma_i,\bar{\gamma})\left(1-\Phi\left(\overline{D}_2^{(i)}(t)\right)\right).$$

In particular, the optimal level of expected utility is given by (see also Jensen and Sørensen (2001)):

$$\widetilde{J}_0(v_i,T,\gamma_i,\bar{\gamma}) = u_i\left(v_ie^{gT}\right)\Phi\left(\bar{d}_1\right) + J_0(xv_i,T,\gamma_i,\bar{\gamma})\left(1-\Phi\left(\bar{d}_2^{(i)}\right)\right), \tag{3.10}$$

where

$$\bar{d}_2^{(i)} = \overline{D}_2^{(i)}(0) = \bar{d}_1 + \frac{\gamma_i-1}{\bar{\gamma}}\lambda\sqrt{T},$$

and $J_0(xv_i,T,\gamma_i,\bar{\gamma})$ is given in (3.5).
Now the certainty equivalent wealth can be computed from (3.10) as follows:

$$\widetilde{\mathrm{CE}}_i(v_i,T,\gamma_i,\bar{\gamma}) = \left((1-\gamma_i)\widetilde{J}_0(v_i,T,\gamma_i,\bar{\gamma})\right)^{\frac{1}{1-\gamma_i}}$$
$$= v_i\left(\left(e^{gT}\right)^{1-\gamma_i}\Phi\left(\bar{d}_1\right)+\left(x\exp\left(\left(r+\frac{\lambda^2}{\bar{\gamma}}\right)T-\frac{\lambda^2}{2\bar{\gamma}^2}\gamma_iT\right)\right)^{1-\gamma_i}\left(1-\Phi\left(\bar{d}_2^{(i)}\right)\right)\right)^{\frac{1}{1-\gamma_i}}.$$

As a consequence, the certainty equivalent return is given by

$$\widetilde{y}_i(v_i,T,\gamma_i,\bar{\gamma}) = \frac{1}{T}\log\left(\frac{\widetilde{\mathrm{CE}}_i(v_i,T,\gamma_i,\bar{\gamma})}{v_i}\right)$$
$$= \frac{1}{T(1-\gamma_i)}\log\left(\left(e^{gT}\right)^{1-\gamma_i}\Phi\left(\bar{d}_1\right)+\left(x\exp\left(\left(r+\frac{\lambda^2}{\bar{\gamma}}\right)T-\frac{\lambda^2}{2\bar{\gamma}^2}\gamma_iT\right)\right)^{1-\gamma_i}\left(1-\Phi\left(\bar{d}_2^{(i)}\right)\right)\right).$$

At the end of the previous section we have already mentioned some advantages of the approach described in this section. If we consider the case without guarantees, one last advantage of this procedure is that the wealth obtained is Pareto efficient among all the constant mix strategies if $\bar{\gamma}$ is chosen between $\min_i \gamma_i$ and $\max_i \gamma_i$ (for a proof see Branger et al. (2018)). However, as we will see in the following chapter, such constant mix strategies are, in general, not Pareto efficient if we do not restrict the optimization problem to constant mix strategies.
Therefore, we will follow Branger et al. (2018) and Jensen and Nielsen (2016) and focus on generally Pareto efficient payoffs in the following section. We will start by giving a formal definition of Pareto efficiency and then show that all the terminal payoffs considered so far, are, in general, not Pareto efficient under our linear sharing rule. Jensen and Nielsen (2016) have already performed a numerical analysis in which they show that such Pareto efficient payoffs yield higher wealth equivalents than the constant mix strategy considered in Section 3.2.1. This effect can be observed most clearly when the risk aversion parameters of the two members in the collective differ widely. For our analysis, we will, of course, rely on the certainty equivalent return, as in the previous sections. Additionally, we will not only compare the Pareto efficient payoff to the terminal wealth resulting from the constant mix strategy but also to the first best payoff (given by the sum of the individual optima), and we will both analyze the case with and without guarantee.

3.3. Pareto Efficient Payoffs

In this section we will focus on Pareto efficient payoffs combined with the linear sharing rule from (3.1). Pareto efficiency here means that no investor can increase her utility without simultaneously decreasing another investor's utility. For a mathematical definition, we will follow Branger et al. (2018): Let $\mathcal{V}$ be the set of all admissible terminal payoffs that can be obtained assuming that the initial wealth is v, that is,

$$\mathcal{V} = \{V_T : \mathbb{E}\left[M_T V_T\right] = v\}.$$

- We assume that the linear sharing rule defined in (3.1) is used. Furthermore, let $V_1, V_2 \in \mathcal{V}$. Then V_1 is called a Pareto improvement of V_2 if

$$\mathbb{E}\left[u_i\left(\frac{v_i}{v}V_1\right)\right] \geq \mathbb{E}\left[u_i\left(\frac{v_i}{v}V_2\right)\right]$$

for all $i = 1, \dots, n$, where strict inequality holds for at least one $i \in \{1, \dots, n\}$.

- The set of Pareto efficient payoffs $\mathcal{V}^{\text{PE}} \subset \mathcal{V}$ is defined as the set of all admissible payoffs such that there does not exist a Pareto improvement:

$$\mathcal{V}^{\text{PE}} = \{V_T \in \mathcal{V} : \text{There exists no } V_T' \in \mathcal{V} \text{ s.t. } V_T' \text{ is a Pareto improvement of } V_T\}.$$

Whenever possible, we should use a Pareto efficient payoff. Therefore, the question arises how such payoffs can be characterized. If we assume that all the investors have positive initial wealth $v_i > 0$ for all $i = 1, \dots, n$, then, according to Branger et al. (2018), the set of Pareto efficient payoffs can be represented as follows:

$$\mathcal{V}^{\text{PE}} = \left\{ V_T^* \in \mathcal{V} : V_T^* = \underset{V_T \in \mathcal{V}}{\operatorname{argmax}} \, \mathbb{E}\left[\sum_{i=1}^{n} \beta_i u_i \left(\frac{v_i}{v} V_T \right) \right], \ \beta_i \geq 0, \ \sum_{i=1}^{n} \beta_i = 1 \right\}. \tag{3.11}$$

3.3.1. Pareto Efficient Payoffs Without Guarantee

Based on the representation of Pareto efficient payoffs in (3.11) we will now follow Branger et al. (2018) and Jensen and Nielsen (2016) and solve the following optimization problem:

$$\max_{V_T} \mathbb{E}\left[\sum_{i=1}^{n} \beta_i \frac{\left(\frac{v_i}{v} V_T\right)^{1-\gamma_i}}{1-\gamma_i} \right] \quad \text{subject to} \quad \mathbb{E}[M_T V_T] = v. \tag{3.12}$$

The first order conditions of this problem are given by

$$\sum_{i=1}^{n} \beta_i \left(\frac{v_i}{v} \right)^{1-\gamma_i} V_T^{-\gamma_i} = \nu M_T, \tag{3.13}$$

$$\mathbb{E}[M_T V_T] = v, \tag{3.14}$$

where $\nu > 0$ is the Lagrangian multiplier which needs to be determined in such a way that the budget constraint (3.14) is fulfilled. Except for a few special cases, we first need to determine ν numerically, and then, using this ν, the optimal V_T can be determined numerically from (3.13) for given values of M_T. These special cases are the following:

- $\beta_i = 1$ for exactly one i and $\beta_j = 0$ for all $j \neq i$: In this case the problem is reduced to Merton's optimization problem with just one investor as described in Section 2.2. Clearly, investor i reaches her individual unrestricted utility in this case, whereas all the other investors suffer a larger loss in utility, the more their risk aversions γ_j deviate from γ_i.

- $\gamma_i = \gamma$ for all $i = 1, \ldots, n$: If all the risk aversion parameters coincide, then the problem is again reduced to the individual optimization problem with one investor having risk aversion γ. All the investors reach their individual optimal wealth and expected utility.

- $n = 2,\ \gamma_2 = 2\gamma_1$: Here equation (3.13) becomes a quadratic polynomial and can be solved analytically using the quadratic formula. The solution is given by (see Jensen and Nielsen (2016)):

$$V_T = \left(\frac{2\beta_2 \left(\frac{v_2}{v}\right)^{1-2\gamma_1}}{\sqrt{\beta_1^2 \left(\frac{v_1}{v}\right)^{2(1-\gamma_1)} + 4\beta_2 \left(\frac{v_2}{v}\right)^{1-2\gamma_1} \nu M_T} - \beta_1 \left(\frac{v_1}{v}\right)^{1-\gamma_1}} \right)^{1/\gamma_1}.$$

For more general cases it is not possible to determine ν or V_T explicitly. Hence, we need to use numerical procedures. The algorithm used here is based on the following idea: For a given value of ν we can determine V_T numerically using the bisection method. Therefore, we could create a set of realizations of M_T and then determine a corresponding set of realizations of V_T. The mean of the product of these realizations gets closer to $\mathbb{E}[M_T V_T]$, the larger the number of realizations N is, due to the law of large numbers. Then we can compare this mean to v and try the next value of ν. To find the right value for ν we can again use the bisection method. We keep looking for the right value of ν until the mean is close enough to v, that is, until the budget constraint is (approximately) fulfilled. A pseudo code of this algorithm is given in the following. From now on, we will denote by tol an arbitrarily small number greater than zero. This is the error by which our numerical result is allowed to deviate from the theoretical result. The corresponding R code can be found in Appendix B.

1. Initialize r, μ, σ, $\lambda = \frac{\mu - r}{\sigma}$, T, tol, N, n, β_i, γ_i, v_i .

2. Use bisection method to obtain ν s.t. the budget constraint is fulfilled: Choose ν_h and ν_l (higher and lower bounds for ν).

3. While $\left| \frac{1}{N} \sum_{j=1}^{N} M_T^j V_T^j - v \right| >$ tol:

   3.1. Create N realizations of a standard normally distributed random variable $X^j \sim \mathcal{N}(0,1),\ i = 1, \ldots, N$.

3.2. Compute N realizations of the state price density at T:
$M_T^j = \exp\left(-rT - \frac{1}{2}\lambda^2 T - \lambda\sqrt{T}X^j\right)$.

3.3. Set $\nu = \frac{1}{2}(\nu_h + \nu_l)$.

3.4. Use bisection method to obtain V_T^j for each $j = 1, \ldots, N$. Choose $V_T^{j,h}$ and $V_T^{j,l}$ (higher and lower bounds for V_T^j).

3.5. While $\max_j \left|\sum_{i=1}^n \beta_i \left(\frac{v_i}{v}\right)^{1-\gamma_i} \left(V_T^j\right)^{-\gamma_i} - \nu M_T^j\right| >$ tol:

3.5.1. $V_T^j = \frac{1}{2}\left(V_T^{j,l} + V_T^{j,h}\right)$ for all $j = 1, \ldots, N$.

3.5.2. For each j with $\sum_{i=1}^n \beta_i \left(\frac{v_i}{v}\right)^{1-\gamma_i} \left(V_T^j\right)^{-\gamma_i} - \nu M_T^j > 0$ set $V_T^{j,l} = V_T^j$.

3.5.3. For each j with $\sum_{i=1}^n \beta_i \left(\frac{v_i}{v}\right)^{1-\gamma_i} \left(V_T^j\right)^{-\gamma_i} - \nu M_T^j < 0$ set $V_T^{j,h} = V_T^j$.

3.6. If $\left(\frac{1}{N}\sum_{j=1}^N M_T^j V_T^j - v > 0\right)$, then $\nu_l = \nu$.

3.7. If $\left(\frac{1}{N}\sum_{j=1}^N M_T^j V_T^j - v < 0\right)$, then $\nu_h = \nu$.

For the simulation of the standard normally distributed random variables X^j we first create a stratified random sample $U^1, \ldots, U^N$ of standard uniformly distributed random variables as described in Appendix A.2. We use the following parameters:

- Total number of realizations: $N = 10^6$.
- Number of strata: $K = 10^4$.
- The strata are defined as $A_i = \left(\frac{i-1}{K}, \frac{i}{K}\right)$ for all $i = 1, \ldots, K$.

Here the proportional allocation is used, so we need $N_i = p_i N$. Since the distribution considered here is uniform, we obtain $p_i = \frac{1}{K}$. Therefore, we get $N_i = \frac{N}{K} = 100$. Having created this set of realizations, we can apply the inverse transformation method (see Appendix A.1) to create a stratified set of realizations of a standard normally distributed random variable.

Having computed η, we can use steps 3.4 and 3.5 of the above algorithm to create realizations of V_T for a given set of realizations of M_T. Then we can use (A.1) to estimate the expected utility and the certainty equivalent wealth and return for each individual in the pool. Since $N_i K = N$ the estimator in (A.1) simplifies as follows:

$$\widehat{\ell} = \frac{1}{N}\sum_{i=1}^{K}\sum_{j=1}^{N_i} f\left(X_j^i\right).$$

This is simply the mean over all the realizations created. Therefore, we obtain the following estimators for the certainty equivalent wealth and return:

$$\widehat{\mathrm{CE}}_i = \left(\frac{1}{N} \sum_{j=1}^{N} \left(\frac{v_i}{v} V_T^j \right)^{1-\gamma_i} \right)^{\frac{1}{1-\gamma_i}},$$
$$\widehat{y}_i = \frac{1}{T} \log \left(\frac{\widehat{\mathrm{CE}}_i}{v_i} \right). \tag{3.15}$$

Now that we are able to compute the Pareto efficient aggregated terminal wealth, we can compare this payoff to the payoffs considered in Sections 3.1 and 3.2. We will see that, in general, neither the first best payoff with linear sharing rule nor the constant mix strategy with $\bar{\gamma} = \left(\prod_{i=1}^{n} \gamma_i\right)^{1/n}$ are Pareto efficient if we allow for all possible investment strategies to be used and do not restrict the optimization problem to constant mix strategies. If the risk aversion parameters differ widely, then the individual losses in utility get larger for these two cases than for the Pareto efficient payoff. A rather similar result could already be seen in Jensen and Nielsen (2016), at least for the constant mix strategy. We will consider the following example:

- There are $n = 2$ investors tied together in the pool,
- they have the same initial wealth $v_1 = v_2 = 1$, and
- their relative risk aversions are given by $\gamma_1 = 1/2, \;\; \gamma_2 = 10$.

In the following we will numerically determine a value for β_1 such that both investors obtain a higher certainty equivalent return from the Pareto efficient payoff than from the other two cases. We can use the bisection method to find such a value for β_1. For the payoff obtained from the constant mix strategy this can, for example, be done as follows:

1. Initialize r, μ, σ, $\lambda = \frac{\mu - r}{\sigma}$, T, tol, N, n, β_i, γ_i, v_i .
2. Choose higher and lower bounds $\beta_1^h = 1$ and $\beta_1^l = 0$ for β_1.
3. While $\widehat{y}_1 < \bar{y}_1$ or $\widehat{y}_2 < \bar{y}_2$:
   3.1. Set $\beta_1 = \frac{1}{2}\left(\beta_1^h + \beta_1^l\right)$.
   3.2. Compute $\widehat{y}_i$ for this β_1 and $\bar{y}_i$.
   3.3. If $\widehat{y}_1 < \bar{y}_1$ and $\widehat{y}_2 > \bar{y}_2$, then $\beta_1^l = \beta_1$.
   3.4. If $\widehat{y}_1 > \bar{y}_1$ and $\widehat{y}_2 < \bar{y}_2$, then $\beta_1^h = \beta_1$.

Because of the criterion in step 2 this algorithm will stop as soon as one value for β_1 is found for which both investors are better off with the Pareto efficient payoff. However, it is important to keep in mind that there exist more than one such values for β_1 which can fulfill this criterion.

Now we first consider the case where the total terminal wealth is given by the sum of the unrestricted optimal payoffs combined with the linear sharing rule. Here we obtain $\beta_1 = 0.99$ using the bisection method. In Figure 3.1 we compare the total terminal wealths. From the figure it is not really clear which payoff the preferable one is. While the Pareto

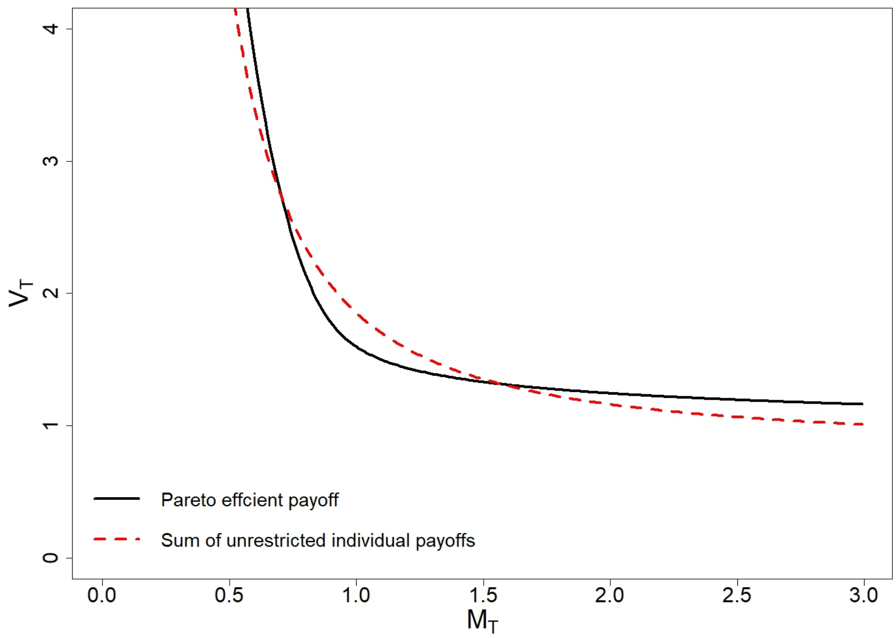


Figure 3.1.: Comparison between Pareto efficient terminal payoff and payoff obtained by adding the individual optimal terminal wealths, where $\gamma_1 = 1/2$, $\gamma_2 = 10$, $v_i = 1$, $\beta_1 = 0.99$.

efficient payoff is higher in more extreme scenarios (both good and bad) the first best payoff is higher in most rather common scenarios. To figure out which payoff is better we need to compute the certainty equivalent return for both investors. For the Pareto efficient payoff this is now straightforward and has already been explained. The estimator for the certainty equivalent return of this payoff is given in (3.15).
The question remains how we can compute the certainty equivalent returns for the payoff that is given by the sum of the individual optimal terminal wealths. We know that investor

i obtains the following payoff at T:

$$V_T^{(i)} = \frac{v_i}{v} \sum_{j=1}^{n} V_T^{(j,*)}.$$

Hence, the certainty equivalent wealth has the following form:

$$\begin{aligned}\mathrm{CE}_i\left(\frac{v_i}{v}\sum_{j=1}^{n} V_T^{(j,*)}\right) &= \mathbb{E}\left[\left(\frac{v_i}{v}\sum_{j=1}^{n} V_T^{(j,*)}\right)^{1-\gamma_i}\right]^{\frac{1}{1-\gamma_i}} \\ &= \frac{v_i}{v}\mathbb{E}\left[\left(\sum_{j=1}^{n} V_T^{(j,*)}\right)^{1-\gamma_i}\right]^{\frac{1}{1-\gamma_i}}.\end{aligned}$$

From (2.13) we know that

$$V_T^{(j,*)} = v_j \exp\left((1-m(\gamma_j))\left(r+\frac{1}{2}m(\gamma_j)\sigma^2\right)T\right)\left(\frac{S_T}{S_0}\right)^{m(\gamma_j)},$$

and from (2.1) we know that

$$S_T = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right).$$

Combining these two equations, we get

$$\begin{aligned}V_T^{(j,*)} &= v_j \exp\left((1-m(\gamma_j))\left(r+\frac{1}{2}m(\gamma_j)\sigma^2\right)T\right)\exp\left(m(\gamma_j)\left(\mu-\frac{1}{2}\sigma^2\right)T + m(\gamma_j)\sigma W_T\right) \\ &= v_j \exp\left((1-m(\gamma_j))\left(r+\frac{1}{2}m(\gamma_j)\sigma^2\right)T + m(\gamma_j)\left(\mu-\frac{1}{2}\sigma^2\right)T + m(\gamma_j)\sigma W_T\right) \\ &= v_j c_j \exp\left(m(\gamma_j)\sigma W_T\right), \qquad (3.16)\end{aligned}$$

where

$$c_j = \exp\left((1-m(\gamma_j))\left(r+\frac{1}{2}m(\gamma_j)\sigma^2\right)T + m(\gamma_j)\left(\mu-\frac{1}{2}\sigma^2\right)T\right) \qquad (3.17)$$

is only a constant. As a consequence, if we assume that X is a standard normally distributed random variable, we can write

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{j=1}^{n} V_T^{(j,*)}\right)^{1-\gamma_i}\right] &= \mathbb{E}\left[\left(\sum_{j=1}^{n} v_j c_j \exp\left(m(\gamma_j)\sigma W_T\right)\right)^{1-\gamma_i}\right] \\ &= \mathbb{E}\left[\left(\sum_{j=1}^{n} v_j c_j \exp\left(m(\gamma_j)\sigma\sqrt{T}X\right)\right)^{1-\gamma_i}\right] \\ &= \int_{-\infty}^{\infty}\left(\sum_{j=1}^{n} v_j c_j \exp\left(m(\gamma_j)\sigma\sqrt{T}x\right)\right)^{1-\gamma_i}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx\,.\end{aligned}$$

This integral can easily be computed numerically (see Appendix B for the corresponding R code). After that, we can use equation (2.24) to determine the corresponding certainty equivalent return. This certainty equivalent return will be denoted by y_i', that is,

$$y_i' = \frac{1}{T}\log\left(\frac{\mathrm{CE}_i\left(\frac{v_i}{v}\sum_{j=1}^{n} V_T^{(j,*)}\right)}{v_i}\right). \tag{3.18}$$

Now we are able to compute the certainty equivalent returns for both investors. The values are provided in Table 3.1. Since both values are greater when the Pareto efficient payoff is used, it follows that the terminal wealth obtained by adding the individual optimal payoffs is not Pareto efficient when the linear sharing rule is used.

	$\gamma_1 = 1/2$	$\gamma_2 = 10$
Pareto efficient	0.1520	-0.2885
Sum of unrestricted optimal payoffs	0.1448	-0.3043

Table 3.1.: Comparison between certainty equivalent returns obtained from Pareto efficient terminal payoff and payoff obtained by adding the individual optimal terminal wealths, where $\gamma_1 = 1/2$, $\gamma_2 = 10$, $v_i = 1$, $\beta_1 = 0.99$.

For the constant mix strategy the first value that we obtain from the bisection method is $\beta_1 = 0.7$. In Figure 3.2 the aggregated terminal wealth of the investors is again plotted against the state price density M_T. We obtain a rather similar result as in Figure 3.1. Here we observe that the Pareto efficient payoff is much higher than the payoff obtained from the constant mix strategy in extremely good and bad scenarios. On the other hand, the constant mix strategy yields a higher payoff in the more common states.

Computing the certainty equivalent returns, we see that the constant mix strategy with

$\bar{\gamma} = \sqrt{\gamma_1\gamma_2}$ is not Pareto efficient because both certainty equivalent returns are smaller than the ones obtained from the Pareto efficient payoff. The actual numbers can be found in Table 3.2. The returns of the Pareto efficient payoff are estimated as in (3.15). The values of the constant mix strategy can be computed using (3.6).

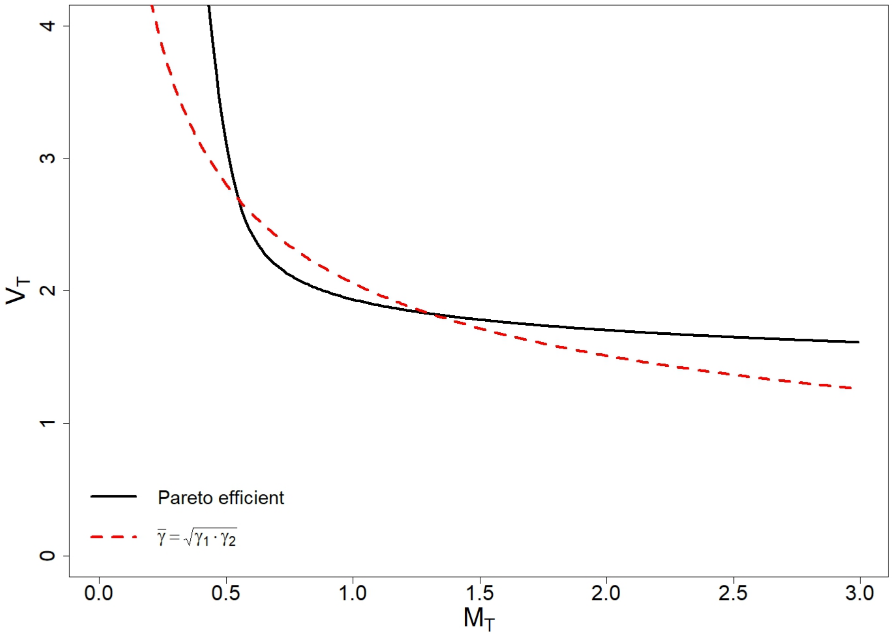


Figure 3.2.: Comparison between Pareto efficient and constant mix terminal payoff, where $\gamma_1 = 1/2$, $\gamma_2 = 10$, $\bar{\gamma} = \sqrt{\gamma_1\gamma_2}$, $v_i = 1$, $\beta_1 = 0.7$.

	$\gamma_1 = 1/2$	$\gamma_2 = 10$
Pareto efficient	0.0855	-0.0312
$\bar{\gamma}$	0.0815	-0.0775

Table 3.2.: Comparison between certainty equivalent returns for Pareto efficient and constant mix payoff, where $\gamma_1 = 1/2$, $\gamma_2 = 10$, $\bar{\gamma} = \sqrt{\gamma_1\gamma_2}$, $v_i = 1$, $\beta_1 = 0.7$.

3.3.2. Pareto Efficient Payoffs With Interest Rate Guarantee

Now we assume that the total wealth invested at the beginning needs to earn a specific interest rate, similar as in Section 2.3. We use again g to denote the guaranteed rate of return, where $g < r$. Then the optimization problem (3.12) can be modified as follows:

$$\max_{\widetilde{V}_T} \mathbb{E}\left[\sum_{i=1}^{n} \beta_i \frac{\left(\frac{v_i}{v}\widetilde{V}_T\right)^{1-\gamma_i}}{1-\gamma_i}\right] \quad \text{subject to} \quad \mathbb{E}[M_T\widetilde{V}_T] = v \ \text{ and } \ \widetilde{V}_T \geq ve^{gT}. \tag{3.19}$$

The first order conditions of this problem are

$$\sum_{i=1}^{n}\beta_i\left(\frac{v_i}{v}\right)^{1-\gamma_i}\widetilde{V}_T^{-\gamma_i}-\nu_1M_T-\nu_2=0,$$
$$\nu_2(\widetilde{V}_T-ve^{gT})=0.$$

or, equivalently,

$$\widetilde{V}_T=\begin{cases}V_T \text{ s.t. } \sum_{i=1}^{n}\beta_i\left(\frac{v_i}{v}\right)^{1-\gamma_i}V_T^{-\gamma_i}-\nu_1M_T=0, & \text{if } V_T\geq ve^{gT}\\ ve^{gT}, & \text{if } V_T<ve^{gT}\end{cases}.$$

We can solve this problem numerically in the same way as problem (3.12). First we determine ν_1, then we can compute V_T for a given value of M_T. After that, we obtain $\widetilde{V}_T=\max\{V_T,ve^{gT}\}$. We can basically use the same algorithm to compute ν_1 as for the previous problem (3.12), we just need to add a small modification to make sure that the guarantee is always met. This can be done by simply executing one additional step before the higher and lower bounds for ν_1 are redefined (the complete R code can be found in Appendix B again):

1. Initialize r, μ, σ, $\lambda=\frac{\mu-r}{\sigma}$, g, T, tol, N, n, β_i, γ_i, v_i .
2. Use bisection method to obtain ν_1 s.t. the budget constraint is fulfilled: Choose ν_1^h and ν_1^l (higher and lower bounds for ν_1).
3. While $\left|\frac{1}{N}\sum_{j=1}^{N}M_T^j\widetilde{V}_T^j-v\right|>$ tol:
   - 3.1. Create N realizations of a standard normally distributed random variable $X^j\sim\mathcal{N}(0,1),\ \ j=1,\dots,N$.
   - 3.2. Compute N realizations of the state price density at T: $M_T^j=\exp\left(-rT-\frac{1}{2}\lambda^2T-\lambda\sqrt{T}X^j\right)$.
   - 3.3. Set $\nu_1=\frac{1}{2}\left(\nu_1^h+\nu_1^l\right)$.
   - 3.4. Use bisection method to obtain V_T^j for each $j=1,\dots,N$. Choose $V_T^{j,h}$ and $V_T^{j,l}$ (higher and lower bounds for V_T^j).
   - 3.5. While $\max_j\left|\sum_{i=1}^{n}\beta_i(\frac{v_i}{v})^{1-\gamma_i}(V_T^j)^{-\gamma_i}-\nu M_T^j\right|>$ tol:
     - 3.5.1. $V_T^j=\frac{1}{2}\left(V_T^{j,l}+V_T^{j,h}\right)$ for all $j=1,\dots,N$.
     - 3.5.2. For each j with $\sum_{i=1}^{n}\beta_i\left(\frac{v_i}{v}\right)^{1-\gamma_i}\left(V_T^j\right)^{-\gamma_i}-\nu M_T^j>0$ set $V_T^{j,l}=V_T^j$.

3.5.3. For each j with $\sum_{i=1}^{n} \beta_i \left(\frac{v_i}{v}\right)^{1-\gamma_i} \left(V_T^j\right)^{-\gamma_i} - \nu M_T^j < 0$ set $V_T^{j,h} = V_T^j$.

3.6. For each j with $V_T^j < ve^{gT}$ set $\widetilde{V}_T^j = ve^{gT}$, for all remaining j set $\widetilde{V}_T^j = V_T^j$.

3.7. If $\left(\frac{1}{N}\sum_{j=1}^{N} M_T^j \widetilde{V}_T^j - v > 0\right)$, then $\nu_1^l = \nu_1$.

3.8. If $\left(\frac{1}{N}\sum_{j=1}^{N} M_T^j \widetilde{V}_T^j - v < 0\right)$, then $\nu_1^h = \nu_1$.

For creating the realizations of a standard normally distributed random variable we use again a stratified sample as in Section 3.3.1. Having determined ν_1, we can again create a set of realizations of M_T. Then we can create a corresponding set of realizations of $\widetilde{V}_T$ and estimate the certainty equivalent return for each investor as in (3.15).

Now we can compare the terminal payoff with guarantee to the payoff without guarantee determined in Section 3.3.1. We consider the following example:

- There are four investors having the same initial wealth $v_1 = v_2 = v_3 = v_4 = 1$,
- their risk aversion parameters are given by $\gamma_1 = 1/2$, $\gamma_2 = 2$, $\gamma_3 = 8$, $\gamma_4 = 10$, and
- the weights β_i are all equal.

Furthermore, we assume that for all the numerical analyses in this section the level of the interest rate guarantee is given by $g = 0.5\%$. In Figure 3.3 the terminal Pareto efficient payoffs with and without guarantee can be seen. We clearly observe that the position in risky assets needs to be reduced in order to meet the prescribed guarantee. Therefore, the quite risk averse investors 3 and 4 should benefit from this restriction. In Table 3.3 we can see that this is, in fact, the case.

	$\gamma_1 = 1/2$	$\gamma_2 = 2$	$\gamma_3 = 8$	$\gamma_4 = 10$
without guarantee	0.0526	0.0442	0.0188	0.0120
with guarantee	0.0252	0.0237	0.0194	0.0184

Table 3.3.: Comparison between certainty equivalent returns for terminal payoffs with and without interest rate guarantee, where $\gamma_1 = 1/2$, $\gamma_2 = 2$, $\gamma_3 = 8$, $\gamma_4 = 10$, $g = 0.5\%$, $v_i = 1$, $\beta_i = 1/4$.

We observe that investors 1 and 2, the ones with the lower risk aversion, suffer a huge loss in utility because of the newly introduced guarantee. On the other hand, the guarantee compensates the more risk averse investors 3 and 4 who would, without the guarantee, be forced to invest in a more risky way than they would usually choose. These results coincide

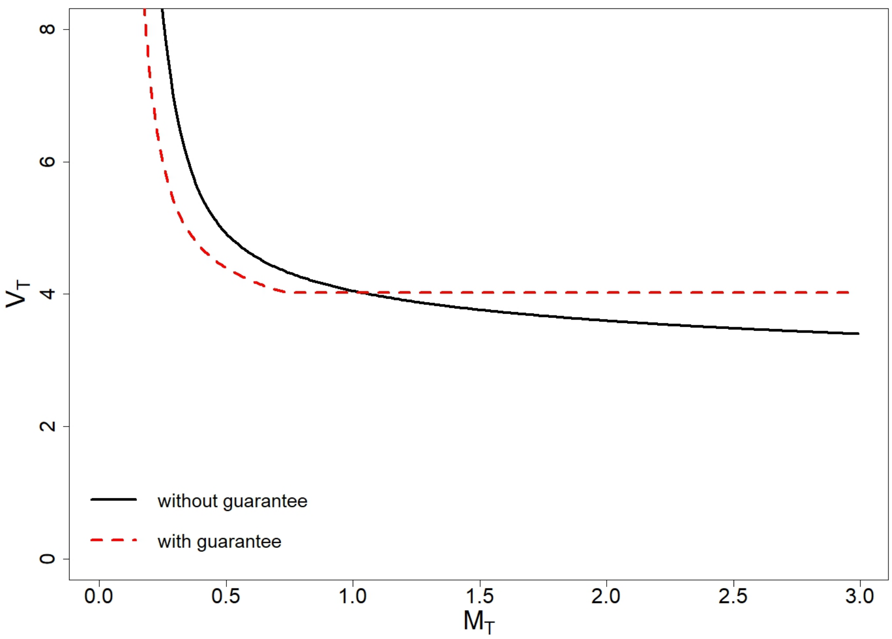


Figure 3.3.: Comparison between Pareto efficient terminal payoffs with and without interest rate guarantee, where $\gamma_1 = 1/2$, $\gamma_2 = 2$, $\gamma_3 = 8$, $\gamma_4 = 10$, $g = 0.5\%$, $v_i = 1$, $\beta_i = 1/4$.

with the results in Jensen and Sørensen (2001) who already performed similar analyses for a pool using an investment strategy corresponding to a joint risk aversion parameter $\bar{\gamma}$ as described in Section 3.2. The advantage of our analysis is, however, that the terminal payoff is Pareto efficient. It is not surprising, though, that the effects of the newly introduced guarantee remain the same, no matter if the terminal payoff is Pareto efficient or not.

We will now show that, in the presence of a guarantee, the payoff obtained by adding the individual optima and the terminal wealth corresponding to $\bar{\gamma} = \sqrt{\gamma_1 \gamma_2}$ are also not Pareto efficient. We will use the exact same example as in Section 3.3.1: There are $n = 2$ investors having the same initial wealth $v_1 = v_2 = 1$, and their risk aversion parameters are $\gamma_1 = 1/2, \ \gamma_2 = 10$. We can proceed in the exact same way as in Section 3.3.1 to find a value for β_1 such that both investors obtain a higher utility from the Pareto efficient payoff than from the the other two approaches.
We will start with the first best payoff which is given by the sum of the individual terminal wealths with guarantee. The certainty equivalent return can be computed in a similar way

as for the case without guarantee. We know that investor i receives

$$V_T^{(i)} = \frac{v_i}{v} \sum_{j=1}^{n} \widetilde{V}_T^{(j,*)} = \frac{v_i}{v} \sum_{j=1}^{n} \max\left\{x_j V_T^{(j,*)}, v_j e^{gT}\right\}$$

with x_j being determined numerically from (2.33) for all $j = 1, \ldots, n$. Hence, the certainty equivalent wealth can be determined as

$$\mathrm{CE}_i\left(\frac{v_i}{v}\sum_{j=1}^{n} V_T^{(j,*)}\right) = \frac{v_i}{v}\mathbb{E}\left[\left(\sum_{j=1}^{n}\max\left\{x_j V_T^{(j,*)}, v_j e^{gT}\right\}\right)^{1-\gamma_i}\right]^{\frac{1}{1-\gamma_i}}.$$

Here we can use (3.16) to obtain

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{j=1}^{n}\widetilde{V}_T^{(j,*)}\right)^{1-\gamma_i}\right] &= \mathbb{E}\left[\left(\sum_{j=1}^{n}\max\left\{x_j v_j c_j \exp\left(m(\gamma_j)\sigma W_T\right), v_j e^{gT}\right\}\right)^{1-\gamma_i}\right] \\ &= \int_{-\infty}^{\infty}\left(\sum_{j=1}^{n}\max\left\{x_j v_j c_j \exp\left(m(\gamma_j)\sigma\sqrt{T}z\right), v_j e^{gT}\right\}\right)^{1-\gamma_i}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)dz,\end{aligned}$$

where c_j is given in (3.17). Of course, this integral needs to be determined numerically again, the corresponding R code is given in Appendix B.
Using the bisection method in a similar way as for the case without guarantee, we obtain one possible value for the first weight as $\beta_1 = 0.55$. The payoffs are given in Figure 3.4. The wealth curves have multiple intersections as in the case without guarantee. From this plot it is not clear at all which payoff should be preferred. To answer this question, we need to take a look at the certainty equivalent returns. They are provided in Table 3.4.

	$\gamma_1 = 1/2$	$\gamma_2 = 10$
Pareto efficient	0.0268	0.0166
Sum of individual payoffs with guarantee	0.0263	0.0160

Table 3.4.: Comparison between certainty equivalent returns obtained from Pareto efficient terminal payoff and the sum of the individual terminal wealths with guarantee, where $\gamma_1 = 1/2$, $\gamma_2 = 10$, $g = 0.5\%$, $v_i = 1$, $\beta_1 = 0.55$.

We can see that the Pareto efficient strategy provides higher returns than the linearly shared sum of the individual payoffs with guarantee to both investors. So we can draw the same conclusion as in the case without guarantee: The first best payoff is, in general, not Pareto optimal under our linear sharing rule.

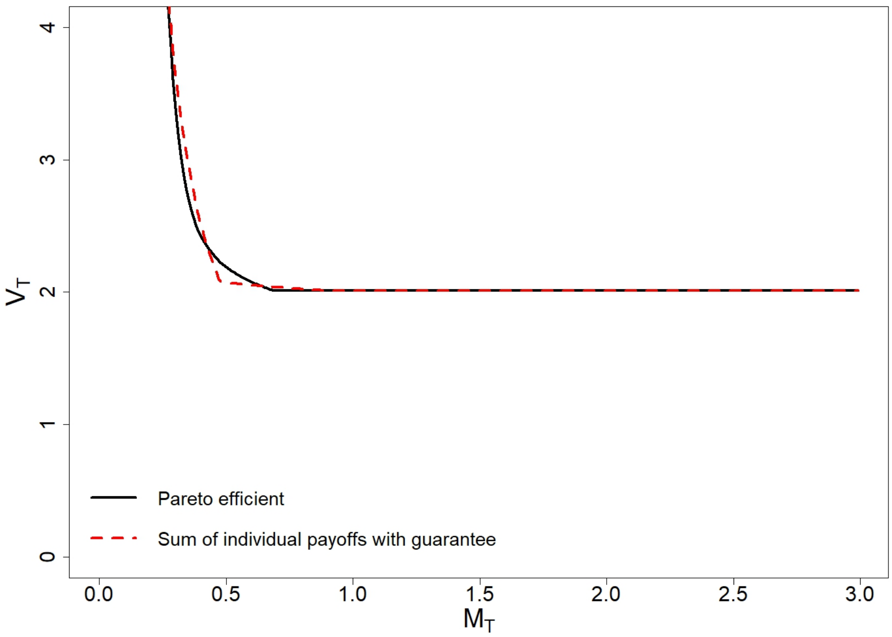


Figure 3.4.: Comparison between Pareto efficient terminal payoff and the sum of the individual terminal wealths with guarantee, where $\gamma_1 = 1/2$, $\gamma_2 = 10$, $g = 0.5\%$, $v_i = 1$, $\beta_1 = 0.55$.

Let us now move on to the joint risk aversion parameter $\bar{\gamma}$. One possible value for β_1 would be 0.525. The corresponding payoff is given in Figure 3.5. To measure the well-being of the two investors we need to take a look at the certainty equivalent return since it is, again, not clear from the plot which payoff preferable is. The results are provided in Table 3.5.

	$\gamma_1 = 1/2$	$\gamma_2 = 10$
Pareto efficient	0.0265	0.0170
$\bar{\gamma}$	0.0263	0.0168

Table 3.5.: Comparison between certainty equivalent returns obtained from Pareto efficient terminal payoff and the payoff obtained from investing according to the joint risk aversion parameter $\bar{\gamma} = \sqrt{\gamma_1 \gamma_2}$ with guarantee, where $\gamma_1 = 1/2$, $\gamma_2 = 10$, $g = 0.5\%$, $v_i = 1$, $\beta_1 = 0.525$.

Once again the Pareto efficient strategy manages to yield a higher certainty equivalent return to both investors in the pool in this specific case. So the presence of a guarantee does not change the fact that the investment policy corresponding to $\bar{\gamma}$ is, generally, not Pareto optimal.

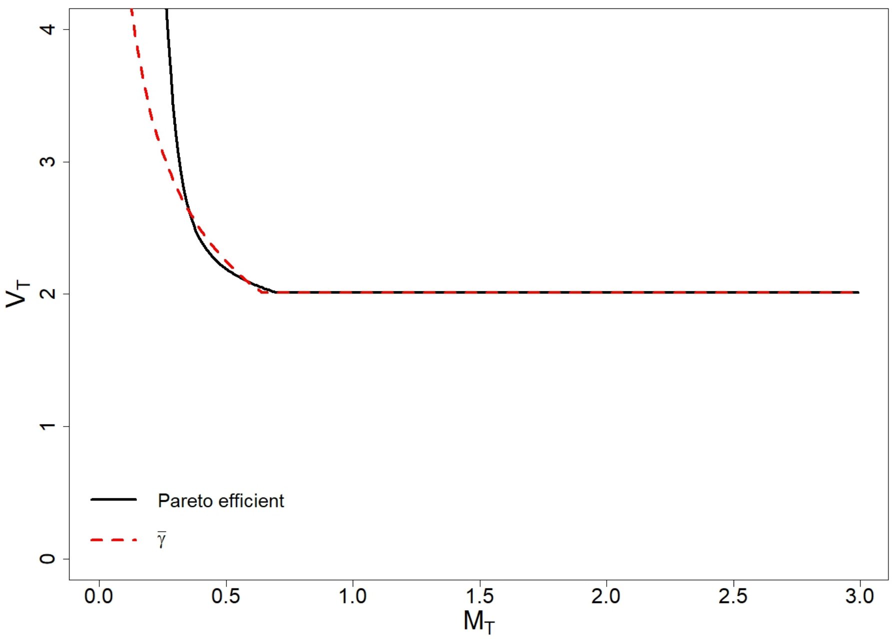


Figure 3.5.: Comparison between Pareto efficient terminal payoff and payoff obtained from investing according to the joint risk aversion parameter $\bar{\gamma} = \sqrt{\gamma_1 \gamma_2}$ with guarantee, where $\gamma_1 = 1/2$, $\gamma_2 = 10$, $g = 0.5\%$, $v_i = 1$, $\beta_1 = 0.525$.

It is, however, important to keep in mind that it might not always be possible to find a vector of weights $(\beta_1, \ldots, \beta_n)$ such that the Pareto efficient strategy provides higher returns to all the members of the collective since the definition of Pareto efficiency clearly does not imply that. Furthermore, the weights presented so far are, in some cases, not really fair to the members of the collective. In section 3.3.1, for example, we used $\beta_1 = 0.99$, so we were taking the less risk averse investor much stronger into account than the more risk averse, although both made the same initial contribution. So we definitely have to ask ourselves the question how the weights should be chosen for any general case. Providing an answer to this question is the purpose of the next section.

3.3.3. One Choice of the Weights

In this section we will use a slightly different method to compare the certainty equivalents of the different investors. If the individual risk aversion parameters are, without loss of generality, chosen to be increasing, that is, $\gamma_1 < \gamma_2 < \cdots < \gamma_n$, it is possible to (approximately) plot the corresponding certainty equivalent returns $\widehat{y}_i$ as a function of the risk aversion parameters γ_i (for the values between γ_i and γ_{i+1} we will use linear

interpolation). We can then have a look at the behavior of this function and try to derive a proper choice for the weights β_i.
In order to investigate the effects of the initial wealth v_i and the risk aversion parameters γ_i on the investment strategy, we will start with $\beta_i = 1/n$. Once we have determined the effects of v_i and γ_i on the certainty equivalent return, we can move on to different choices of the weights β_i. Since the Pareto efficient terminal payoff behaves similarly with and without guarantee, we will, for simplicity, focus only on the case without guarantee in this section. We will use the following example:

- There are $n = 10$ participants in the pool.
- The corresponding risk aversion parameters γ_i are spread equally in the interval $[1/2, 10]$, that is,

$$\gamma_i = 1/2 + \frac{9.5(i-1)}{n-1}, \quad i = 1, \dots, n.$$

We will consider three different cases of initial wealth:

(a) The initial wealth increases linearly in the risk aversion:

$$v_i = 1 + \frac{9(i-1)}{n-1} \quad \text{for all } i = 1, \dots, n.$$

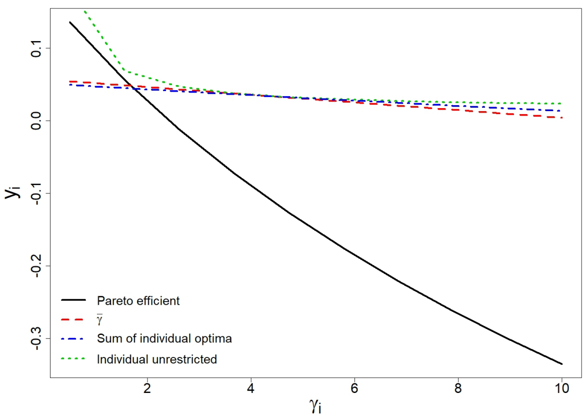
(b) The initial wealth is the same for all the members:

$$v_i = 10 \quad \text{for all } i = 1, \dots, n.$$

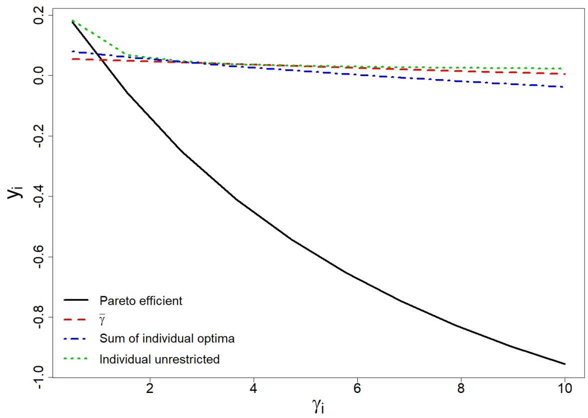
(c) The initial wealth decreases linearly in the risk aversion:

$$v_i = 10 - \frac{9(i-1)}{n-1} \quad \text{for all } i = 1, \dots, n.$$

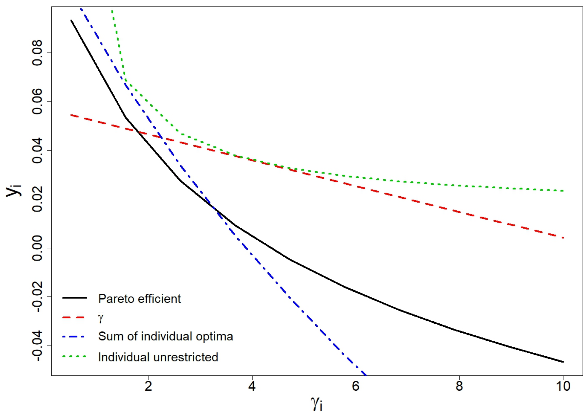
In Figure 3.6 we plot the individual certainty equivalent returns y_i for four payoffs against the risk aversion parameters γ_i. Here we consider the cases already analyzed before, the Pareto efficient payoff, the first best terminal wealth for the collective (with linear sharing rule), the constant mix strategy with $\bar{\gamma} = (\prod_{i=1}^{n} \gamma_i)^{1/n}$ and the individual unrestricted payoff. In other words, the certainty equivalent returns $\hat{y}_i$ from (3.15), y_i' from (3.18), $\bar{y}_i$ from (3.6) and y_i^* from (2.25) are plotted against γ_i. We make the following observations:



(a) $v_i = 1 + \frac{9(i-1)}{n-1}$ for $i = 1, \ldots, n$.



(b) $v_i = 10$ for $i = 1, \ldots, n$.



(c) $v_i = 10 - \frac{9(i-1)}{n-1}$ for $i = 1, \ldots, n$.

Figure 3.6.: Comparison of the certainty equivalent returns with $\beta_i = 1/n$ for three different cases of the initial wealth.

- **In all the cases the Pareto efficient strategy yields negative returns to at least some of the investors.** This effect can be seen in all the figures. Especially in Figures (a) and (b) these negative returns are drastically low. The reason for this is that the Pareto efficient strategy here only favors the least risk averse investors.
- **The Pareto efficient strategy clearly does not seem to be the best option for the collective.** In Figures (a) and (b) clearly both the first best payoff and the constant mix strategy seem to be preferable over the Pareto efficient payoff since we would like to avoid drastically low rates of return for such large groups of the collective. The curves obtained from the first best approach and the constant mix strategy are much closer to the unrestricted individual curve than the Pareto

efficient in these two cases. In Figure (c) it is not that clear which approach the most preferable one is. However, at least the constant mix strategy seems to still be a better choice than the Pareto efficient strategy.

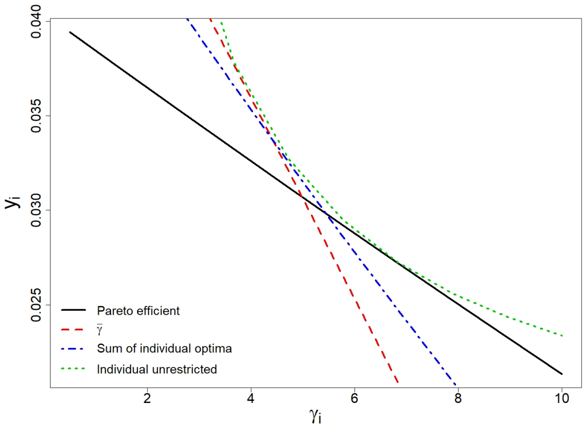
- **The rather risk tolerant investors have a stronger impact on the Pareto efficient strategy than the more risk averse individuals.** This becomes clear in Figures (a) and, particularly, (b). In the latter one all the investors have the same initial wealth. However, the Pareto efficient payoff basically only takes into account the least risk averse investor in the pool.
- **People with a lower initial wealth are taken into account stronger in the investment strategy than more wealthy people.** This effect becomes clear if we compare all the Figures to each other. Usually, the less risk averse people have more influence on the investment strategy (see Figures (a) and (b)). If, however, the more risk averse people have less wealth than the rather risk tolerant investors, the investment strategy takes these risk averse people also into account, as can be seen in Figure (c).

Clearly we would like to manipulate the Pareto efficient payoff in such a way that there are no negative certainty equivalent returns anymore. In all the cases considered here, β_i did not depend on i and could therefore be ignored in the optimization problem. If we want the investment strategy to be more suitable for all the investors in the pool, we need to choose the weights β_i accordingly. From the above observations it is clear that the higher the initial wealth v_i and the risk aversion γ_i are, the higher β_i should be. A possible approach would therefore be the following:

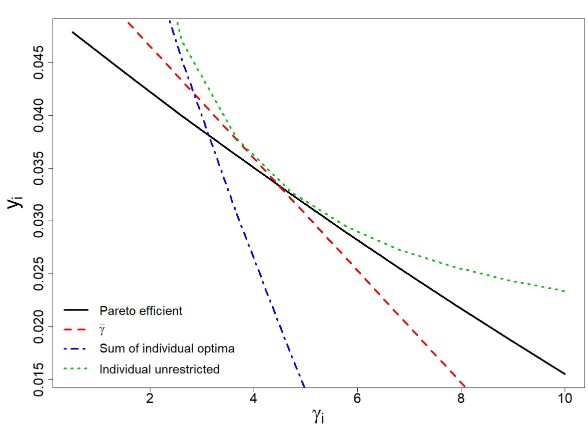
$$\beta_i = \frac{v_i^{\gamma_i}}{\sum_{j=1}^{n} v_j^{\gamma_j}} \, . \tag{3.20}$$

Now we can compute the certainty equivalent returns for all the different examples from before again and perform similar analyses with these new weights. The results are given in Figure 3.7. Here we observe the following:

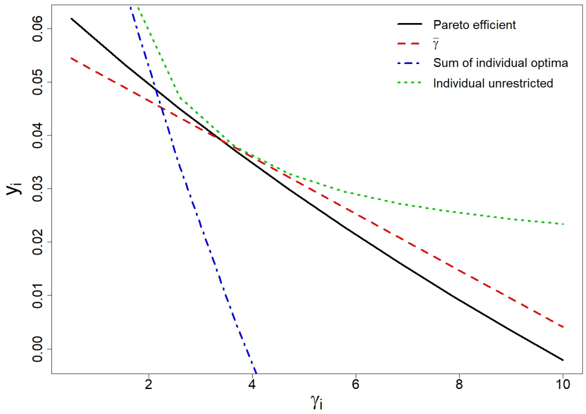
- **The newly chosen weights improve the situation significantly. In particular, the Pareto efficient strategy does not yield negative returns anymore.** This can be observed in all the figures. Especially for Figures (a) and (b) the situation improves drastically. In Figure (c) there are no negative returns anymore as well. All in all, the Pareto efficient returns are now much closer to the individual



(a) $v_i = 1 + \frac{9(i-1)}{n-1}$ for $i = 1, \ldots, n$.



(b) $v_i = 10$ for $i = 1, \ldots, n$.



(c) $v_i = 10 - \frac{9(i-1)}{n-1}$ for $i = 1, \ldots, n$.

Figure 3.7.: Comparison of the certainty equivalent returns with β_i given in (3.20) for three cases of initial wealth.

optimal returns. The "price" of this is, of course, that the risk tolerant investors receive much lower rates of return than before.

- **With the new weights the Pareto efficient strategy takes wealthier people slightly stronger into account than less wealthy individuals.** This is of course due to the fact that β_i is increasing in v_i. This behavior seems fair since people making higher payments should be considered stronger than investors only paying very small contributions. An important observation here is also that this effect is not too strong. The most wealthy investor does not completely dominate the investment decision, all the individuals are still taken into account. We can clearly observe that the more wealthy people are taken stronger into consideration but investors with lower initial wealth are not completely ignored and, in particular, no such drastic losses as in Figure 3.6 are suffered by anyone here.

4. Summary

The individual optimization problem considered here has already been studied thoroughly in the literature. For the unrestricted case, it is well known that the optimal terminal payoff is given by the inverse marginal utility function and that the corresponding self-financing investment strategy is constant over time. Under portfolio insurance, the optimal terminal payoff is the maximum of the guaranteed amount and a fraction of the individual optimal terminal wealth obtained from the optimization problem without restrictions. The corresponding self-financing investment strategy is no longer constant but varies over time. We also observed the well-known result that the guarantee leads to a loss in utility for every investor where the loss is decreasing in the risk aversion parameter γ. This result is quite natural as more risk averse people tend to invest most of their capital in risk-free assets, even if there is no guarantee imposed. For relatively risk tolerant investors, on the other hand, the guarantee can be a rather strong restriction that forces the investor to invest in a much less risky way than would be optimal for her.

Having considered the individual optimization problem, we have presented various approaches to solve the problems of how to invest the total initial wealth of the investors and how to share the total terminal wealth obtained from this strategy. In the first best approach the terminal wealth is obtained from adding the individual unrestricted terminal payoffs for each investor. We can then use a non-linear sharing rule to return to each participant her unrestricted optimal payoff. However, since this sharing rule is non-linear and state dependent, it is usually not used. Therefore, we have decided to only focus on a specific linear sharing rule that is easy to communicate.

Further natural approaches would then be to use the first best payoff in combination with this linear sharing rule or to invest the total initial wealth in the same way as an investor with some risk aversion parameter would do. Usually, this risk aversion parameter lies between the smallest and the largest risk aversion parameters of all the investors in the pool. The latter of these procedures leads to a Pareto efficient payoff if we restrict the

analysis to constant mix strategies, at least for the case without guarantees.
We have seen, however, that none of these two approaches is generally Pareto optimal if we allow for any investment strategy to be used. For such Pareto optimal payoffs we need to maximize the weighted sum of the individual utility functions.
Since we would like to use a Pareto efficient payoff whenever possible, we have to answer the question how the weights should be chosen here. In the numerical analyses performed we have seen that a bad choice of them might result in drastic losses for large groups of the collective. We have therefore derived a choice of the weights which avoids these drastic losses and yields returns that are, in total, closer to the individual optima.
It is also important to note that for a pool of investors the situation with the guarantee is not as clear anymore as in the case with a single investor. For the Pareto efficient payoff with linear sharing rule the more risk averse individuals in the pool benefit from the guarantee while the less risk averse suffer an additional loss compared to the case without guarantee. These results coincide with the results presented in Jensen and Sørensen (2001) who conducted a similar analysis for the payoff obtained by a joint risk aversion parameter.

A. Methods of Monte Carlo Simulation

A.1. Inverse Transformation Method

Let

$$F_X^{-1}(u) = \inf\{x : F_X(x) \geq u\}$$

be the quantile function of a random variable X. Further assume that U is uniformly distributed on $[0,1]$. Then the distribution function of the random variable $F_X^{-1}(U)$ is given by F_X. The proof of this well-known result can be found for example in Ross (2014) or Kroese et al. (2013).
From this observation we can derive the inverse transform algorithm:

1. Simulate $U \sim \mathcal{U}[0,1]$.
2. Return $X = F_X^{-1}(U)$.

A.2. Stratified Sampling

Stratified sampling is a method used to increase the precision of estimators by making sure that the samples used for the estimation are spread sufficiently across the state space. In this section we will briefly review the main idea of stratified sampling in the context of Monte Carlo estimation. For further details regarding this topic we refer to Kroese et al. (2013) or Lohr (2009).
Assume that X is a random variable, $f(\cdot)$ is a function and we would like to estimate $\ell = \mathbb{E}[f(X)]$. We further assume that the distribution of X is known and that we can

divide the range of X into disjoint sets $A_1, \ldots, A_K$ (also called strata). Note that, under these assumptions, we have

$$\ell = \mathbb{E}[f(X)] = \sum_{i=1}^{K} \mathbb{E}[f(X) \mid X \in A_i] P(X \in A_i) \,.$$

Let us introduce the following notation:

- Let $p_i = P(X \in A_i)$ for all $i = 1, \ldots, K$.
- Further let $\{X_j^i\}_{j=1,\ldots,N_i}$ be a sequence of independent and identically distributed random variables drawn from the conditional distribution of X given that $X \in A_i$ with N_i being a positive integer for $i = 1, \ldots, K$.

Then ℓ can be estimated by

$$\widehat{\ell} = \sum_{i=1}^{K} p_i \frac{1}{N_i} \sum_{j=1}^{N_i} f\left(X_j^i\right) . \tag{A.1}$$

The question remains how the strata can be chosen. For our simulation algorithms we will always use the (rather simple) proportional allocation. To be more precise, denoting by N the total number of realizations, we assume that

$$N_i = p_i N \,.$$

It is proven in Kroese et al. (2013) that from using proportional sampling one always obtains an estimator having a smaller or equal variance than the standard Monte Carlo estimator.

B. R Codes

In this section the R codes used for the numerical analyses performed in the thesis are provided. Since all the codes used for the plots and computations of certainty equivalent returns in Chapter 3 solve either optimization problem (3.12) or (3.19), we omit the codes for Table 3.3 and Figures 3.3, 3.6 and 3.7. Clearly, it is possible to change the parameters in the codes provided below in any way we would like to in order to obtain the results given in these figures.

Listing B.1: R code used for comparing the individual terminal payoff, the investment strategy and the certainty equivalent return with and without guarantee (see Figures 2.1, 2.2 and 2.3).

```
######################### Parameters
r<-0.015
mu<-0.06
sigma<-0.11
lambda<-(mu-r)/sigma
mat<-1
gamma<-4
v<-1
g<-0.005

######################### Plot Payoff #########################
######################### Unrestricted
M_T<-seq(0.01,3,0.01)
V_T <- v*exp((r+1/2*lambda^2)*(1-1/gamma)*mat - 1/2*lambda^2*(1-1/gamma)^2*
mat)*M_T^(-1/gamma)
par(mar=c(4.5,5.5,2,2))
plot(M_T,V_T,type="l",xlab="",ylab="",lwd=4,yaxt="n",xaxt="n")
axis(1,cex.axis=2)
axis(2,cex.axis=2)
mtext(expression(V[T]), side=2, line=3, cex=2.5)
mtext(expression(M[T]), side=1, line=3.5, cex=2.5)
```

```
########################## With guarantee
tol<-10^(-12)
sigma_MT<-lambda*sqrt(mat)
x_h<-1
x_l<-0
x<-(x_l + x_h)/2

d_1 <- gamma*((g-r)*mat-log(x))/sigma_MT + (1/(2*gamma)-1)*sigma_MT
d_2 <- d_1+(1-1/gamma)*sigma_MT

while (abs(x*pnorm(-d_2) + exp((g-r)*mat)*pnorm(d_1+sigma_MT) - 1) > tol) {
  if (x*pnorm(-d_2) + exp((g-r)*mat)*pnorm(d_1+sigma_MT) > 1) {
    x_h <- x
  } else {
    x_l <- x
  }
  x<-(x_l + x_h)/2
  d_1 <- gamma*((g-r)*mat-log(x))/sigma_MT + (1/(2*gamma)-1)*sigma_MT
  d_2 <- d_1+(1-1/gamma)*sigma_MT
}

V_T_g <- pmax(v*exp(g*mat),x*V_T)
lines(M_T,V_T_g,lty=2,lwd=4,col=2)
legend("topright",c("without guarantee","with guarantee"),
lty=c(1,2),bty="n",lwd=c(4,4),col=c(1,2),cex=1.8)



########################## Plot strategy ##############################
########################## Unrestricted
m<-10^3
t<-seq(1/m,mat,1/m)
pi <- (mu-r)/(gamma*sigma^2)
Pi<-rep(pi,length(t))
plot(t,Pi,xlab="",ylab="",type="l",lwd=3,yaxt="n",xaxt="n",
ylim=range(c(0,1.5)))
par(mar=c(4.5,5.5,2,2))
axis(1,cex.axis=2)
axis(2,cex.axis=2)
mtext(expression(pi[t]), side=2, line=3, cex=2.5)
mtext(expression(t), side=1, line=3.5, cex=2.5)

########################## With guarantee
```

```
W<-cumsum(sqrt(1/m)*rnorm(m))
#plot(t,W,type="l")

V_t <- v*exp(r*t + lambda^2*t*(1/gamma-1/(2*gamma^2)) + lambda/gamma*W)
D_1 <- (gamma*((g-r)*mat-log(x)) + (1/(2*gamma)-1)*lambda^2*mat -
lambda*W)/(lambda*sqrt(mat-t))

D_2 <- D_1+lambda*sqrt(mat-t)*(1-1/gamma)

delta_f <- x*lambda/gamma*V_t - v*exp(g*mat)*exp(-r*(mat-t))*
dnorm(D_1+lambda*sqrt(mat-t))/sqrt(mat-t) - x*lambda/gamma*V_t*pnorm(D_2) +
x*V_t*dnorm(D_2)/sqrt(mat-t)

V_t_tilde <- x*V_t + v*exp(g*mat)*exp(-r*(mat-t))*pnorm(D_1+lambda*
sqrt(mat-t)) - x*V_t*pnorm(D_2)

pi_tilde<-delta_f/(sigma*V_t_tilde)

lines(t,pi_tilde,lty=2,col=2,lwd=3)
legend("topleft",c("without guarantee","with guarantee"),lty=c(1,2),
bty="n",lwd=c(3,3),col=c(1,2),cex=1.8)

########################## Plot CE return ##########################
n<-100
gamma<-seq(1/2,10,length.out=n)

########################## Unrestricted
y<-r+1/2*(mu-r)^2/(gamma*sigma^2)

########################## With guarantee
y_g<-rep(0,n)
for (i in 1:n) {
  x_h<-1
  x_l<-0
  x<-(x_l + x_h)/2

  d_1 <- gamma[i]*((g-r)*mat-log(x))/sigma_MT + (1/(2*gamma[i])-1)*sigma_MT
  d_2 <- d_1+(1-1/gamma[i])*sigma_MT

  while (abs(x*pnorm(-d_2) + exp((g-r)*mat)*pnorm(d_1+sigma_MT)-1)>tol) {
    if (x*pnorm(-d_2) + exp((g-r)*mat)*pnorm(d_1+sigma_MT) > 1) {
      x_h <- x
```

```
    } else {
      x_l <- x
    }
    x<-(x_l + x_h)/2
    d_1 <- gamma[i]*((g-r)*mat-log(x))/sigma_MT + (1/(2*gamma[i])-1)*
    sigma_MT
    d_2 <- d_1+(1-1/gamma[i])*sigma_MT
  }
  y_g[i]<-1/(mat*(1-gamma[i]))*log(exp(g*mat*(1-gamma[i]))*pnorm(d_1) +
  x^(1-gamma[i])*exp((1-gamma[i])*(r+1/2*lambda^2/gamma[i])*mat)*
  (1-pnorm(d_2)))
}

par(mar=c(4.5,5.5,2,2))
plot(gamma,y,type="l",xlab="",ylab="",lwd=4,yaxt="n",xaxt="n")
axis(1,cex.axis=2)
axis(2,cex.axis=2)
mtext(expression(y), side=2, line=3, cex=2.5)
mtext(expression(gamma), side=1, line=3.5, cex=2.5)

lines(gamma,y_g,lty=2,lwd=4,col=2)
legend("topright",c("without guarantee","with guarantee"),lty=c(1,2),
bty="n",lwd=c(4,4),col=c(1,2),cex=1.8)
```

Listing B.2: R code used for comparing the Pareto optimal terminal payoff and certainty equivalent return with the first best payoff (see Figure 3.1 and Table 3.1) and the payoff resulting from a constant mix strategy (see Figure 3.2 and Table 3.2).

```
########################## Parameters
r<-0.015
mu<-0.06
sigma<-0.11
lambda<-(mu-r)/sigma
mat<-1
n<-2
N<-10^6
C<-250

gamma<-c(1/2,10)
v<-rep(1,n)
b<-0.99
```

```
beta<-c(b,1-b)

V<-sum(v)

######################### Pareto efficient #########################
######################### Compute Lagrangian multiplier
tol_1<-10^(-12)
tol_2<-10^(-12)
tol_3<-10^(-20)
tol_4<-10^(-20)

U<-rep(0,N)
K<-N/100
for (j in 1:K) {
  U[((j-1)*N/K+1):(j*N/K)]<-runif(N/K,(j-1)/K,j/K)
}
X<-qnorm(U)

M_T<-exp(-r*mat-1/2*lambda^2*mat-lambda*sqrt(mat)*X)

V_T<-rep(1,N)
zeta_h<-V*10
zeta_l<-0

while(abs(mean(M_T*V_T)-V) > tol_1) {

  U<-rep(0,N)
  K<-N/100
  for (j in 1:K) {
    U[((j-1)*N/K+1):(j*N/K)]<-runif(N/K,(j-1)/K,j/K)
  }
  X<-qnorm(U)
  M_T<-exp(-r*mat-1/2*lambda^2*mat-lambda*sqrt(mat)*X)

  zeta<-(zeta_l+zeta_h)/2
  print(zeta)

  V_h <- rep(C*V,N)
  V_l <- rep(0,N)
  V_T<-(V_l + V_h)/2

  A<-matrix(0,N,n)
```

```
  for (i in 1:n) {
    A[,i]<-beta[i]*(v[i]/V)^(1-gamma[i])*V_T^(-gamma[i])
  }

  while(max(abs(rowSums(A) - zeta*M_T)) > tol_2) {

    V_T<-(V_h + V_l)/2

    A<-matrix(0,N,n)
    for (i in 1:n) {
      A[,i]<-beta[i]*(v[i]/V)^(1-gamma[i])*V_T^(-gamma[i])
    }

    V_l[rowSums(A) - zeta*M_T > 0] <- V_T[rowSums(A) - zeta*M_T > 0]

    V_h[rowSums(A) - zeta*M_T < 0] <- V_T[rowSums(A) - zeta*M_T < 0]

    if (max(V_h-V_l) < tol_3) {
      break
    }
    #print(max(V_h-V_l))
  }

  if (mean(M_T*V_T)-V > 0) {
    zeta_l<-zeta
  } else {
    zeta_h<-zeta
  }

  if (zeta_h-zeta_l < tol_4) {
    break
  }
}

print(zeta)

########################## Compute set of realizations
U<-rep(0,N)
K<-N/100
for (j in 1:K) {
  U[((j-1)*N/K+1):(j*N/K)]<-runif(N/K,(j-1)/K,j/K)
}
```

```
X<-qnorm(U)

M_T<-exp(-r*mat-1/2*lambda^2*mat-lambda*sqrt(mat)*X)

V_h <- rep(C*V,N)
V_l <- rep(0,N)
V_T<-(V_l + V_h)/2

A<-matrix(0,N,n)
for (i in 1:n) {
  A[,i]<-beta[i]*(v[i]/V)^(1-gamma[i])*V_T^(-gamma[i])
}

while(max(abs(rowSums(A) - zeta*M_T)) > tol_1) {

  V_T<-(V_h + V_l)/2

  A<-matrix(0,N,n)
  for (i in 1:n) {
    A[,i]<-beta[i]*(v[i]/V)^(1-gamma[i])*V_T^(-gamma[i])
  }

  V_l[rowSums(A) - zeta*M_T > 0] <- V_T[rowSums(A) - zeta*M_T > 0]

  V_h[rowSums(A) - zeta*M_T < 0] <- V_T[rowSums(A) - zeta*M_T < 0]

  if (max(V_h-V_l) < tol_3) {
    break
  }
}

print(mean(M_T*V_T))
print(max(abs(rowSums(A) - zeta*M_T)))

########################## Compute CE
EU<-rep(0,length(gamma))
CE<-EU
y<-EU
for (i in 1:length(gamma)) {
  EU[i] <- 1/(1-gamma[i])*mean((v[i]/V * V_T)^(1-gamma[i]))
  CE[i] <- (mean(((v[i]/V)*V_T)^(1-gamma[i])))^(1/(1-gamma[i]))
  y[i] <- 1/mat * log(CE[i]/v[i])
```

```
}

######################## Plot payoff
 tol<-10^(-12)
 tol_2<-10^(-20)

M_T<-seq(0.001,3,0.01)
V_T<-rep(1,length(M_T))
C<-10^8

V_h <- rep(C*V,length(M_T))
V_l <- rep(0,length(M_T))
A<-matrix(0,length(M_T),n)
for (i in 1:n) {
  A[,i]<-beta[i]*(v[i]/V)^(1-gamma[i])*V_T^(-gamma[i])
}

while(max(abs(rowSums(A) - zeta*M_T)) > tol) {

  V_T<-(V_h + V_l)/2

  A<-matrix(0,length(M_T),n)
  for (i in 1:n) {
    A[,i]<-beta[i]*(v[i]/V)^(1-gamma[i])*V_T^(-gamma[i])
  }

  V_l[rowSums(A) - zeta*M_T > 0] <- V_T[rowSums(A) - zeta*M_T > 0]

  V_h[rowSums(A) - zeta*M_T < 0] <- V_T[rowSums(A) - zeta*M_T < 0]

  if (max(V_h-V_l) < tol_2) {
    break
  }
  #print(max(V_h-V_l))
}

par(mar=c(4.5,5.5,2,2))
plot(M_T,V_T,type="l",ylim=range(c(0,2*V)),xlab="",ylab="",lwd=4,yaxt="n",
xaxt="n")
axis(1,cex.axis=2)
axis(2,cex.axis=2)
mtext(expression(V[T]), side=2, line=3, cex=2.5)
```

```
mtext(expression(M[T]), side=1, line=3.5, cex=2.5)

########################### First best ##########################
########################### Plot payoff
V_ag <- v[1]*exp((r+1/2*lambda^2)*(1-1/gamma[1])*mat - 1/2*lambda^2*
(1-1/gamma[1])^2*mat)*M_T^(-1/gamma[1]) + v[2]*exp((r+1/2*lambda^2)*
(1-1/gamma[2])*mat - 1/2*lambda^2*(1-1/gamma[2])^2*mat)*M_T^(-1/gamma[2])

V_1<-v[1]*exp((r+1/2*lambda^2)*(1-1/gamma[1])*mat - 1/2*lambda^2*
(1-1/gamma[1])^2*mat)*M_T^(-1/gamma[1])
V_2<-v[2]*exp((r+1/2*lambda^2)*(1-1/gamma[2])*mat - 1/2*lambda^2*
(1-1/gamma[2])^2*mat)*M_T^(-1/gamma[2])

lines(M_T,V_ag,lty=2,lwd=4,col=2)
legend("bottomleft",c("Pareto effcient payoff",
"Sum of unrestricted optimal payoffs"),lty=c(1,2),bty="n",lwd=c(4,4),
col=c(1,2),cex=1.8)

########################### Compute CE
CE_1<-rep(0,length(gamma))
y_1<-CE_1
m<-c((mu-r)/(gamma*sigma^2))

c <- exp((1-m) * (r+1/2*m*sigma^2)*mat + m*(mu-1/2*sigma^2)*mat)

for (j in 1:length(gamma)) {

  h<-function(x) {
    z<-(sum(v*c*exp(m*sigma*sqrt(mat)*x)))^(1-gamma[j])*
    1/sqrt(2*pi)*exp(-x^2/2)
    return(z)
  }

  h<-Vectorize(h,"x")
  E<-integral(h,-Inf,Inf,reltol=1e-30)
  CE_1[j] <- v[j]/sum(v) * E^(1/(1-gamma[j]))
  y_1[j]<-1/mat*log(CE_1[j]/v[j])
}

print(y)
print(y_1)
```

```
########################## Constant mix strategy ##########################
########################## Plot payoff
J_bar <- rep(0,length(gamma))
CE_bar <- J_bar
y_bar <- J_bar

gamma_bar<-prod(gamma)^(1/length(gamma))
W_T <- V*exp((r+1/2*lambda^2)*(1-1/gamma_bar)*mat-1/2*
lambda^2*(1-1/gamma_bar)^2*mat)*M_T^(-1/gamma_bar)

lines(M_T,W_T,lty=2,lwd=4,col=2)
legend("bottomleft",c("Pareto efficient",expression(
bar(gamma) == sqrt(gamma[1] %.% gamma[2]))),
lty=c(1,2),bty="n",lwd=c(4,4),col=c(1,2),cex=1.8)

########################## Compute CE
for (j in 1:length(gamma)) {
  J_bar[j] <- (v[j]*exp(r*mat + 1/(2*gamma[j])*
  (1-((gamma_bar-gamma[j])/gamma_bar)^2) *
  lambda^2*mat))^(1-gamma[j]) / (1-gamma[j])

  CE_bar[j] <- ((1-gamma[j]) * J_bar[j])^(1/(1-gamma[j]))
  y_bar[j] <- 1/mat * log(CE_bar[j]/(v[j]))
}

print(y)
print(y_bar)
```

Listing B.3: R code used for comparing the Pareto efficient payoff with guarantee to the payoff given by the sum of the individual terminal wealths with guarantee (see Figure 3.4 and Table 3.4) and to the payoff obtained from a joint risk aversion parameter with guarantee (see Figure 3.5 and Table 3.5).

```
########################## Parameters
r<-0.015
mu<-0.06
sigma<-0.11
lambda<-(mu-r)/sigma
mat<-1
n<-2
N<-10^6
C<-250
```

```
gamma<-c(1/2,10)
v<-rep(1,n)
b<-0.525
beta<-c(b,1-b)

V<-sum(v)

g<-0.005

######################### Pareto efficient #########################
######################### Compute Lagrangian multiplier
tol_1<-10^(-12)
tol_2<-10^(-12)
tol_3<-10^(-20)
tol_4<-10^(-20)

U<-rep(0,N)
K<-N/100
for (j in 1:K) {
  U[((j-1)*N/K+1):(j*N/K)]<-runif(N/K,(j-1)/K,j/K)
}
X<-qnorm(U)

M_T<-exp(-r*mat-1/2*lambda^2*mat-lambda*sqrt(mat)*X)

V_T<-rep(1,N)
zeta_h<-V*5
zeta_l<-0

while(abs(mean(M_T*V_T)-V) > tol_1) {

  U<-rep(0,N)
  K<-N/100
  for (j in 1:K) {
    U[((j-1)*N/K+1):(j*N/K)]<-runif(N/K,(j-1)/K,j/K)
  }
  X<-qnorm(U)
  M_T<-exp(-r*mat-1/2*lambda^2*mat-lambda*sqrt(mat)*X)

  zeta<-(zeta_l+zeta_h)/2
  print(zeta)
```

```
  V_h <- rep(C*V,N)
  V_l <- rep(0,N)
  V_T<-(V_l + V_h)/2

  A<-matrix(0,N,n)
  for (i in 1:n) {
    A[,i]<-beta[i]*(v[i]/V)^(1-gamma[i])*V_T^(-gamma[i])
  }

  while(max(abs(rowSums(A) - zeta*M_T)) > tol_2) {

    V_T<-(V_h + V_l)/2

    A<-matrix(0,N,n)
    for (i in 1:n) {
      A[,i]<-beta[i]*(v[i]/V)^(1-gamma[i])*V_T^(-gamma[i])
    }

    V_l[rowSums(A) - zeta*M_T > 0] <- V_T[rowSums(A) - zeta*M_T > 0]

    V_h[rowSums(A) - zeta*M_T < 0] <- V_T[rowSums(A) - zeta*M_T < 0]

    if (max(V_h-V_l) < tol_3) {
      break
    }
    #print(max(V_h-V_l))
  }
  V_T[V_T < V*exp(g*mat)] <- V*exp(g*mat)

  if (mean(M_T*V_T)-V > 0) {
    zeta_l<-zeta
  } else {
    zeta_h<-zeta
  }

  if (zeta_h-zeta_l < tol_4) {
    break
  }
}

print(zeta)
```

```
########################## Compute set of realizations
U<-rep(0,N)
K<-N/100
for (j in 1:K) {
  U[((j-1)*N/K+1):(j*N/K)]<-runif(N/K,(j-1)/K,j/K)
}
X<-qnorm(U)

M_T<-exp(-r*mat-1/2*lambda^2*mat-lambda*sqrt(mat)*X)

V_h <- rep(C*V,N)
V_l <- rep(0,N)
V_T<-(V_l + V_h)/2

A<-matrix(0,N,n)
for (i in 1:n) {
  A[,i]<-beta[i]*(v[i]/V)^(1-gamma[i])*V_T^(-gamma[i])
}

while(max(abs(rowSums(A) - zeta*M_T)) > tol_1) {

  V_T<-(V_h + V_l)/2

  A<-matrix(0,N,n)
  for (i in 1:n) {
    A[,i]<-beta[i]*(v[i]/V)^(1-gamma[i])*V_T^(-gamma[i])
  }

  V_l[rowSums(A) - zeta*M_T > 0] <- V_T[rowSums(A) - zeta*M_T > 0]

  V_h[rowSums(A) - zeta*M_T < 0] <- V_T[rowSums(A) - zeta*M_T < 0]

  if (max(V_h-V_l) < tol_3) {
    break
  }
}

V_T[V_T < V*exp(g*mat)] <- V*exp(g*mat)

print(mean(M_T*V_T))
print(max(abs(rowSums(A) - zeta*M_T)))
```

```
########################## Compute CE
EU_g<-rep(0,length(gamma))
CE_g<-EU_g
y_g<-EU_g
for (i in 1:length(gamma)) {
  EU_g[i] <- 1/(1-gamma[i])*mean((v[i]/V * V_T)^(1-gamma[i]))
  CE_g[i] <- (mean(((v[i]/V)*V_T)^(1-gamma[i])))^(1/(1-gamma[i]))
  y_g[i] <- 1/mat * log(CE_g[i]/v[i])
}

########################## Plot payoff
tol<-10^(-12)
tol_2<-10^(-20)

M_T<-seq(0.001,3,0.01)
V_T<-rep(1,length(M_T))
C<-10^8

V_h <- rep(C*V,length(M_T))
V_l <- rep(0,length(M_T))
A<-matrix(0,length(M_T),n)
for (i in 1:n) {
  A[,i]<-beta[i]*(v[i]/V)^(1-gamma[i])*V_T^(-gamma[i])
}

while(max(abs(rowSums(A) - zeta*M_T)) > tol) {

  V_T<-(V_h + V_l)/2

  A<-matrix(0,length(M_T),n)
  for (i in 1:n) {
    A[,i]<-beta[i]*(v[i]/V)^(1-gamma[i])*V_T^(-gamma[i])
  }

  V_l[rowSums(A) - zeta*M_T > 0] <- V_T[rowSums(A) - zeta*M_T > 0]

  V_h[rowSums(A) - zeta*M_T < 0] <- V_T[rowSums(A) - zeta*M_T < 0]

  if (max(V_h-V_l) < tol_2) {
    break
  }
```

```
  #print(max(V_h-V_l))
}
V_T[V_T < V*exp(g*mat)] <- V*exp(g*mat)

par(mar=c(4.5,5.5,2,2))
plot(M_T,V_T,type="l",ylim=range(c(0,2*V)),xlab="",ylab="",lwd=4,yaxt="n",
xaxt="n")
axis(1,cex.axis=2)
axis(2,cex.axis=2)
mtext(expression(V[T]), side=2, line=3, cex=2.5)
mtext(expression(M[T]), side=1, line=3.5, cex=2.5)

######################### First best with guarantee ########################
######################### Plot payoff
M_T<-seq(0.001,3,0.01)
V_T<-matrix(0,length(M_T),n)
V_T_g<-V_T
for (i in 1:n) {
  V_T[,i] <- v[i]*exp((r+1/2*lambda^2)*(1-1/gamma[i])*mat - 1/2*lambda^2*
  (1-1/gamma[i])^2*mat)*M_T^(-1/gamma[i])
}

tol<-10^(-12)
sigma_MT<-lambda*sqrt(mat)
for (i in 1:n) {
  x_h<-1
  x_l<-0
  x<-(x_l + x_h)/2

  d_1 <- gamma[i]*((g-r)*mat-log(x))/sigma_MT + (1/(2*gamma[i])-1)*
  sigma_MT
  d_2 <- d_1+(1-1/gamma[i])*sigma_MT

  while (abs(x*pnorm(-d_2) + exp((g-r)*mat)*pnorm(d_1+sigma_MT) - 1) > tol)
  {
    if (x*pnorm(-d_2) + exp((g-r)*mat)*pnorm(d_1+sigma_MT) > 1) {
      x_h <- x
    } else {
      x_l <- x
    }
    x<-(x_l + x_h)/2
    d_1 <- gamma[i]*((g-r)*mat-log(x))/sigma_MT + (1/(2*gamma[i])-1)*
```

```
    sigma_MT
    d_2 <- d_1+(1-1/gamma[i])*sigma_MT
  }
  V_T_g[,i] <- pmax(v[i]*exp(g*mat),x*V_T[,i])
}

V_T_g_sum <- rowSums(V_T_g)

lines(M_T,V_T_g_sum,lty=2,lwd=4,col=2)
legend("bottomleft",c("Pareto efficient","Sum of individual payoffs with
guarantee"),lty=c(1,2),bty="n",lwd=c(4,4),col=c(1,2),cex=1.8)

########################## Compute CE
CE_1<-rep(0,length(gamma))
y_1<-CE_1
m<-(mu-r)/(gamma*sigma^2)
g<-0.005
sigma_MT<-lambda*sqrt(mat)
tol<-10^(-14)

c <- exp((1-m) * (r+1/2*m*sigma^2)*mat + m*(mu-1/2*sigma^2)*mat)
x_0<-rep(0,n)

for (i in 1:length(gamma)) {

  x_h<-1
  x_l<-0
  x<-(x_l + x_h)/2

  d_1 <- gamma[i]*((g-r)*mat-log(x))/sigma_MT + (1/(2*gamma[i])-1)*
  sigma_MT
  d_2 <- d_1+(1-1/gamma[i])*sigma_MT

  while (abs(x*pnorm(-d_2) + exp((g-r)*mat)*pnorm(d_1+sigma_MT) - 1) > tol)
  {
    if (x*pnorm(-d_2) + exp((g-r)*mat)*pnorm(d_1+sigma_MT) > 1) {
      x_h <- x
    } else {
      x_l <- x
    }
    x<-(x_l + x_h)/2
    d_1 <- gamma[i]*((g-r)*mat-log(x))/sigma_MT + (1/(2*gamma[i])-1)*
```

```
    sigma_MT
    d_2 <- d_1+(1-1/gamma[i])*sigma_MT
  }
  x_0[i]<-x
}

z_crit <- 1/(m*sigma*sqrt(mat)) * log(1/(x_0*c) * exp(g*mat))

for (i in 1:length(gamma)) {

  summand_1 <- function(j) {
    v[j]*exp(g*mat)
  }

  summand_2 <- function(j,z) {
    x_0[j]*v[j]*c[j]*exp(m[j]*sigma*sqrt(mat)*z)
  }

  integrand_1 <- function(z) {
    (sum(sapply(seq(1,n),summand_1)))^(1-gamma[i]) * 1/sqrt(2*pi) *
    exp(-z^2/2)
  }

  integrand_2 <- function(z) {
    (summand_1(which.max(z_crit)) + summand_2(which.min(z_crit),z))^
    (1-gamma[i])*1/sqrt(2*pi) * exp(-z^2/2)
  }

  integrand_3 <- function(z) {
    (sum(sapply(seq(1,n),summand_2,z)))^(1-gamma[i]) * 1/sqrt(2*pi)*
    exp(-z^2/2)
  }

  E_1<-integrate(Vectorize(integrand_1),-Inf,min(z_crit))
  E_2<-integrate(Vectorize(integrand_2),min(z_crit),max(z_crit))
  E_3<-integrate(Vectorize(integrand_3),max(z_crit),10^2)

  CE_1[i] <- v[i]/sum(v) * (E_1[1]$value + E_2[1]$value + E_3[1]$value)^
  (1/(1-gamma[i]))
  y_1[i]<-1/mat*log(CE_1[i]/v[i])
}
```

```
print(y_g)
print(y_1)

######################### Joint gamma #########################
######################### Plot payoff
tol<-10^(-12)
sigma_MT<-sqrt(lambda^2*mat)
W_T<-rep(0,length(M_T))
gamma_bar<-prod(gamma)^(1/length(gamma))

W_T <- V*exp((r+1/2*lambda^2)*(1-1/gamma_bar)*mat-1/2*lambda^2*
(1-1/gamma_bar)^2*mat)*M_T^(-1/gamma_bar)

x_h<-1
x_l<-0
x<-(x_l + x_h)/2

d_1 <- gamma_bar*((g-r)*mat-log(x))/sigma_MT + (1/(2*gamma_bar)-1)*
sigma_MT
d_2 <- d_1+(1-1/gamma_bar)*sigma_MT

while (abs(x*pnorm(-d_2) + exp((g-r)*mat)*pnorm(d_1+sigma_MT) - 1) > tol) {
  if (x*pnorm(-d_2) + exp((g-r)*mat)*pnorm(d_1+sigma_MT) > 1) {
    x_h <- x
  } else {
    x_l <- x
  }
  x<-(x_l + x_h)/2
  d_1 <- gamma_bar*((g-r)*mat-log(x))/sigma_MT + (1/(2*gamma_bar)-1)*
  sigma_MT
  d_2 <- d_1+(1-1/gamma_bar)*sigma_MT
}

W_T_g <- pmax(V*exp(g*mat),x*W_T)

lines(M_T,W_T_g,lty=2,lwd=4,col=2)
legend("bottomleft",c("Pareto efficient",expression(bar(gamma))),
lty=c(1,2),bty="n",lwd=c(4,4),col=c(1,2),cex=1.8)

######################### Compute CE
d1_bar<-gamma_bar*((g-r)*mat-log(x))/sigma_MT + (1/(2*gamma_bar)-1)*
sigma_MT
```

```
d2_bar<-d1_bar + (gamma-1)/gamma_bar * sigma_MT

J <- (x*v*exp(r*mat + 1/(2*gamma)*(1-((gamma_bar-gamma)/gamma_bar)^2)*
sigma_MT^2))^(1-gamma)/(1-gamma)
EU_1 <- (v*exp(g*mat))^(1-gamma)/(1-gamma)*pnorm(d1_bar) +
J*(1-pnorm(d2_bar))

CE_1_u<-((1-gamma)*EU_1)^(1/(1-gamma))
y_1_u <- 1/mat*log(CE_1_u/v)

print(y_g)
print(y_1_u)
```

Bibliography

Amershi, A. H. and Stoeckenius, J. H. (1983). The theory of syndicates and linear sharing rules. *Econometrica: Journal of the Econometric Society*, pages 1407–1416.

Basak, S. (1995). A general equilibrium model of portfolio insurance. *The Review of Financial Studies*, 8(4):1059–1090.

Branger, N., Chen, A., Gatzert, N., and Mahayni, A. (2018). Optimal investments under linear sharing rules. *Working paper*.

Brigo, D. and Mercurio, F. (2007). *Interest rate models-theory and practice: with smile, inflation and credit*. Springer Science & Business Media.

Chen, A. and Nguyen, T. (2018). Pareto optimal asset allocation under financial fairness. *Working paper*.

Cox, J. C. and Huang, C.-f. (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of economic theory*, 49(1):33–83.

Dumas, B. (1989). Two-person dynamic equilibrium in the capital market. *The Review of Financial Studies*, 2(2):157–188.

Gabih, A., Sass, J., and Wunderlich, R. (2009). Utility maximization under bounded expected loss. *Stochastic Models*, 25(3):375–407.

Grossman, S. J. and Vila, J.-L. (1989). Portfolio insurance in complete markets: A note. *Journal of Business*, pages 473–476.

Grossman, S. J. and Zhou, Z. (1996). Equilibrium analysis of portfolio insurance. *The Journal of Finance*, 51(4):1379–1403.

Huang, C.-f. and Litzenberger, R. (1985). On the necessary condition for linear sharing and separation: a note. *Journal of Financial and Quantitative Analysis*, 20(3):381–384.

Jensen, B. A. and Nielsen, J. A. (2016). How suboptimal are linear sharing rules? *Annals of Finance*, 12(2):221–243.

Jensen, B. A. and Sørensen, C. (2001). Paying for minimum interest rate guarantees: Who should compensate who? *European Financial Management*, 7(2):183–211.

Karatzas, I., Lehoczky, J. P., and Shreve, S. E. (1987). Optimal portfolio and consumption decisions for a “small investor” on a finite horizon. *SIAM journal on control and optimization*, 25(6):1557–1586.

Karatzas, I. and Shreve, S. E. (1998). *Methods of mathematical finance*, volume 39. Springer.

Korn, R. (2014). *Moderne Finanzmathematik–Theorie und praktische Anwendung: Band 1–Optionsbewertung und Portfolio-Optimierung*. Springer-Verlag.

Kreps, D. M. (2012). *Microeconomic foundations I: choice and competitive markets*. Princeton university press.

Kroese, D. P., Taimre, T., and Botev, Z. I. (2013). *Handbook of monte carlo methods*, volume 706. John Wiley & Sons.

Lohr, S. (2009). *Sampling: design and analysis*. Nelson Education.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The review of Economics and Statistics*, pages 247–257.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of economic theory*, 3(4):373–413.

Pliska, S. R. (1986). A stochastic calculus model of continuous trading: optimal portfolios. *Mathematics of Operations Research*, 11(2):371–382.

Rogers, L. C. (2013). *Optimal investment*. Springer.

Ross, S. M. (2014). *Introduction to probability models*. Academic press.

Wang, J. (1996). The term structure of interest rates in a pure exchange economy with heterogeneous investors. *Journal of Financial Economics*, 41(1):75–110.

Weinbaum, D. (2009). Investor heterogeneity, asset pricing and volatility dynamics. *Journal of Economic Dynamics and Control*, 33(7):1379–1397.

Wilson, R. (1968). The theory of syndicates. *Econometrica: journal of the Econometric Society*, pages 119–132.

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Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbstständig angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht.

Ich bin mir bewusst, dass eine unwahre Erklärung rechtliche Folgen haben wird.

Ulm, den 05.06.2018 ______________________________

(Unterschrift)