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TESI DI LAUREA

**Market and non-life premium risk in a dynamic  
insurance portfolio**

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*To my family...*

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# Introduction

*The future is uncertain and full of risk. Risk is the chance that an undesirable event will occur, but risk is also opportunity. That's where we come in.*

Be An Actuary.

This simple and recent sentence tries to explain what actuaries do. Actuaries traditionally deal with the measurement and management of risk and uncertainty, and nowadays they are more and more involved in this. For example, actuaries can be responsible for the calculation of capital requirements to reduce the risk of insolvency. In this actuarial thesis we thus want to compute the capital requirements for market and non-life premium risk of an insurance company. We point out that the calculation of capital requirements has become more important since the introduction of Solvency II (Directive 2009/138/EC).

We now provide a general description of the structure and main contents of this thesis.

In the first chapter we will describe the risk reserve equation that we will use in this thesis and the related assumptions. The funds accumulated in the risk reserve will depend on the underwriting and financial results. As a consequence, we will illustrate the annual net cash flows produced by the insurance company and the asset portfolio that we will use, which is composed of stock and zero-coupon bond investments. Subsequently, we will describe some important risk measures, such as the minimum Risk-Based Capital, and some important performance measures, such as the expected

spot Return on Equity. Since we will deal with investments, we will illustrate the interest rate immunization strategy, through the Fisher-Weil theorem and Redington theorem. Finally, we will describe the copula functions, that we will use to model some dependence structures.

In the second chapter we will describe Solvency II and the so called three-pillar structure. We will focus on the illustration of the Solvency Capital Requirement and its calculation according to the standard formula. In particular, we will describe the non-life underwriting risk and market risk, that we will consider in this thesis. Finally, we will illustrate other topics, such as the risk management system and the Own Risk and Solvency Assessment.

In the third chapter we will describe the investment models that we will use in this thesis, which are based on continuous-time stochastic processes. The stock model will be a geometric Brownian motion and the zero-coupon bond model will be based on a one-factor short rate model, i.e. Vasicek model or Cox-Ingersoll-Ross model. Finally, we will illustrate Itô's lemma, we will explain the differences between the risk-neutral world and the real world and we will introduce the market price of risk.

In the fourth chapter we will describe the aggregate claim model that we will use in this thesis, i.e. the collective risk model. Hence, we will describe the distribution of the number of claims, the distribution of the single claim amount and the resulting distribution of the aggregate claim amount. Finally, we will give some comments on the descriptive statistics of the distributions.

In the fifth chapter we will introduce a case study on a single-line insurance company. Firstly, we will produce the distributions of the stock and zero-coupon bond investments, so that we will obtain the distribution of the annual rate of return. For this reason, we will apply a portfolio optimization strategy. Subsequently, we will produce the distribution of the aggregate claim amount. We thus will calculate the capital requirements for market risk, for non-life premium risk, and for market and non-life premium risk. Finally, we will describe some sensitivity analysis and we will investigate the interest rate exposure of the insurance company.

In the sixth and last chapter we will extend the previous case study to



the case of a multi-line insurance company. In doing so, we will keep the distribution of the annual rate of return that we have previously obtained. We will describe the dependence structure of the lines of business by using Gaussian copulas or Gumbel copulas. Moreover, we will produce the distribution of the aggregate claim amount of each line of business, so that we will obtain the distribution of the total aggregate claim amount. Once again, we will calculate the capital requirements for market risk, for non-life premium risk, and for market and non-life premium risk. Finally, we will describe some sensitivity analysis.

# Chapter 1

## Insurance risk management

In this thesis we deal with different instruments to properly manage the risk of insurance companies, such as risk and performance measures. Moreover, we consider some financial indicators, i.e. internal rate of return, duration and convexity, and we use copula functions, because of the peculiarity of our model.

### 1.1 Risk reserve

The risk reserve represents the funds accumulated by the insurance company time by time.<sup>1</sup> In this thesis we neglect the reserve risk, dropping the claims reserving run-off, and reinsurance. Moreover, we assume that taxes and dividends are absent. As a result, we only consider the non-life underwriting and market risk, assuming that the stochastic risk reserve at the end of time  $t$  is given by:

$$\tilde{U}_t = (1 + \tilde{j}_t) \cdot \tilde{U}_{t-1} + \left( \pi_t - \tilde{X}_t - E_t \right) + \tilde{j}_t \cdot L_{t-1}$$

where  $\tilde{j}_t$  is the stochastic annual rate of return of the investments of the insurance company,  $\pi_t$  is the gross premium amount,  $\tilde{X}_t$  is the stochastic aggregate claim amount,  $E_t$  is the expense amount and  $L_{t-1}$  is the claims

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<sup>1</sup>See Daykin, Pentikäinen and Pesonen [6] for further details.

reserve (also called loss reserve) at the end of the previous year. The gross premium amount, stochastic aggregate claim amount and expense amount are not capitalized, since they are accounted as realized at the end of the year.

The gross premium amount is given by:

$$\pi_t = P_t + \varphi \cdot P_t + c \cdot \pi_t \quad (1.1)$$

where  $P_t$  is the risk premium amount,  $\varphi$  is the safety loading coefficient and  $c$  is the expense loading coefficient.

We now assume that the expense amount is equal to the expense loadings, because empirically it is not highly volatile:

$$E_t = c \cdot \pi_t \quad (1.2)$$

We can observe that the ratio of claims reserve and gross premium amount is empirically highly influenced by the line of business. Hence, we assume that the claims reserve is equal to a constant percentage  $\delta$  of the gross premium amount:

$$L_t = \delta \cdot \pi_t \quad (1.3)$$

As a result, the risk reserve is found to be:

$$\tilde{U}_t = (1 + \tilde{j}_t) \cdot \tilde{U}_{t-1} + \left[ (1 + \varphi) \cdot P_t - \tilde{X}_t \right] + \tilde{j}_t \cdot \delta \cdot \pi_{t-1} \quad (1.4)$$

In conclusion, since the insurance portfolio is dynamic, we assume that the risk premium amount increases every year:

$$P_t = P_{t-1} \cdot (1 + i) \cdot (1 + g) = P_0 \cdot (1 + i)^t \cdot (1 + g)^t$$

and the gross premium amount as well:

$$\pi_t = \pi_{t-1} \cdot (1 + i) \cdot (1 + g) = \pi_0 \cdot (1 + i)^t \cdot (1 + g)^t \quad (1.5)$$

where  $i$  is the claims inflation rate and  $g$  is the real growth rate. We might observe that empirically these rates differ for different lines of business.

### 1.1.1 Annual net cash flows

The stochastic annual net cash flows originated by the insurance business at the end of time  $t$  are given by:

$$\tilde{F}_t = \pi_t - E_t - \left( \tilde{C}_t^{CY} + \tilde{C}_t^{PY} \right) \quad (1.6)$$

where  $\tilde{C}_t^{CY}$  is the amount paid for the claims occurred in the current year and settled in the same year and  $\tilde{C}_t^{PY}$  is the amount paid for the claims occurred in the previous years and settled in the current year.

Since the claims reserving run-off is neglected, the claims reserve can be written as follows:

$$L_t = L_t^{CY} + L_t^{PY} = \tilde{X}_t - \tilde{C}_t^{CY} + L_{t-1} - \tilde{C}_t^{PY}$$

where  $L_t^{CY}$  is the claims reserve for the claims occurred in the current year and  $L_t^{PY}$  is the claims reserve for the claims occurred in the previous years. Hence, using equations (1.3) and (1.5):

$$\tilde{C}_t^{CY} + \tilde{C}_t^{PY} = \tilde{X}_t - L_t + L_{t-1} = \tilde{X}_t - \delta \cdot \pi_t \cdot \left( 1 - \frac{1}{(1+i) \cdot (1+g)} \right)$$

then, using equation (1.2), the stochastic annual net cash flows originated by the insurance business are found to be:

$$\tilde{F}_t = \pi_t \cdot \left[ (1-c) + \delta \cdot \left( 1 - \frac{1}{(1+i) \cdot (1+g)} \right) \right] - \tilde{X}_t \quad (1.7)$$

In conclusion, the claims reserve is found to be:

$$L_t = L_{t-1} + \delta \cdot \pi_t \cdot \left( 1 - \frac{1}{(1+i) \cdot (1+g)} \right)$$

### 1.1.2 Asset portfolio

In this thesis we deal with three investments in stocks and five investments in zero-coupon bonds with time to maturity  $i = 1, 2, 3, 5, 10$ , even though there are a lot of other investments in the market. Furthermore, we assume

that the asset allocation is kept constant over time.

The stochastic asset value of the portfolio at the end of time  $t$  is obtained from the combination of the stochastic values of the stock and bond portfolios:

$$\tilde{A}_t = \tilde{A}_t^S + \tilde{A}_t^B$$

The stochastic value of the stock portfolio is given by:

$$\tilde{A}_t^S = \alpha \cdot \left( \tilde{A}_{t-1} \cdot \sum_{h=1}^3 \beta_h \cdot \frac{\tilde{S}_h(t)}{\tilde{S}_h(t-1)} + \tilde{F}_t \right) = \alpha \cdot \left( \tilde{A}_{t-1} \cdot \frac{\tilde{S}_t}{\tilde{S}_{t-1}} + \tilde{F}_t \right) \quad (1.8)$$

so that the stochastic value of a single stock investment is found to be:

$$\tilde{A}_t^{S_h} = \alpha \cdot \beta_h \cdot \left( \tilde{A}_{t-1} \cdot \frac{\tilde{S}_h(t)}{\tilde{S}_h(t-1)} + \tilde{F}_t \right) \quad \text{with } h = 1, 2, 3 \quad (1.9)$$

and the stochastic value of the bond portfolio is given by:

$$\begin{aligned} \tilde{A}_t^B &= (1 - \alpha) \cdot \left( \tilde{A}_{t-1} \cdot \sum_{i \in \{1, 2, 3, 5, 10\}} \gamma_i \cdot \frac{\tilde{B}(t, t-1+i)}{\tilde{B}(t-1, t-1+i)} + \tilde{F}_t \right) \\ &= (1 - \alpha) \cdot \left( \tilde{A}_{t-1} \cdot \frac{\tilde{B}_t}{\tilde{B}_{t-1}} + \tilde{F}_t \right) \end{aligned} \quad (1.10)$$

so that the stochastic value of a single bond investment is found to be:

$$\tilde{A}_t^{B_i} = (1 - \alpha) \cdot \gamma_i \cdot \left( \tilde{A}_{t-1} \cdot \frac{\tilde{B}(t, t-1+i)}{\tilde{B}(t-1, t-1+i)} + \tilde{F}_t \right) \quad (1.11)$$

$$\text{with } i = 1, 2, 3, 5, 10$$

where  $\alpha$  and  $1 - \alpha$  are the percentages invested in the stock and bond portfolios respectively. Moreover,  $\beta_h$  is the percentage invested in the  $h$ -th stock, so that  $\tilde{S}_t$  is the stochastic average stock price, and  $\gamma_i$  is the percentage invested in the bond with time to maturity  $i$ , so that  $\tilde{B}_t$  is the stochastic average bond price.

As a result, the stochastic asset value of the portfolio at the end of time  $t$  is found to be:

$$\tilde{A}_t = \alpha \cdot \left( \tilde{A}_{t-1} \cdot \frac{\tilde{S}_t}{\tilde{S}_{t-1}} + \tilde{F}_t \right) + (1 - \alpha) \cdot \left( \tilde{A}_{t-1} \cdot \frac{\tilde{B}_t}{\tilde{B}_{t-1}} + \tilde{F}_t \right)$$

Furthermore, the initial asset value of the portfolio is given by:

$$A_0 = U_0 + L_0$$

In conclusion, the stochastic annual rate of return of the investments of the insurance company, used in equation (1.4), is given by:

$$\tilde{j}_t = \frac{(\tilde{A}_t - \tilde{F}_t) - \tilde{A}_{t-1}}{\tilde{A}_{t-1}} = \alpha \cdot \frac{\tilde{S}_t}{\tilde{S}_{t-1}} + (1 - \alpha) \cdot \frac{\tilde{B}_t}{\tilde{B}_{t-1}} - 1 \quad (1.12)$$

## 1.2 Risk reserve ratio

The risk reserve is an absolute amount, that depends more on the dimension of the insurance company than on the goodness of its result. Actually we could have a very high risk reserve, which is very low compared with the dimension of the insurance company. Hence, we usually prefer to deal with relative amounts.<sup>2</sup>

The risk reserve ratio at the end of time  $t$  is given by:

$$\tilde{u}_t = \frac{\tilde{U}_t}{\pi_t} \quad (1.13)$$

Using equations (1.4) and (1.5), the risk reserve ratio is found to be:

$$\tilde{u}_t = \frac{(1 + \tilde{j}_t)}{(1 + i) \cdot (1 + g)} \cdot \tilde{u}_{t-1} + \frac{P_t}{\pi_t} \cdot \left[ (1 + \varphi) - \frac{\tilde{X}_t}{P_t} \right] + \frac{\tilde{j}_t \cdot \delta}{(1 + i) \cdot (1 + g)}$$

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<sup>2</sup>See Savelli [28] for further details.

Using equation (1.1), we have:

$$\frac{P_t}{\pi_t} = \frac{1 - c}{1 + \varphi} \quad (1.14)$$

then:

$$\tilde{u}_t = \frac{(1 + \tilde{j}_t)}{(1 + i) \cdot (1 + g)} \cdot \tilde{u}_{t-1} + \frac{1 - c}{1 + \varphi} \cdot \left[ (1 + \varphi) - \frac{\tilde{X}_t}{P_t} \right] + \frac{\tilde{j}_t \cdot \delta}{(1 + i) \cdot (1 + g)} \quad (1.15)$$

## 1.3 Risk measures

Risk measures are statistical measures used to assess risks. They are usually combined with performance measures, in order to select an appropriate management strategy. The most traditional approach is to compare the variance and the mean, minimizing the first and maximizing the last one.<sup>3</sup> Nevertheless, the variance does not detect the downside risk only, hence other risk measures are typically preferred.

In this section we deal with some risk measures applied to the risk reserve distribution.<sup>4</sup>

### 1.3.1 Capital-at-Risk

The Capital-at-Risk is the risk measure that represents the maximum loss for an insurance company over a time horizon within a given confidence level.

Let  $(0, t)$  be the time horizon and let  $1 - \varepsilon$  be the confidence level. Then, the Capital-at-Risk (see Figure 1.1) is given by:

$$CaR(0, t) = U_0 - U_\varepsilon(t) \quad (1.16)$$

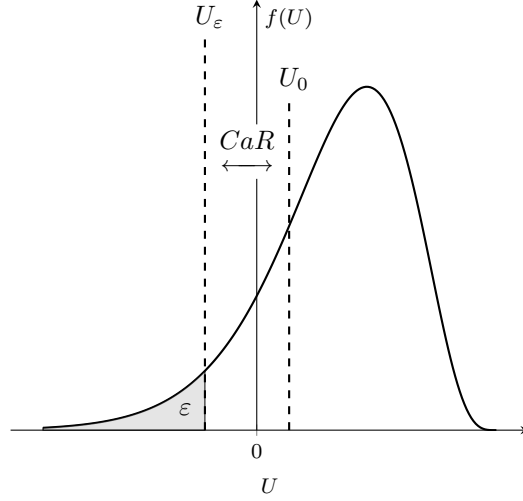
where  $U_0$  is the initial risk reserve and  $U_\varepsilon(t)$  is the  $\varepsilon$ -th order quantile of the current risk reserve. If the Capital-at-Risk is higher than the initial risk

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<sup>3</sup>See Markowitz [23] for further details.

<sup>4</sup>For this section see Daykin, Pentikäinen and Pesonen [6], and Savelli [29].

reserve, the insurance company should increase the initial risk reserve, adding fresh capital.



**Figure 1.1:** Capital-at-Risk assuming that the initial risk reserve exists

We usually prefer to express the Capital-at-Risk as a percentage of the initial gross premium amount, because it is easier to interpretate and compare. Using equation (1.5), it is found to be:

$$u_{CaR}(0, t) = \frac{CaR(0, t)}{\pi_0} = u_0 - u_\varepsilon(t) \cdot \frac{\pi_t}{\pi_0} = u_0 - u_\varepsilon(t) \cdot (1 + i)^t \cdot (1 + g)^t \quad (1.17)$$

where  $u_\varepsilon(t)$  is the  $\varepsilon$ -th order quantile of the current risk reserve ratio, that is given by:

$$u_\varepsilon(t) = \frac{U_\varepsilon(t)}{\pi_t}$$

Alternatively, we can express the Capital-at-Risk as a percentage of the initial risk reserve. Using equation (1.5), it is found to be:

$$u_{CaR}(0, t) = \frac{CaR(0, t)}{U_0} = 1 - u_\varepsilon(t) \cdot \frac{\pi_t}{U_0} = 1 - \frac{u_\varepsilon(t)}{u_0} \cdot (1 + i)^t \cdot (1 + g)^t$$

In this case the insurance company should increase the initial risk reserve if the ratio of Capital-at-Risk and initial risk reserve is higher than one hundred

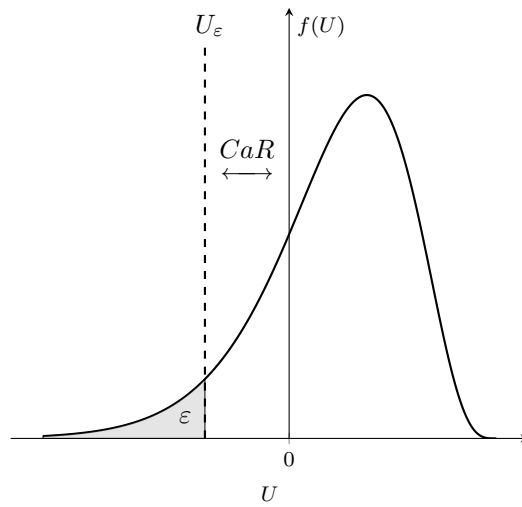


per cent.

Actually we can assume that the initial risk reserve does not exist, hence the Capital-at-Risk (see Figure 1.2) is given by:

$$CaR(0, t) = -U_\varepsilon(t) \quad (1.18)$$

In this case the insurance company should constitute a risk reserve if the Capital-at-Risk is higher than zero.



**Figure 1.2:** Capital-at-Risk assuming that the initial risk reserve does not exist

As a result, the ratio of Capital-at-Risk and initial gross premium amount is found to be:

$$u_{CaR}(0, t) = \frac{CaR(0, t)}{\pi_0} = -u_\varepsilon(t) \cdot \frac{\pi_t}{\pi_0} = -u_\varepsilon(t) \cdot (1 + i)^t \cdot (1 + g)^t \quad (1.19)$$

Obviously we cannot express the Capital-at-Risk as a percentage of the initial risk reserve, because the latter is assumed to be inexistent.

### 1.3.2 Minimum Risk-Based Capital

The minimum Risk-Based Capital is a risk measure that differs from the Capital-at-Risk, because it also takes into account the expected return produced by the investment of the resources.

Let  $(0, t)$  be the time horizon and let  $1 - \varepsilon$  be the confidence level. Then, the minimum Risk-Based Capital has to fulfill the following equation:

$$RBC(0, t) \cdot \prod_{k=1}^t 1 + E(\tilde{j}_k) = \left[ U_0 \cdot \prod_{k=1}^t 1 + E(\tilde{j}_k) \right] - U_\varepsilon(t)$$

then:

$$RBC(0, t) = U_0 - \frac{U_\varepsilon(t)}{\prod_{k=1}^t 1 + E(\tilde{j}_k)} \quad (1.20)$$

As a result, the ratio of minimum Risk-Based Capital and initial gross premium amount is found to be:

$$u_{RBC}(0, t) = \frac{RBC(0, t)}{\pi_0} = u_0 - u_\varepsilon(t) \cdot \frac{(1+i)^t \cdot (1+g)^t}{\prod_{k=1}^t 1 + E(\tilde{j}_k)} \quad (1.21)$$

Furthermore, the ratio of minimum Risk-Based Capital and initial risk reserve is found to be:

$$u_{RBC}(0, t) = \frac{RBC(0, t)}{U_0} = 1 - \frac{u_\varepsilon(t)}{u_0} \cdot \frac{(1+i)^t \cdot (1+g)^t}{\prod_{k=1}^t 1 + E(\tilde{j}_k)}$$

Actually we can assume that the initial risk reserve does not exist, hence the minimum Risk-Based Capital is given by:

$$RBC(0, t) = - \frac{U_\varepsilon(t)}{\prod_{k=1}^t 1 + E(\tilde{j}_k)}$$

As a result, the ratio of minimum Risk-Based Capital and initial gross premium amount is found to be:

$$u_{RBC}(0, t) = \frac{RBC(0, t)}{\pi_0} = - u_\varepsilon(t) \cdot \frac{(1+i)^t \cdot (1+g)^t}{\prod_{k=1}^t 1 + E(\tilde{j}_k)}$$

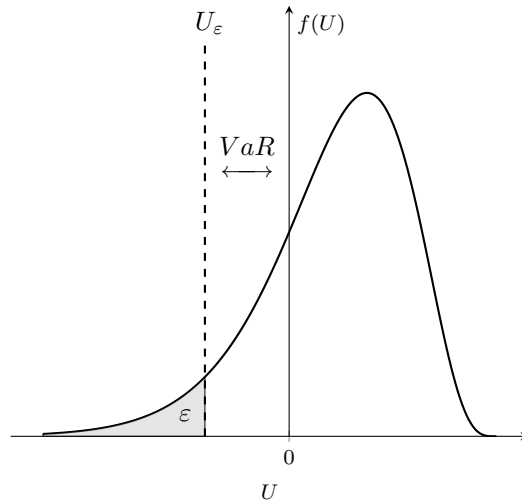
We can make the same comments as in the case of the Capital-at-Risk to express if the minimum Risk-Based Capital is insufficient.

### 1.3.3 Value-at-Risk

The Value-at-Risk is the risk measure that represents the value of a given random variable in case of maximum loss over a time horizon within a given confidence level. This risk measure has become very popular and it is fundamental in the calculation of capital requirements according to Solvency II, in which the time horizon is one year and the confidence level is 99.5%.

Let  $(0, t)$  be the time horizon and let  $1 - \varepsilon$  be the confidence level. Then, the Value-at-Risk (see Figure 1.3) is given by:

$$VaR(0, t) = -U_\varepsilon(t)$$



**Figure 1.3:** Value-at-Risk

As a result, the ratio of Value-at-Risk and initial gross premium amount is found to be:

$$u_{VaR}(0, t) = \frac{VaR(0, t)}{\pi_0} = -u_\varepsilon(t) \cdot (1 + i)^t \cdot (1 + g)^t$$

Hence, when the initial risk reserve is equal to zero, the results above are the same as those of the Capital-at-Risk, as shown in equations (1.18) and (1.19). On the contrary, when the initial risk reserve exists, the results are

different from equations (1.16) and (1.17).

Furthermore, the ratio of Value-at-Risk and initial risk reserve is found to be:

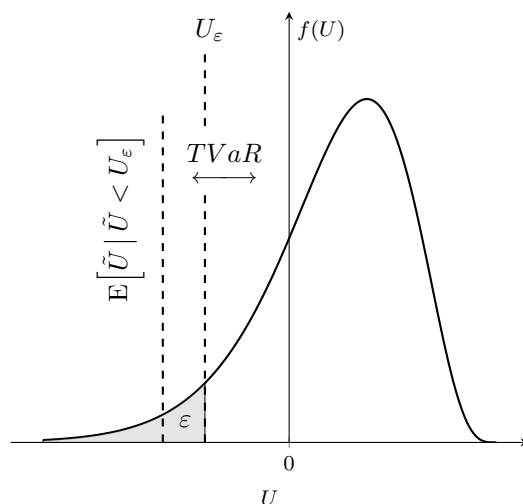
$$u_{VaR}(0, t) = \frac{VaR(0, t)}{U_0} = -\frac{u_\varepsilon(t)}{u_0} \cdot (1 + i)^t \cdot (1 + g)^t$$

### 1.3.4 Tail Value-at-Risk

The Tail Value-at-Risk (also called tail conditional expectation or conditional tail expectation) is the risk measure that represents the expected value of a given random variable over a time horizon, conditional on the loss exceeding the Value-at-Risk over the same time horizon within a given confidence level. This risk measure has become very popular and it is fundamental in the calculation of capital requirements according to the Swiss Solvency Test, in which the time horizon is one year and the confidence level is 99%.

Let  $(0, t)$  be the time horizon and let  $1 - \varepsilon$  be the confidence level. Then, the Tail Value-at-Risk (see Figure 1.4) is given by:

$$TVaR(0, t) = -E\left[\tilde{U}_t \mid \tilde{U}_t < U_\varepsilon(t)\right]$$



**Figure 1.4:** Tail Value-at-Risk

Also in this case we can express the Tail Value-at-Risk as a percentage of the initial gross premium amount and initial risk reserve.

### 1.3.5 Ruin probabilities

The Capital-at-Risk and the minimum Risk-Based Capital, as well as the Value-at-Risk and the Tail Value-at-Risk, deal with the cumulative result at time  $t$ . As a result, capital requirements do not increase when the risk reserve is negative before time  $t$  and it becomes positive at time  $t$ . The finite time ruin probability could be useful to make up for this drawback.

Ruin probabilities are used to assess the likelihood of being or not in the state of ruin after some years.

The probability of being in the state of ruin at time  $t$  is given by:

$$\varphi(U; t) = \Pr(\tilde{U}_t < 0 \mid U_0 = U)$$

It does not consider the occurrence or not of the state of ruin before time  $t$ .

Let  $(0, t)$  be the time horizon. Then, the finite time ruin probability is given by:

$$\psi(U; t) = \Pr(\tilde{U}_k < 0 \text{ for at least one } k = 1, 2, \dots, t \mid U_0 = U)$$

As a result, the survival probability at time  $t$  is found to be:

$$\Phi(U; t) = 1 - \psi(U; t) = \Pr(\tilde{U}_k \geq 0 \text{ for each } k = 1, 2, \dots, t \mid U_0 = U)$$

The finite time ruin probability considers the occurrence of the state of ruin happening at least once between time 1 and time  $t$ . On the contrary, the survival probability considers the non-occurrence of the state of ruin all over the period.

Let  $(0, t)$  be the time horizon. Then, the one-year ruin probability at time  $t$  is given by:

$$\psi(U; t - 1, t) = \Pr(\tilde{U}_t < 0 \text{ and } \tilde{U}_k \geq 0 \text{ for } k = 1, 2, \dots, t - 1 \mid U_0 = U)$$

It considers the state of ruin at time  $t$ , previously not being in the state of ruin.

The one-year ruin probability is also given by:

$$\psi(U; t - 1, t) = 1 - \frac{1 - \psi(U; t)}{1 - \psi(U; t - 1)}$$

Actually we can assume a ruin barrier different from zero. According to Solvency 0 or Solvency I, it could be equal to the Required Solvency Margin or Guarantee Fund. According to Solvency II, it can be equal to the Solvency Capital Requirement or Minimum Capital Requirement.

## 1.4 Performance measures

Performance measures are indicators used to assess the performance.

In this section we focus on the expected spot or forward Return on Equity and we assume that dividends are absent.<sup>5</sup>

### 1.4.1 Expected spot Return on Equity

The expected spot Return on Equity is the performance measure that represents the expected profitability of the stockholders' equity over a time horizon starting from the present time.

Let  $(0, t)$  be the time horizon. Then, using equations (1.5) and (1.13), the expected spot Return on Equity is given by:

$$\begin{aligned} \overline{RoE}(0, t) &= \mathbb{E}\left(\frac{\tilde{U}_t - U_0}{U_0}\right) = \frac{\mathbb{E}(\tilde{U}_t)}{U_0} - 1 = \frac{\pi_t \cdot \mathbb{E}(\tilde{u}_t)}{\pi_0 \cdot u_0} - 1 \\ &= \frac{\pi_0 \cdot (1 + g)^t \cdot (1 + i)^t \cdot \mathbb{E}(\tilde{u}_t)}{\pi_0 \cdot u_0} - 1 \\ &= (1 + g)^t \cdot (1 + i)^t \cdot \frac{\mathbb{E}(\tilde{u}_t)}{u_0} - 1 \end{aligned} \tag{1.22}$$

In case we assume that the initial risk reserve does not exist, we are not able

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<sup>5</sup>For this section see Savelli [29].

to compute the expected spot Return on Equity, hence we can just take into account the expected value of the risk reserve.

### 1.4.2 Expected forward Return on Equity

The expected forward Return on Equity is the performance measure that represents the expected profitability of the future stockholders' equity over a time horizon starting from a future time.

Let  $(t - 1, t)$  be the time horizon. Then, using equations (1.4), (1.5), (1.13) and (1.14), the expected forward Return on Equity is given by:

$$\begin{aligned}\overline{RoE}(t - 1, t) &= \mathbb{E}\left(\frac{\tilde{U}_t - \tilde{U}_{t-1}}{\tilde{U}_{t-1}}\right) = \mathbb{E}\left(\frac{\tilde{U}_t}{\tilde{U}_{t-1}}\right) - 1 \\ &= \mathbb{E}(\tilde{j}_t) + \frac{P_t + \varphi \cdot P_t - \mathbb{E}(\tilde{X}_t)}{\pi_t} \cdot \frac{(1 + g) \cdot (1 + i)}{\mathbb{E}(\tilde{u}_{t-1})} + \frac{\delta \cdot \mathbb{E}(\tilde{j}_t)}{\mathbb{E}(\tilde{u}_{t-1})} \\ &= \mathbb{E}(\tilde{j}_t) + \varphi \cdot \frac{1 - c}{1 + \varphi} \cdot \frac{(1 + g) \cdot (1 + i)}{\mathbb{E}(\tilde{u}_{t-1})} + \frac{\delta \cdot \mathbb{E}(\tilde{j}_t)}{\mathbb{E}(\tilde{u}_{t-1})}\end{aligned}$$

In case we assume that the initial risk reserve does not exist, in order to be consistent with the expected spot Return on Equity, we can just take into account the expected value of the difference between the risk reserve of two consecutive years.

The expected forward Return on Equity is also given by:

$$\overline{RoE}(t - 1, t) = \frac{1 + \overline{RoE}(0, t)}{1 + \overline{RoE}(0, t - 1)} - 1$$

On the contrary, if we assume that dividends are present, shareholders can pay them in order to decrease equity and keep the expected forward Return on Equity constant, in case it decreases over time. As a result, the Solvency Ratio decreases as well.

## 1.5 Interest rate immunization

The interest rate immunization is a strategy that ensures that changes in interest rates do not affect the value of a portfolio, i.e. the asset value remains equal or higher than the liability value. In this section we deal with the classical interest rate immunization, that is based on the assumption of a flat term structure and a parallel shift in interest rates. Actually there are also other interest rate movements, such as the twist or butterfly.

### 1.5.1 Duration and convexity

Let  $F_1, F_2, \dots, F_m$  be the cash flows at time  $t_1, t_2, \dots, t_m$ , then the value of an investment, such as a bond, is given by:

$$V = \sum_{k=1}^m F_k \cdot (1 + R)^{-t_k}$$

where  $R$  is the interest rate and  $t_m$  (frequently denoted with  $T$ ) is the maturity date of the investment.<sup>6</sup>

The duration is a measure of the time to wait before receiving the present value of the fixed cash payments of an investment. In other words, it is the weighted average of the times when payments are made. More in detail, the duration is given by:

$$D = \frac{\sum_{k=1}^m t_k \cdot F_k \cdot (1 + R)^{-t_k}}{\sum_{k=1}^m F_k \cdot (1 + R)^{-t_k}}$$

At the same time the duration measures the price sensitivity to the interest rate, i.e. the percentage change in the investment value for a small parallel shift in interest rates  $\Delta R$ , so that:

$$\frac{\Delta V}{V} \approx -\Delta R \cdot \frac{D}{(1 + R)} \quad (1.23)$$

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<sup>6</sup>For this subsection see Hull [16], and Schultz [33].



Furthermore, the modified duration is found to be:

$$D^* = \frac{D}{(1 + R)}$$

The convexity is given by:

$$C = \frac{1}{(1 + R)^2} \cdot \frac{\sum_{k=1}^m (t_k^2 + t_k) \cdot F_k \cdot (1 + R)^{-t_k}}{\sum_{k=1}^m F_k \cdot (1 + R)^{-t_k}}$$

Relation (1.23) only applies to small changes in the interest rate. Hence, we can improve it using convexity, so that:

$$\frac{\Delta V}{V} \approx -\Delta R \cdot \frac{D}{1 + R} + \frac{1}{2} \cdot (\Delta R)^2 \cdot C \quad (1.24)$$

We point out that the duration of a zero-coupon bond is equal to the time to maturity and convexity increases with the square of its time to maturity. On the other side, the duration of a fixed-coupon bond is lower than the time to maturity, even though it is usually not so distant, because coupons are small compared with the par value.

## 1.5.2 Fisher-Weil theorem

The Fisher-Weil theorem deals with an immunization strategy of an asset portfolio. This is an improvement of the obvious one of buying a bond which matures at the horizon of the portfolio, referred to as the maturity matching.

The portfolio is said to be immunized against a parallel shift in interest rates for a holding period  $H$  if the duration of the portfolio is equal to the length of the holding period, namely:

$$D = H$$

It follows that the value of the portfolio at the end of the holding period, if the shift has occurred, is at least as large as it would have been otherwise.<sup>7</sup>

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<sup>7</sup>See De Felice and Moriconi [7], and Fisher and Weil [14].

### 1.5.3 Redington theorem

The Redington theorem deals with an immunization strategy of an asset and liability portfolio.

The portfolio is said to be immunized against a small parallel shift in interest rates for the holding period if the asset value equals the liability value, the asset duration equals the liability duration and the asset convexity is higher or equal to the liability convexity, namely:

$$V_A = V_L$$

and:

$$D_A = D_L$$

and:

$$C_A \geq C_L$$

It follows that the asset value at the end of the holding period, if the shift has occurred, is at least as large as the liability value, so that the value of the portfolio is non-negative.<sup>8</sup>

We point out that the Fisher-Weil theorem can be seen as a particular case of the Redington theorem. Furthermore, they both have the purpose to find a local minimum in the interest rate of the function that describes the value of the portfolio.

## 1.6 Copula functions

A  $n$ -dimensional copula  $C : [0, 1]^n \rightarrow [0, 1]$  is a multivariate cumulative distribution function of uniformly distributed marginals and it satisfies the following properties:<sup>9</sup>

1.  $C(u_1, \dots, u_n)$  is non-decreasing in each component  $u_i$ .

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<sup>8</sup>See De Felice and Moriconi [7], and Redington [27].

<sup>9</sup>For this section see Embrechts, Lindskog and McNeil [10], and Nelsen [24].

2.  $C(u_1, \dots, u_n) = 0$  if at least one component  $u_i$  is equal to zero.
3.  $C(u_1, \dots, u_n) = u_i$  if all the components are equal to one, except  $u_i$ .
4. The  $C$ -volume of each hyperrectangle inside the domain of the copula is non-negative, i.e.  $C$  is  $n$ -non-decreasing.

Sklar's theorem states that every  $n$ -dimensional multivariate cumulative distribution function  $F$  can be expressed in terms of its marginals  $F_1, \dots, F_n$  and a  $n$ -dimensional copula  $C$ , such that:

$$F(x_1, \dots, x_n) = C[F_1(x_1), \dots, F_n(x_n)]$$

Furthermore, if the marginals  $F_1, \dots, F_n$  are continuous, then the copula  $C$  is unique and we have:<sup>10</sup>

$$C(u_1, \dots, u_n) = F[F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)]$$

We now list the steps to generate pseudo-random samples from general classes of multivariate distributions, using copulas.

1. Generate a sample  $(U_1, \dots, U_n)$  from the copula.
2. Obtain the required sample  $(X_1, \dots, X_n)$  by the inverse of the marginals:

$$(X_1, \dots, X_n) = [F_1^{-1}(U_1), \dots, F_n^{-1}(U_n)]$$

We point out that we do not need to know the multivariate probability distribution, because copulas contain all the information on the dependence structure between the marginals.

In conclusion, through the relationship between the probability density function and cumulative distribution function and through the Sklar's theorem, the multivariate probability density function related to the copula

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<sup>10</sup>See Sklar [34].

is found to be:

$$c(u_1, \dots, u_n) = \frac{f[F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)]}{\prod_{i=1}^n f_i[F_i^{-1}(u_i)]}$$

The most popular families of copulas are the elliptical copulas and the Archimedean copulas.

### 1.6.1 Measures of dependence

Copulas represent the most general way of modeling the dependence between random variables and they are able to describe a wide range of dependence structures, including but not limited to linear dependence. The principal measures of dependence are the Pearson correlation coefficient, the Kendall's and Spearman's rank correlation coefficients and the coefficients of tail dependence.

The Pearson correlation coefficient between the random variables  $\tilde{X}$  and  $\tilde{Y}$  is given by:

$$\text{Corr}(\tilde{X}, \tilde{Y}) = \frac{\text{Cov}(\tilde{X}, \tilde{Y})}{\sigma_{\tilde{X}} \cdot \sigma_{\tilde{Y}}}$$

The Pearson correlation coefficient is the linear correlation coefficient and it is the canonical measure for spherical and elliptical distributions. We point out that we need finite variance for its calculation, hence heavy-tailed distributions present computational difficulties. The Pearson correlation coefficient is invariant under positive linear transformations, but not under general strictly increasing transformations. Moreover, the uncorrelation, i.e. a Pearson correlation coefficient equal to zero, implies full independence in the case of Normal random variables only.

The Kendall's rank correlation coefficient between the random variables  $\tilde{X}$  and  $\tilde{Y}$  is given by:

$$\tau(\tilde{X}, \tilde{Y}) = P[(\tilde{X}_1 - \tilde{X}_2) \cdot (\tilde{Y}_1 - \tilde{Y}_2) > 0] - P[(\tilde{X}_1 - \tilde{X}_2) \cdot (\tilde{Y}_1 - \tilde{Y}_2) < 0]$$

with  $(\tilde{X}_1, \tilde{Y}_1)$  independent of  $(\tilde{X}_2, \tilde{Y}_2)$

and identically distributed with respect to  $(\tilde{X}, \tilde{Y})$

Moreover, if  $\tilde{X}$  and  $\tilde{Y}$  have continuous marginals, i.e. the bivariate copula  $C$  is unique, then we have:

$$\tau(\tilde{X}, \tilde{Y}) = 4 \cdot \int_0^1 \int_0^1 C(u, v) \cdot dC(u, v) - 1$$

For a given sample of  $n$  observations from a bivariate random vector, an estimate of the Kendall's rank correlation coefficient is found to be:

$$\hat{\tau}(\tilde{X}, \tilde{Y}) = \frac{2}{n \cdot (n-1)} \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^n \text{sign}(x_i - x_j) \cdot \text{sign}(y_i - y_j)$$

The Spearman's rank correlation coefficient between the random variables  $\tilde{X}$  and  $\tilde{Y}$  is given by:

$$\rho_s(\tilde{X}, \tilde{Y}) = \text{Corr}[F_{\tilde{X}}(x), F_{\tilde{Y}}(y)]$$

where  $F_{\tilde{X}}$  and  $F_{\tilde{Y}}$  are the marginal cumulative distribution functions. Once again, if  $\tilde{X}$  and  $\tilde{Y}$  have continuous marginals, i.e. the bivariate copula  $C$  is unique, then we have:

$$\rho_s(\tilde{X}, \tilde{Y}) = 12 \cdot \int_0^1 \int_0^1 [C(u, v) - u \cdot v] \cdot du \cdot dv$$

For a given sample of  $n$  observations from a bivariate random vector, an estimate of the Spearman's rank correlation coefficient is found to be:

$$\hat{\rho}_s(\tilde{X}, \tilde{Y}) = 1 - \frac{6 \cdot \sum_{i=1}^n [\text{rank}(x_i) - \text{rank}(y_i)]^2}{n \cdot (n^2 - 1)}$$

In conclusion, the upper and lower coefficients of tail dependence between the random variables  $\tilde{X}$  and  $\tilde{Y}$  are given by:

$$\lambda_U = \lim_{u \rightarrow 1^-} P[F_{\tilde{X}}(x) > u \mid F_{\tilde{Y}}(y) > u]$$

and:

$$\lambda_L = \lim_{u \rightarrow 0^+} P[F_{\tilde{X}}(x) \leq u \mid F_{\tilde{Y}}(y) \leq u]$$

## 1.6.2 Elliptical copulas

Elliptical copulas are based on multivariate elliptical distributions, which have some properties in common with the multivariate Normal distribution. There is no simple analytical formula for the elliptical copulas, hence they can be approximated using numerical integration. Some popular elliptical copulas are the Gaussian and Student's t copulas. Unlike the first one, the Student's t copula can also be used to model the extreme dependence, i.e. the dependence on the distribution tails.

### Gaussian copula

For a given correlation matrix  $P$ , the Gaussian copula is given by:

$$C_P^{Gaussian}(u_1, \dots, u_n) = \phi_P[\phi^{-1}(u_1), \dots, \phi^{-1}(u_n)]$$

where  $\phi_P$  is the joint Normal cumulative distribution function and  $\phi$  is the univariate Normal cumulative distribution function.

The multivariate probability density function related to the Gaussian copula is found to be:

$$c_P^{Gaussian}(u_1, \dots, u_n) = |P|^{-1/2} \cdot \exp\left[-\frac{1}{2} \cdot \zeta^T \cdot (P^{-1} - I) \cdot \zeta\right]$$

where:

$$\zeta^T = \left(\phi^{-1}(u_1) \cdots \phi^{-1}(u_n)\right)$$

Moreover, for the bivariate case, the Kendall's rank correlation coefficient is found to be:

$$\tau = \frac{2}{\pi} \cdot \arcsin \rho \quad (1.25)$$

where  $\rho$  is the correlation coefficient between the two distributions.

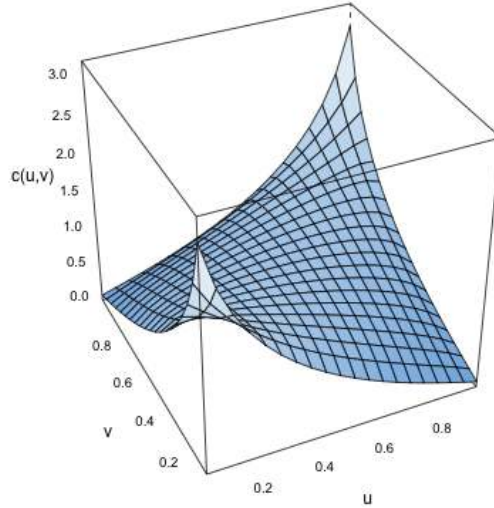
In conclusion, unless the correlation matrix exhibits perfect positive or negative dependence, the upper and lower coefficients of tail dependence are found to be:

$$\lambda_U = 0$$

and:

$$\lambda_L = 0$$

Figure 1.5 shows the bivariate probability density function related to the Gaussian copula.



**Figure 1.5:** Probability density function of a Gaussian copula with parameter 0.5

### Student's t copula

For a given correlation matrix  $P$  and a number of  $\nu$  degrees of freedom, the Student's t copula is given by:

$$C_{\nu,P}^t(u_1, \dots, u_n) = t_{\nu,P} [t_{\nu}^{-1}(u_1), \dots, t_{\nu}^{-1}(u_n)]$$

where  $t_{\nu,P}$  is the joint Student's t cumulative distribution function and  $t_{\nu}$  is the univariate Student's t cumulative distribution function.

The multivariate probability density function related to the Student's t copula is found to be:

$$c_{\nu,P}^t(u_1, \dots, u_n) = |P|^{-1/2} \cdot \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})} \cdot \left[ \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+1}{2})} \right]^n \cdot \frac{\left(1 + \frac{1}{\nu} \cdot \zeta^T \cdot P^{-1} \cdot \zeta\right)^{-(\nu+n)/2}}{\prod_{i=1}^n \left(1 + \frac{1}{\nu} \cdot \zeta_i^2\right)^{-(\nu+1)/2}}$$

where:

$$\zeta^T = \left( t_\nu^{-1}(u_1) \cdots t_\nu^{-1}(u_n) \right)$$

and:

$$\zeta_i = t_\nu^{-1}(u_i)$$

Moreover, for the bivariate case, the Kendall's rank correlation coefficient is found to be:

$$\tau = \frac{2}{\pi} \cdot \arcsin \rho$$

where  $\rho$  is the correlation coefficient between the two distributions.

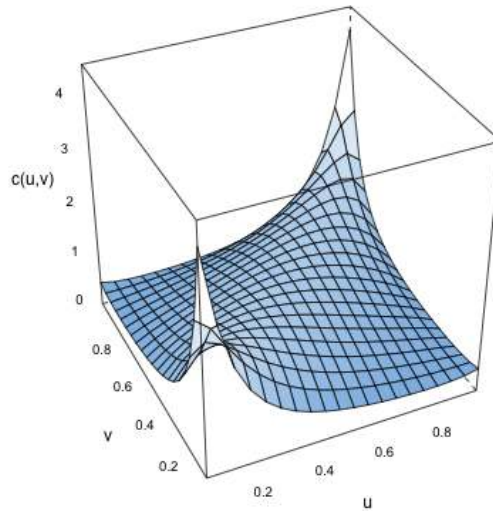
In conclusion, the upper and lower coefficients of tail dependence are found to be:

$$\lambda_U = 2 \cdot t_{\nu+1} \left( -\sqrt{\frac{(\nu+1) \cdot (1-\rho)}{1+\rho}} \right)$$

and:

$$\lambda_L = 2 \cdot t_{\nu+1} \left( -\sqrt{\frac{(\nu+1) \cdot (1-\rho)}{1+\rho}} \right)$$

Figure 1.6 shows the bivariate probability density function related to the Student's t copula.



**Figure 1.6:** Probability density function of a Student's t copula with parameters 0.5 and 3



### 1.6.3 Archimedean copulas

The generator function  $\psi : [0, 1] \rightarrow [0, \infty)$  is a continuous, strictly decreasing and convex function, such that  $\psi(1) = 0$ . Its pseudo-inverse is given by:

$$\psi^{[-1]}(t) = \begin{cases} \psi^{-1}(t) & \text{if } 0 \leq t \leq \psi(0) \\ 0 & \text{if } \psi(0) \leq t \leq \infty \end{cases}$$

A copula is said to be Archimedean if it can be written as follows:<sup>11</sup>

$$C(u_1, \dots, u_n) = \psi^{[-1]}[\psi(u_1) + \dots + \psi(u_n)]$$

and:

$$(-1)^k \cdot \frac{\partial^k \psi^{-1}(t)}{\partial t^k} \geq 0 \quad \text{for } k \in \mathbb{N}$$

Archimedean copulas are able to describe a lot of dependence structures. Moreover, there are simple analytical formulas for them. Some popular Archimedean copulas are the Gumbel and Clayton copulas.

#### Gumbel copula

For a given parameter  $\theta \geq 1$ , the bivariate Gumbel copula is given by:

$$C_\theta^{Gumbel}(u, v) = \exp\left[-\left((-\ln u)^\theta + (-\ln v)^\theta\right)^{1/\theta}\right]$$

Moreover, for the bivariate case, the Kendall's rank correlation coefficient is found to be:

$$\tau = 1 - \frac{1}{\theta} \tag{1.26}$$

In conclusion, the upper and lower coefficients of tail dependence are found to be:

$$\lambda_U = 2 - 2^{1/\theta}$$

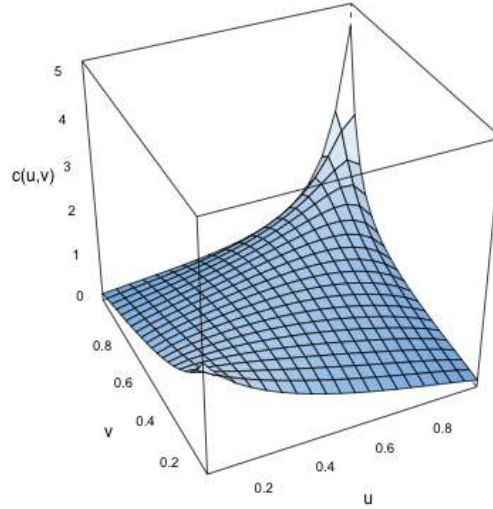
and:

$$\lambda_L = 0$$

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<sup>11</sup>See Kimberling [21].

Figure 1.7 shows the bivariate probability density function related to the Gumbel copula.



**Figure 1.7:** Probability density function of a Gumbel copula with parameter 1.5

### Clayton copula

For a given parameter  $\theta \geq -1$ , the bivariate Clayton copula is given by:

$$C_{\theta}^{Clayton}(u, v) = \max\left[(u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, 0\right]$$

If  $\theta > 0$ , the bivariate Clayton copula is found to be:

$$C_{\theta}^{Clayton}(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$$

Moreover, for the bivariate case, the Kendall's rank correlation coefficient is found to be:

$$\tau = \frac{\theta}{2 + \theta}$$

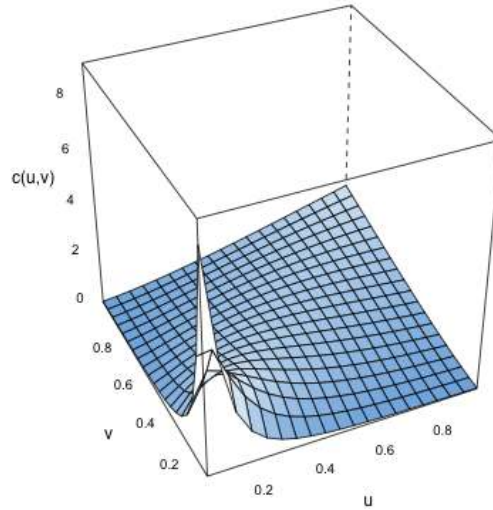
In conclusion, the upper and lower coefficients of tail dependence are found to be:

$$\lambda_U = 0$$

and:

$$\lambda_L = 2^{-1/\theta}$$

Figure 1.8 shows the bivariate probability density function related to the Clayton copula.



**Figure 1.8:** Probability density function of a Clayton copula with parameter 1.5

# Chapter 2

## Solvency II

Solvency II (Directive 2009/138/EC) is an EU directive which codifies and harmonises the EU insurance regulation and it was implemented in Italy in 2015 by D. Lgs. 74/2015. Primarily this concerns the amount of capital that EU insurance companies must hold to reduce the risk of insolvency.<sup>1</sup>

The process of regulatory harmonisation distinguishes some levels. We now list the main ones.

1. Solvency II (Directive 2009/138/EC).
2. Delegated Regulation (Commission Delegated Regulation (EU) 2015/35).
- 2.5. Technical standards, proposed by EIOPA, that involve:
  - regulatory technical standard;
  - implementing technical standard.
3. Guidelines, issued by EIOPA, that were implemented in Italy by IVASS Regulations.
4. Rigorous enforcement of community legislation by the Commission.

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<sup>1</sup>Different parts of this chapter are taken from Solvency II [13], and from the Delegated Regulation [12].

Solvency II has a three-pillar structure. We now list the main items involved in each pillar.

1. Quantitative requirements.
  - 1.1. Economic balance sheet.
  - 1.2. Eligible own funds.
  - 1.3. Solvency Capital Requirement (SCR).
  - 1.4. Minimum Capital Requirement (MCR).
2. Qualitative requirements.
  - 2.1. Supervisory review process, that involves:
    - capital add-on.
  - 2.2. System of governance.
  - 2.3. Risk management system, that involves:
    - Own Risk and Solvency Assessment (ORSA).
  - 2.4. Control functions, that involve:
    - risk management function;
    - compliance function;
    - internal audit function;
    - actuarial function.
3. Reporting and disclosures requirements.
  - 3.1. Solvency and financial condition report (SFCR).
  - 3.2. Regular supervisory report (RSR).
  - 3.3. Quantitative reporting templates (QRTs).

## 2.1 Technical provisions

Solvency II defines a market consistent valuation of technical provisions and it makes a distinction for hedgeable and non-hedgeable technical provisions.<sup>2</sup>

*The value of technical provisions shall correspond to the current amount insurance and reinsurance undertakings would have to pay if they were to transfer their insurance and reinsurance obligations immediately to another insurance or reinsurance undertaking.*

*The value of technical provisions shall be equal to the sum of a best estimate and a risk margin. The best estimate shall correspond to the probability-weighted average of future cash-flows, taking account of the time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure.*

*Where future cash flows associated with insurance or reinsurance obligations can be replicated reliably using financial instruments for which a reliable market value is observable, the value of technical provisions associated with those future cash flows shall be determined on the basis of the market value of those financial instruments. In this case separate calculations of the best estimate and the risk margin shall not be required.*

We point out that the majority of technical provisions are non-hedgeable. Moreover, the hedgeable technical provisions are those related to unit-linked or index-linked contracts without guarantees.

The best estimate is calculated gross of recoverables and it takes account of all the cash flows of the insurance and reinsurance obligations over their lifetime. Moreover, the basic risk-free interest rates are derived on the basis of interest rate swap rates, adjusted to take account of credit risk. When it is

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<sup>2</sup>The following part is taken from is taken from Art. 76 and 77 of Solvency II [13].

not possible, they are derived on the basis of government bonds. Solvency II allows the application of a volatility adjustment or a matching adjustment to the basic risk-free interest rates. The first one does not require any prior approval by the supervisory authority, differently from the last one. In conclusion, the risk margin is calculated by determining the cost of providing an amount of eligible own funds equal to the Solvency Capital Requirement necessary to support the insurance and reinsurance obligations over their lifetime. The risk margin is computed on the whole portfolio and the so-called Cost-of-Capital rate is the same for each insurance company. We point out that Solvency II allows some simplifications for the calculation of the risk margin.

## 2.2 Solvency Capital Requirement

Solvency II demands some requirements for the calculation of the Solvency Capital Requirement (see Figure 2.1).<sup>3</sup>

*The Solvency Capital Requirement shall be calculated on the presumption that the undertaking will pursue its business as a going concern.*

*The Solvency Capital Requirement shall be calibrated so as to ensure that all quantifiable risks to which an insurance or reinsurance undertaking is exposed are taken into account. It shall cover existing business, as well as the new business expected to be written over the following 12 months. With respect to existing business, it shall cover only unexpected losses.*

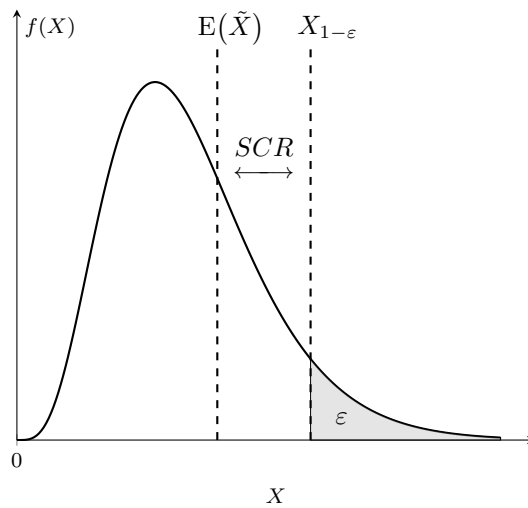
*It shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5% over a one-year period.*

The Solvency Capital Requirement shall cover at least the following risks:

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<sup>3</sup>The following part is taken from Art. 101 of Solvency II [13].

1. Non-life underwriting risk.
2. Life underwriting risk.
3. Health underwriting risk.
4. Market risk.
5. Credit risk.
6. Operational risk.



**Figure 2.1:** Solvency Capital Requirement

## 2.3 Standard formula

The Solvency Capital Requirement calculated on the basis of the standard formula shall be equal to the following:

$$SCR = BasicSCR + SCR_{op} - Adj$$

where  $BasicSCR$  is the Basic Solvency Capital Requirement,  $SCR_{op}$  is the capital requirement for the operational risk and  $Adj$  is the adjustment for the loss-absorbing capacity of technical provisions and deferred taxes.



The Basic Solvency Capital Requirement shall consist of at least the following risk modules:

1. Non-life underwriting risk, that involves:

- non-life premium and reserve risk;
- non-life catastrophe risk;
- non-life lapse risk.

2. Life underwriting risk, that involves:

- mortality risk;
- longevity risk;
- disability risk;
- life expense risk;
- revision risk;
- lapse risk;
- life catastrophe risk.

3. Health underwriting risk.

4. Market risk, that involves:

- interest rate risk;
- equity risk;
- property risk;
- spread risk;
- currency risk;
- market risk concentrations.

5. Counterparty default risk.

We point out that the value of a risk module or sub-module of the Basic Solvency Capital Requirement cannot be negative.

The Basic Solvency Capital Requirement shall be equal to the following:

$$BasicSCR = \sqrt{\sum_{i,j} Corr_{(i,j)} \cdot SCR_i \cdot SCR_j} + SCR_{intangibles}$$

where  $Corr_{(i,j)}$  is the correlation parameter for the Basic Solvency Capital Requirement for modules  $i$  and  $j$  (see Table 2.1),  $SCR_i$  and  $SCR_j$  are the capital requirements for modules  $i$  and  $j$  respectively and  $SCR_{intangibles}$  is the capital requirement for intangible asset risk.

**Table 2.1:** Correlation matrix for the Basic Solvency Capital Requirement

$i \setminus j$	Market	Default	Life	Health	Non-life
Market	1	0.25	0.25	0.25	0.25
Default	0.25	1	0.25	0.25	0.5
Life	0.25	0.25	1	0.25	0
Health	0.25	0.25	0.25	1	0
Non-life	0.25	0.5	0	0	1

We point out that the capital requirements that are aggregated in the standard formula are not standard deviations, but quantiles of probability distributions. For multivariate Normal distributions (or more in general for elliptical distributions), the aggregation with correlation matrices produces a correct aggregate of quantiles. On the other hand, only for a restricted class of distributions, the aggregation with linear correlation coefficients produces the correct result. Nevertheless, the shape of the marginal distributions could be significantly different from the Normal distribution (e.g. if distributions are skewed) and, moreover, the dependence between the distributions could be non-linear (e.g. if tail dependencies are present).

The Solvency Capital Requirement is calculated separately for each risk module or sub-module through a factor based approach or a scenario based approach. In the first case, the capital requirements are determined by using single risk exposures and risk factors, calibrated by considering the tail of the

distribution and by taking into account both volatility and trend effects. In the second case, the capital requirements shall be equal to the loss in basic own funds caused by a stressed scenario.

The Delegated Regulation demands some requirements for the scenario based approach.<sup>4</sup>

*Where the calculation of a module or sub-module of the Basic Solvency Capital Requirement is based on the impact of a scenario on the basic own funds of insurance and reinsurance undertakings, all of the following assumptions shall be made in that calculation:*

- (a) the scenario does not change the amount of the risk margin included in technical provisions;*
- (b) the scenario does not change the value of deferred tax assets and liabilities;*
- (c) the scenario does not change the value of future discretionary benefits included in technical provisions;*
- (d) no management actions are taken by the undertaking during the scenario.*

### **2.3.1 Non-life underwriting risk**

The non-life underwriting risk module shall consist of all of the following sub-modules:

1. Non-life premium and reserve risk sub-module.
2. Non-life catastrophe risk sub-module.
3. Non-life lapse risk sub-module.

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<sup>4</sup>The following part is taken from Art. 83 of the Delegated Regulation [12].

The capital requirement for non-life underwriting risk shall be equal to the following:

$$SCR_{non-life} = \sqrt{\sum_{i,j} CorrNL_{(i,j)} \cdot SCR_i \cdot SCR_j}$$

where  $CorrNL_{(i,j)}$  is the correlation parameter for non-life underwriting risk for sub-modules  $i$  and  $j$  (see Table 2.2) and  $SCR_i$  and  $SCR_j$  are the capital requirements for risk sub-modules  $i$  and  $j$  respectively.

**Table 2.2:** Correlation matrix for non-life underwriting risk

$i \setminus j$	Non-life premium and reserve	Non-life catastrophe	Non-life lapse
Non-life premium and reserve	1	0.25	0
Non-life catastrophe	0.25	1	0
Non-life lapse	0	0	1

### Non-life premium and reserve risk

Solvency II defines the non-life premium and reserve risk sub-module.<sup>5</sup>

*The risk of loss, or of adverse change in the value of insurance liabilities, resulting from fluctuations in the timing, frequency and severity of insured events, and in the timing and amount of claim settlements.*

The capital requirement for non-life premium and reserve risk shall be equal to the following:

$$SCR_{nl\ prem\ res} = 3 \cdot \sigma_{nl} \cdot V_{nl} \quad (2.1)$$

where  $\sigma_{nl}$  and  $V_{nl}$  are the standard deviation, in relative terms, and the volume measure for non-life premium and reserve risk.

The standard deviation for non-life premium and reserve risk shall be

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<sup>5</sup>The following part is taken from Art. 105 of Solvency II [13].

equal to the following:

$$\sigma_{nl} = \frac{1}{V_{nl}} \cdot \sqrt{\sum_{s,t} CorrS_{(s,t)} \cdot \sigma_s \cdot V_s \cdot \sigma_t \cdot V_t}$$

where  $CorrS_{(s,t)}$  is the correlation parameter for non-life premium and reserve risk for segments  $s$  and  $t$  (see Table 2.4) and  $\sigma_s$  and  $\sigma_t$  are the standard deviations for non-life premium and reserve risk of segments  $s$  and  $t$  respectively. They shall be equal to the following:

$$\sigma_s = \frac{\sqrt{\sigma_{(prem,s)}^2 \cdot V_{(prem,s)}^2 + \sigma_{(res,s)} \cdot V_{(prem,s)} \cdot \sigma_{(res,s)} \cdot V_{(res,s)} + \sigma_{(res,s)}^2 \cdot V_{(res,s)}^2}}{V_{(prem,s)} + V_{(res,s)}}$$

where  $\sigma_{(prem,s)}$  and  $\sigma_{(res,s)}$  are the standard deviations for non-life premium risk and reserve risk of segment  $s$  (see Table 2.3) and  $V_{(prem,s)}$  and  $V_{(res,s)}$  are the volume measures for non-life premium risk and reserve risk of segment  $s$ .<sup>6</sup>

We point out that Table 2.3 contains standard deviations, estimated through the market wide approach. Furthermore, Tables 2.3 and 2.4 take into account the proportional and non proportional direct reinsurance. The proportional direct reinsurance is treated as direct insurance, because the relative volatility of a particular segment is the same as in the case of the insurance company. Furthermore, the standard deviation for non-life premium risk of a segment shall be equal to the product of the standard deviation for non-life gross premium risk of the segment and the adjustment factor for non-proportional excess of loss and stop loss reinsurance. For segments 1, 4 and 5 the adjustment factor for non-proportional reinsurance shall be equal to 80%. For all the other segments the adjustment factor for non-proportional reinsurance shall be equal to 100%.

The volume measure for non-life premium and reserve risk shall be equal to the following:

$$V_{nl} = \sum_s V_s$$

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<sup>6</sup>The standard deviations for non-life premium and reserve risk were amended in 2019.

**Table 2.3:** Segmentation of non-life insurance and reinsurance obligations and standard deviations for non-life premium and reserve risk

	Segment $s$	$\sigma_{(prem,s)}$	$\sigma_{(res,s)}$
1	Motor vehicle liability insurance and proportional reinsurance	10%	9%
2	Other motor insurance and proportional reinsurance	8%	8%
3	Marine, aviation and transport insurance and proportional reinsurance	15%	11%
4	Fire and other damage to property insurance and proportional reinsurance	8%	10%
5	General liability insurance and proportional reinsurance	14%	11%
6	Credit and suretyship insurance and proportional reinsurance	19%	17.2%
7	Legal expenses insurance and proportional reinsurance	8.3%	5.5%
8	Assistance and its proportional reinsurance	6.4%	22%
9	Miscellaneous financial loss insurance and proportional reinsurance	13%	20%
10	Non-proportional casualty reinsurance	17%	20%
11	Non-proportional marine, aviation and transport reinsurance	17%	20%
12	Non-proportional property reinsurance	17%	20%

**Table 2.4:** Correlation matrix for non-life premium and reserve risk

$s \setminus t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	0.5	0.5	0.25	0.5	0.25	0.5	0.25	0.5	0.25	0.25	0.25
2	0.5	1	0.25	0.25	0.25	0.25	0.5	0.5	0.5	0.25	0.25	0.25
3	0.5	0.25	1	0.25	0.25	0.25	0.25	0.5	0.5	0.25	0.5	0.25
4	0.25	0.25	0.25	1	0.25	0.25	0.25	0.5	0.5	0.25	0.5	0.5
5	0.5	0.25	0.25	0.25	1	0.5	0.5	0.25	0.5	0.5	0.25	0.25
6	0.25	0.25	0.25	0.25	0.5	1	0.5	0.25	0.5	0.5	0.25	0.25
7	0.5	0.5	0.25	0.25	0.5	0.5	1	0.25	0.5	0.5	0.25	0.25
8	0.25	0.5	0.5	0.5	0.25	0.25	0.25	1	0.5	0.25	0.25	0.5
9	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	0.25	0.5	0.25
10	0.25	0.25	0.25	0.25	0.5	0.5	0.5	0.25	0.25	1	0.25	0.25
11	0.25	0.25	0.5	0.5	0.25	0.25	0.25	0.25	0.5	0.25	1	0.25
12	0.25	0.25	0.25	0.5	0.25	0.25	0.25	0.5	0.25	0.25	0.25	1

where  $V_s$  is the volume measure of a particular segment  $s$ , adjusted for the geographical diversification. It shall be equal to the following:

$$V_s = (V_{(prem,s)} + V_{(res,s)}) \cdot (0.75 + 0.25 \cdot DIV_s)$$

where  $V_{(prem,s)}$  and  $V_{(res,s)}$  are the volume measures for non-life premium risk and reserve risk of segment  $s$  and  $DIV_s$  is the factor for geographical diversification of segment  $s$ .

The volume measure for non-life premium risk of segment  $s$  shall be equal to the following:

$$V_{(prem,s)} = \max(P_s, P_{(last,s)}) + FP_{(existing,s)} + FP_{(future,s)}$$

where  $P_s$  is an estimate of the premiums to be earned by the insurance or reinsurance undertaking in the segment  $s$  during the following 12 months,  $P_{(last,s)}$  are the premiums earned by the insurance or reinsurance undertaking in the segment  $s$  during the last 12 months,  $FP_{(existing,s)}$  is the expected present value of premiums to be earned by the insurance or reinsurance undertaking in the segment  $s$  after the following 12 months for existing contracts and  $FP_{(future,s)}$  refers to contracts where the initial recognition date falls in the following 12 months. For all such contracts whose initial term is one year or less, it is the expected present value of premiums to be earned by the insurance or reinsurance undertaking in the segment  $s$ , but excluding the premiums to be earned during the 12 months after the initial recognition date. For all such contracts whose initial term is more than one year, it is the amount equal to 30% of the expected present value of premiums to be earned by the insurance or reinsurance undertaking in the segment  $s$  after the following 12 months.<sup>7</sup>

The volume measure for reserve risk of segment  $s$  shall be equal to the following:

$$V_{(res,s)} = PCO_s$$

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<sup>7</sup>The definition of the last item was amended in 2019.

where  $PCO_s$  is the best estimate (without risk margin) of the provisions for claims outstanding for the segment  $s$ , after deduction of the amounts recoverable from reinsurance contracts and special purpose vehicles.

The factor for geographical diversification of segment  $s$  shall be equal to the following:

$$DIV_s = \frac{\sum_j (V_{(prem,j,s)} + V_{(res,j,s)})^2}{\left(\sum_j V_{(prem,j,s)} + V_{(res,j,s)}\right)^2}$$

where  $V_{(prem,j,s)}$  and  $V_{(res,j,s)}$  are the volume measures for non-life premium risk and reserve risk of segment  $s$  and region  $j$ . We point out that the geographical diversification is neglected for credit and suretyship, non-proportional reinsurance and if the insurers use an undertaking-specific parameter for the standard deviation for non-life premium risk or reserve risk.

The main drawbacks of the non-life premium and reserve risk sub-module of the standard formula are:

1. The multiplier 3 comes from the assumption of Lognormal distribution for non-life premium and reserve risk, with standard deviation equal to 14.47% in relative terms. Nevertheless, the lognormality assumption could be erroneous as well as the multiplier, i.e. the standard deviation could be different from 14.47%. The lower the standard deviation is, the lower the skewness of the Lognormal distribution is and the lower the multiplier should be. Therefore, the multiplier 3 penalizes the insurance companies with extremely low relative volatility, i.e. large companies, and facilitates those with extremely high relative volatility, i.e. small companies.
2. The safety loadings are neglected. The higher the safety loadings are, the lower the risk is and the lower the capital requirement should be. On the other hand, if the safety loadings are negative (e.g. because of marketing reasons), the capital requirement should increase. We point out that the expected profits, for the existing business only, are already included in the own funds, because the Solvency II premium reserve



does not include any profit. Hence, the own funds are higher and make up for the Solvency Capital Requirement for non-life premium and reserve risk, that is generally overestimated in the standard formula.

3. The risk margin is neglected. The capital requirement should decrease by the portion of risk margin released during the year.
4. The size factor is neglected, then the standard deviations in the non-life premium and reserve risk sub-module are the same for each insurance company. This measure, that exists to keep competition, penalizes the insurance companies with extremely low relative volatility, i.e. large companies, and facilitates those with extremely high relative volatility, i.e. small companies.
5. The adjustment factor for non-proportional excess of loss and stop loss reinsurance does not depend on the characteristics of the reinsurance treaty, nor on the insurance company relative volatility.
6. The cost of proportional reinsurance is neglected. The higher the reinsurance cost is, the higher the capital requirement should be.

An insurance company could make up for some of these drawbacks, asking the supervisory authority for the undertaking-specific parameters or an internal model. We point out that the underestimation of capital requirements is prevented by the Own Risk and Solvency Assessment.

### **2.3.2 Market risk**

The market risk module shall consist of all of the following sub-modules:

1. Interest rate risk sub-module.
2. Equity risk sub-module.
3. Property risk sub-module.
4. Spread risk sub-module.

5. Currency risk sub-module.

6. Market risk concentrations sub-module.

The capital requirement for market risk shall be equal to the following:

$$SCR_{market} = \sqrt{\sum_{i,j} CorrM_{(i,j)} \cdot SCR_i \cdot SCR_j}$$

where  $CorrM_{(i,j)}$  is the correlation parameter for market risk for sub-modules  $i$  and  $j$  (see Table 2.5) and  $SCR_i$  and  $SCR_j$  are the capital requirements for risk sub-modules  $i$  and  $j$  respectively.

**Table 2.5:** Correlation matrix for market risk

$i \setminus j$	Interest rate	Equity	Property	Spread	Currency	Concentration
Interest rate	1	A	A	A	0.25	0
Equity	A	1	0.75	0.75	0.25	0
Property	A	0.75	1	0.5	0.25	0
Spread	A	0.75	0.5	1	0.25	0
Currency	0.25	0.25	0.25	0.25	1	0
Concentration	0	0	0	0	0	1

We point out that the parameter A in Table 2.5 shall be equal to 0 where the capital requirement for interest rate risk depends on the risk of an increase in the term structure of interest rates. In all other cases, the parameter A shall be equal to 0.5.

We stress that the market risk module may affect both assets and liabilities. Hence, the effect on the asset side can be partially compensated by the effect on the liability side, and vice versa. Let us take the example of an increase in the term structure of interest rates. The bond investments are found to decrease as well as the technical provisions, because of a higher discounting effect, so that the own funds can increase or decrease, depending on which of the two drops more. Moreover, we point out that, according to the standard formula, the government bonds are not exposed to spread risk and market risk concentrations.

## Interest rate risk

Solvency II defines the interest rate risk sub-module.<sup>8</sup>

*The sensitivity of the values of assets, liabilities and financial instruments to changes in the term structure of interest rates, or in the volatility of interest rates.*

The capital requirement for interest rate risk shall be equal to the larger of the following:<sup>9</sup>

- the sum, over all currencies, of the capital requirements for the risk of an increase in the term structure of interest rates;
- the sum, over all currencies, of the capital requirements for the risk of a decrease in the term structure of interest rates.

The capital requirement for the risk of an increase or decrease in the term structure of interest rates for a given currency shall be equal to the loss in the basic own funds that would result from an instantaneous increase or decrease in basic risk-free interest rates for that currency at different maturities in accordance with Table 2.6.

For maturities not specified in Table 2.6, the value of the increase and decrease shall be linearly interpolated. For maturities shorter than 1 year, the increase and decrease shall be 70% and 75% respectively. For maturities longer than 90 years, the increase and decrease shall be 20%. Furthermore, the increase in basic risk-free interest rates at any maturity shall be at least one percentage point. For negative basic risk-free interest rates the decrease shall be nil.

We point out that nowadays the short-term basic risk-free interest rates are negative.

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<sup>8</sup>The following part is taken from Art. 105 of Solvency II [13].

<sup>9</sup>The scenario shall be coherent with the scenario of the largest net Basic Solvency Capital Requirement, related to the adjustment for the loss-absorbing capacity of technical provisions, described in Art. 206 of the Delegated Regulation [12].

**Table 2.6:** Increase and decrease in the term structure of interest rates

Maturity (years)	Increase	Decrease
1	70%	75%
2	70%	65%
3	64%	56%
4	59%	50%
5	55%	46%
6	52%	42%
7	49%	39%
8	47%	36%
9	44%	33%
10	42%	31%
11	39%	30%
12	37%	29%
13	35%	28%
14	34%	28%
15	33%	27%
16	31%	28%
17	30%	28%
18	29%	28%
19	27%	29%
20	26%	29%
90	20%	20%

## Equity risk

Solvency II defines the equity risk sub-module.<sup>10</sup>

*The sensitivity of the values of assets, liabilities and financial instruments to changes in the level or in the volatility of market prices of equities.*

The equity risk sub-module shall consist of all of the following sub-modules:

1. Sub-module for type 1 equities.

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<sup>10</sup>The following part is taken from Art. 105 of Solvency II [13].

## 2. Sub-module for type 2 equities.

Type 1 equities shall comprise equities listed in regulated markets in the countries which are members of the European Economic Area (EEA) or the Organisation for Economic Cooperation and Development (OECD). On the other hand, type 2 equities shall comprise equities listed in stock exchanges in countries which are not members of the EEA or the OECD, equities which are not listed, commodities and other alternative investments. They shall also comprise all assets other than those covered in the interest rate risk sub-module, the property risk sub-module or the spread risk sub-module, including the assets and indirect exposures where a look-through approach is not possible.

The capital requirement for the equity risk shall be equal to the following:

$$SCR_{equity} = \sqrt{SCR_{type1}^2 + 2 \cdot 0.75 \cdot SCR_{type1} \cdot SCR_{type2} + SCR_{type2}^2}$$

where  $SCR_{type1}$  and  $SCR_{type2}$  are the capital requirements for sub-modules for type 1 and type 2 equities respectively. We point out that the correlation parameter is assumed to be 0.75.

The capital requirement for type 1 equities shall be equal to the loss in the basic own funds that would result from the following instantaneous decreases:

- an instantaneous decrease equal to 22% in the value of type 1 equity investments in related undertakings, where these investments are of a strategic nature;
- an instantaneous decrease equal to 22% in the value of type 1 equity investments that are treated as long-term equity investments;
- an instantaneous decrease equal to the sum of 39% and the symmetric adjustment in the value of other type 1 equities.

The capital requirement for type 2 equities shall be equal to the loss in the basic own funds that would result from the following instantaneous decreases:

- an instantaneous decrease equal to 22% in the value of type 2 equity investments in related undertakings, where these investments are of a strategic nature;
- an instantaneous decrease equal to 22% in the value of type 2 equity investments that are treated as long-term equity investments;
- an instantaneous decrease equal to the sum of 49% and the symmetric adjustment in the value of other type 2 equities.

The capital requirement for qualifying infrastructure equities and qualifying infrastructure corporate equities is based on favourable measures about the instantaneous decrease. Furthermore, in some cases, equities can be subject to the duration-based equity risk.<sup>11</sup>

The Delegated Regulation describes the equity investments of a strategic nature.<sup>12</sup>

*Equity investments of a strategic nature shall mean equity investments for which the participating insurance or reinsurance undertaking demonstrates the following:*

- (a) *that the value of the equity investment is likely to be materially less volatile for the following 12 months than the value of other equities over the same period as a result of both the nature of the investment and the influence exercised by the participating undertaking in the related undertaking;*
- (b) *that the nature of the investment is strategic, taking into account all relevant factors, including:*
  - (i) *the existence of a clear decisive strategy to continue holding the participation for long period;*

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<sup>11</sup>The criteria for the calculation of the capital requirements for type 1 equities, type 2 equities and qualifying infrastructure equities were amended in 2019.

<sup>12</sup>The following part is taken from Art. 171 of the Delegated Regulation [12].

- (ii) *the consistency of the strategy referred to in point (a) with the main policies guiding or limiting the actions of the undertaking;*
- (iii) *the participating undertaking's ability to continue holding the participation in the related undertaking;*
- (iv) *the existence of a durable link;*
- (v) *where the insurance or reinsurance participating company is part of a group, the consistency of such strategy with the main policies guiding or limiting the actions of the group.*

The Delegated Regulation describes the long-term equity investments.<sup>13</sup>

*Equity investments may be treated as long-term equity investments if the insurance or reinsurance undertaking demonstrates, to the satisfaction of the supervisory authority, that all of the following conditions are met:*

- (a) *the sub-set of equity investments as well as the holding period of each equity investment within the sub-set are clearly identified;*
- (b) *the sub-set of equity investment is included within a portfolio of assets which is assigned to cover the best estimate of a portfolio of insurance or reinsurance obligations corresponding to one or several clearly identified businesses, and the undertaking maintains that assignment over the lifetime of the obligations;*
- (c) *the portfolio of insurance or reinsurance obligations, and the assigned portfolio of assets referred to in point (b) are identified, managed and organised separately from the other activities of the undertaking, and the assigned portfolio of*

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<sup>13</sup>The following part is taken from Art. 171a (introduced in 2019) of the Delegated Regulation [12].

- assets cannot be used to cover losses arising from other activities of the undertaking;*
- (d) the technical provisions within the portfolio of insurance or reinsurance obligations referred to in point (b) only represent a part of the total technical provisions of the insurance or reinsurance undertaking;*
  - (e) the average holding period of equity investments in the sub-set exceeds 5 years, or where the average holding period of the sub-set is lower than 5 years, the insurance or reinsurance undertaking does not sell any equity investments within the sub-set until the average holding period exceeds 5 years;*
  - (f) the sub-set of equity investments consists only of equities that are listed in the EEA or of unlisted equities of companies that have their head offices in countries that are members of the EEA;*
  - (g) the solvency and liquidity position of the insurance or reinsurance undertaking, as well as its strategies, processes and reporting procedures with respect to asset-liability management, are such as to ensure, on an ongoing basis and under stressed conditions, that it is able to avoid forced sales of each equity investments within the sub-set for at least 10 years;*
  - (h) the risk management, asset-liability management and investment policies of the insurance or reinsurance undertaking reflects the undertaking's intention to hold the sub-set of equity investments for a period that is compatible with the requirement of point (e) and its ability to meet the requirement of point (g).*

We point out that the treatment of equity investments as long-term equity investments shall not be reverted back to an approach that does not



include long-term equity investments. Furthermore, where an insurance or reinsurance undertaking that treats a sub-set of equity investments as long-term equity investments is no longer able to comply with the conditions, it shall immediately inform the supervisory authority.

The symmetric adjustment is an anti-procyclicality measure, since the stress parameters (i.e. 39% and 49%) are reduced when the market drops, in order to avoid that insurance companies sell equities and make the market drop further. The symmetric adjustment shall be equal to the following:

$$SA = \frac{1}{2} \cdot \left( \frac{CI - AI}{AI} - 8\% \right)$$

where  $CI$  is the current level of the equity index and  $AI$  is the weighted average of the daily levels of the equity index over the last 36 months, where the weights for all daily levels shall be equal. We point out that the symmetric adjustment shall not be lower than  $-10\%$  or higher than  $10\%$ .

In conclusion, a transitional measure for standard equity risk shall only be applied to type 1 equities that were purchased on or before January 1, 2016 and which are not subject to the duration-based equity risk.

## 2.4 Internal models

Solvency II describes the internal models.<sup>14</sup>

*The Solvency Capital Requirement shall be calculated, either in accordance with the standard formula [...] or using an internal model [...].*

*Member States shall ensure that insurance or reinsurance undertakings may calculate the Solvency Capital Requirement using a full or partial internal model as approved by the supervisory authorities.*

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<sup>14</sup>The following part is taken from Subsection 3 (Art. 112-127) of Solvency II [13].

*Insurance and reinsurance undertakings may use partial internal models for the calculation of one or more of the following:*

- (a) one or more risk modules, or sub-modules, of the Basic Solvency Capital Requirement [...];*
- (b) the capital requirement for operational risk [...];*
- (c) the adjustment [...].*

*In addition, partial modelling may be applied to the whole business of insurance and reinsurance undertakings, or only to one or more major business units.*

*After having received approval [...], insurance and reinsurance undertakings shall not revert to calculating the whole or any part of the Solvency Capital Requirement in accordance with the standard formula, [...] except in duly justified circumstances and subject to the approval of the supervisory authorities.*

*Where it is inappropriate to calculate the Solvency Capital Requirement in accordance with the standard formula, [...] because the risk profile of the insurance or reinsurance undertaking concerned deviates significantly from the assumptions underlying the standard formula calculation, the supervisory authorities may, by means of a decision stating the reasons, require the undertaking concerned to use an internal model to calculate the Solvency Capital Requirement, or the relevant risk modules thereof.*

The requirements for an internal model shall consist of the following:

- Use test.

*Insurance and reinsurance undertakings shall demonstrate that the internal model is widely used in and plays an important role in their system of governance.*

- Statistical quality standards.

*The methods used to calculate the probability distribution forecast shall be based on adequate, applicable and relevant actuarial and statistical techniques and shall be consistent with the methods used to calculate technical provisions.*

- Calibration standards.

*Insurance and reinsurance undertakings may use a different time period or risk measure [...] for internal modelling purposes as long as the outputs of the internal model can be used by those undertakings to calculate the Solvency Capital Requirement in a manner that provides policy holders and beneficiaries with a level of protection equivalent [...].*

- Profit and loss attribution.

*They shall demonstrate how the categorisation of risk chosen in the internal model explains the causes and sources of profits and losses. The categorisation of risk and attribution of profits and losses shall reflect the risk profile of the insurance and reinsurance undertakings.*

- Validation standards.

*The model validation process shall include an effective statistical process for validating the internal model which enables the insurance and reinsurance undertakings to demonstrate to their supervisory authorities that the resulting capital requirements are appropriate.*

- Documentation standards.

*The documentation shall provide a detailed outline of the theory, assumptions, and mathematical and empirical bases underlying the internal model.*

We point out that a full internal model shall consider all the risk modules and sub-modules, differently from a partial internal model. The choice to adopt a full or partial internal model is not as simple as it seems, because it depends on several reasons, such as commercial reasons.

## 2.5 Supervisory review process

Solvency II describes the supervisory review process.<sup>15</sup>

*Member States shall ensure that the supervisory authorities review and evaluate the strategies, processes and reporting procedures which are established by the insurance and reinsurance undertakings to comply with the laws, regulations and administrative provisions adopted pursuant to this Directive.*

*The supervisory authorities shall in particular review and evaluate compliance with the following:*

- (a) the system of governance, including the own-risk and solvency assessment [...];*
- (b) the technical provisions [...];*
- (c) the capital requirements [...];*
- (d) the investment rules [...];*
- (e) the quality and quantity of own funds [...];*
- (f) where the insurance or reinsurance undertaking uses a full or partial internal model, on-going compliance with the requirements for full and partial internal models [...].*

The supervisory authorities shall approve internal models (see section 2.4) and evaluate compliance with their requirements. We point out that the supervisory review process is important to ensure that insurance companies comply with the law. In the case that insurance companies do not comply with the law, the supervisory authorities may set a capital add-on.

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<sup>15</sup>The following part is taken from Art. 36 of Solvency II [13].

## 2.5.1 Capital add-on

Solvency II describes the capital add-on.<sup>16</sup>

*Following the supervisory review process supervisory authorities may in exceptional circumstances set a capital add-on for an insurance or reinsurance undertaking by a decision stating the reasons. That possibility shall exist only in the following cases:*

- (a) the supervisory authority concludes that the risk profile of the insurance or reinsurance undertaking deviates significantly from the assumptions underlying the Solvency Capital Requirement, as calculated using the standard formula [...];*
- (b) the supervisory authority concludes that the risk profile of the insurance or reinsurance undertaking deviates significantly from the assumptions underlying the Solvency Capital Requirement, as calculated using an internal model or partial internal model [...], because certain quantifiable risks are captured insufficiently and the adaptation of the model to better reflect the given risk profile has failed within an appropriate timeframe;*
- (c) the supervisory authority concludes that the system of governance of an insurance or reinsurance undertaking deviates significantly from the standards [...], that those deviations prevent it from being able to properly identify, measure, monitor, manage and report the risks that it is or could be exposed to and that the application of other measures is in itself unlikely to improve the deficiencies sufficiently within an appropriate time frame;*  
*[...]*

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<sup>16</sup>The following part is taken from Art. 37 of Solvency II [13].

We point out that the supervisory review process and the capital add-on prevent insurance companies not to be careful.

## 2.6 System of governance

Solvency II describes the system of governance.<sup>17</sup>

*Member States shall require all insurance and reinsurance undertakings to have in place an effective system of governance which provides for sound and prudent management of the business.*

We point out that the system of governance is important to ensure that insurance companies work well.

## 2.7 Risk management system

Solvency II describes the risk management system.<sup>18</sup>

*Insurance and reinsurance undertakings shall have in place an effective risk-management system comprising strategies, processes and reporting procedures necessary to identify, measure, monitor, manage and report, on a continuous basis the risks, at an individual and at an aggregated level, to which they are or could be exposed, and their interdependencies.*

*That risk-management system shall be effective and well integrated into the organisational structure and in the decision-making processes of the insurance or reinsurance undertaking with proper consideration of the persons who effectively run the undertaking or have other key functions.*

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<sup>17</sup>The following part is taken from Art. 41 of Solvency II [13].

<sup>18</sup>The following part is taken from Art. 44 of Solvency II [13].

*The risk-management system shall cover the risks to be included in the calculation of the Solvency Capital Requirement [...] as well as the risks which are not or not fully included in the calculation thereof.*

*For insurance and reinsurance undertakings using a partial or full internal model [...] the risk-management function shall cover the following additional tasks:*

- (a) to design and implement the internal model;*
- (b) to test and validate the internal model;*
- (c) to document the internal model and any subsequent changes made to it;*
- (d) to analyse the performance of the internal model and to produce summary reports thereof;*
- (e) to inform the administrative, management or supervisory body about the performance of the internal model, suggesting areas needing improvement, and up-dating that body on the status of efforts to improve previously identified weaknesses.*

We point out that insurance company work with risks. Hence, the risk management system is important to ensure that insurance companies work well.

### **2.7.1 Own Risk and Solvency Assessment**

Solvency II describes the Own Risk and Solvency Assessment.<sup>19</sup>

*As part of its risk-management system every insurance undertaking and reinsurance undertaking shall conduct its own risk and solvency assessment.*

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<sup>19</sup>The following part is taken from Art. 45 of Solvency II [13].

*That assessment shall include at least the following:*

- (a) the overall solvency needs taking into account the specific risk profile, approved risk tolerance limits and the business strategy of the undertaking;*
- (b) the compliance, on a continuous basis, with the capital requirements [...] and with the requirements regarding technical provisions [...];*
- (c) the significance with which the risk profile of the undertaking concerned deviates from the assumptions underlying the Solvency Capital Requirement [...], calculated with the standard formula [...] or with its partial or full internal model [...].*

*The own-risk and solvency assessment shall be an integral part of the business strategy and shall be taken into account on an ongoing basis in the strategic decisions of the undertaking.*

*Insurance and reinsurance undertakings shall perform the assessment [...] regularly and without any delay following any significant change in their risk profile.*

*The insurance and reinsurance undertakings shall inform the supervisory authorities of the results of each own-risk and solvency assessment [...].*

We point out that the Solvency Capital Requirement shall be calculated over a one-year period, hence the view according to Solvency II is short-term. In this situation, the risk is not properly managed, since problems could emerge in the future. Nevertheless, the Own Risk and Solvency Assessment encourages insurance companies to have a medium-term view as well.

### **Forward Looking Assessment of Own Risks**

Solvency II requires the undertaking to perform a regular Forward Looking Assessment of the undertaking's Own Risks (FLAOR) as part of the risk



management system. The main purpose of the Forward Looking Assessment of the undertaking's Own Risks is to ensure that the undertaking engages in the process of assessing all the risks inherent to its business and determines the corresponding capital needs. To achieve this, an undertaking needs adequate and robust processes to assess, monitor and measure its risks and overall solvency needs, and also to ensure that the output from the assessment forms an important part of the decision making processes of the undertaking.<sup>20</sup>

The EIOPA Guidelines on Forward Looking Assessment of Own Risks demand some provisions.<sup>21</sup>

*In accordance with Article 45 of Solvency II, national competent authorities should ensure that the undertaking's assessment of the overall solvency needs is forward-looking, including a medium term or long term perspective as appropriate.*

The analysis of the undertaking's ability to continue as a going concern and the financial resources needed to do so over a time horizon of more than one year is an important part of the Forward Looking Assessment of the undertaking's Own Risks.

Unless an undertaking is in a winding-up situation, it has to consider how it can ensure that it can continue as a going concern. In order to do this successfully, not only does it have to assess its current risks, but also the risks it will or could face in the long term. That means that, depending on the complexity of the undertaking's business, it may be appropriate to perform long term projections of the business, which are in any case a key part of any undertaking's financial planning. This might include business plans and projections of the economic balance sheet as well as variation analysis to reconcile these two items. These projections are required to feed into the Forward Looking Assessment of the undertaking's Own Risks in order to enable the undertaking to form an opinion on its overall solvency needs and own funds in a forward looking perspective.

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<sup>20</sup>Different parts of this paragraph are taken from the EIOPA Explanatory Text [8].

<sup>21</sup>The following part is taken from Guideline 13 of the EIOPA Guidelines [9].

An undertaking also identifies and takes into account external factors that could have an adverse impact on its overall solvency needs or on its own funds. Such external factors could include changes in the economic conditions, the legal framework, the fiscal environment, the insurance market, technical developments that have an impact on underwriting risk, or any other probable relevant event. The undertaking will need to consider as part of its capital management plans and capital projections how it might respond to unexpected changes in external factors.

# Chapter 3

## Investment models

The inversion of the production cycle is a peculiar and important feature of insurance companies and it means that policyholders pay premiums in advance and contractual benefits are paid later, only when an accident occurs. This characteristic implies that insurance companies have a lot of resources, other than their own equity, to be invested, in order to make profits.

Using equation (1.12), we are able to obtain the distribution of the annual rate of return time by time, once we have described over time the distributions of the average stock and bond prices.

In this chapter we deal with models based on differential equations. The random variables are not indicated with the tilde, in order to avoid the complexity of the mathematical notation. Moreover, we assume that the market is frictionless, meaning that all securities are perfectly divisible and that no short-sale restrictions, transaction costs, or taxes are present. The security trading is continuous and there are no riskless arbitrage opportunities.

### 3.1 Continuous-time stochastic processes

A stochastic process is a process that describes the evolution in time of a random phenomenon. From a mathematical point of view, a stochastic process is a collection of random variables defined on a common probability

space, taking values in a common state space and indexed by some set, that usually represents time. In a continuous-time stochastic process, the index set of the stochastic process is continuous.<sup>1</sup>

### 3.1.1 Brownian motion

A stochastic process is said to be a Brownian motion  $W(t)$  (also called Wiener process) if it satisfies the following properties:

1.  $W(0) = 0$
2.  $W$  has increments independent of the past, namely:

$$\forall t > 0 \quad \text{then} \quad W(t + \Delta t) - W(t) \perp W(s)$$

$$\text{with} \quad \Delta t \geq 0 \quad \text{and} \quad s < t$$

3.  $W$  has Normal increments, such that:

$$\forall t > 0 \quad \text{then} \quad W(t + \Delta t) - W(t) \sim \mathcal{N}(0, \Delta t)$$

$$\text{or} \quad \Delta W(t) = \epsilon \cdot \sqrt{\Delta t} \tag{3.1}$$

$$\text{with} \quad \Delta t \geq 0 \quad \text{and} \quad \epsilon \sim \mathcal{N}(0, 1)$$

4.  $W$  has continuous paths, namely  $W(t)$  is continuous in  $t$ .

As a result, the auto-covariance function is found to be:

$$\text{Cov}[W(t), W(s)] = \min(t, s)$$

Furthemore, the Brownian motion itself has a Normal distribution as well, such that:

$$W(t) \sim \mathcal{N}(0, t)$$

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<sup>1</sup>For this section see Ballotta and Fusai [2], and Hull [16].

This result enables to simulate a Brownian motion at some point in time. In order to simulate its path, we must iteratively use equation (3.1).

### Correlated Brownian motions

In some cases we may need to construct a pair of Brownian motions  $W_1(t)$  and  $W_2(t)$ , such that their correlation coefficient is given by:

$$\text{Corr}[W_1(t), W_2(t)] = \rho$$

For this reason, we take the two Brownian motions  $Z_1(t)$  and  $Z_2(t)$ , independent of each other, so that  $W_1(t)$  and  $W_2(t)$  are found to be:

$$W_1(t) = Z_1(t)$$

and:

$$W_2(t) = \rho \cdot Z_1(t) + \sqrt{1 - \rho^2} \cdot Z_2(t)$$

then:

$$\text{Corr}[W_1(t), W_2(t)] = \frac{\text{Cov}[Z_1(t), \rho \cdot Z_1(t) + \sqrt{1 - \rho^2} \cdot Z_2(t)]}{\sqrt{\text{Var}[Z_1(t)]} \cdot \sqrt{\text{Var}[Z_2(t)]}} = \rho$$

More in general we may need to construct some different Brownian motions  $W_1(t), W_2(t), \dots, W_h(t)$ , such that their correlation matrix is given by:

$$\text{Corr}_W(t) = \begin{bmatrix} 1 & \rho_{21} & \dots & \rho_{h1} \\ \rho_{21} & 1 & \dots & \rho_{h2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{h1} & \rho_{h2} & \dots & 1 \end{bmatrix}$$

The Cholesky decomposition (also called Cholesky factorization) shows that the correlation matrix can be decomposed as follows:<sup>2</sup>

$$\text{Corr}_W(t) = \begin{bmatrix} L_{11} & 0 & \dots & 0 \\ L_{21} & L_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{h1} & L_{h2} & \dots & L_{hh} \end{bmatrix} \begin{bmatrix} L_{11} & L_{21} & \dots & L_{h1} \\ 0 & L_{22} & \dots & L_{h2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & L_{hh} \end{bmatrix}$$

where:

$$L_{jj} = \sqrt{\rho_{jj} - \sum_{k=1}^{j-1} L_{jk}^2}$$

and:

$$L_{ij} = \frac{1}{L_{jj}} \cdot \left( \rho_{ij} - \sum_{k=1}^{j-1} L_{ik} \cdot L_{jk} \right) \quad \text{for } i > j$$

Hence, we take the  $h$  Brownian motions  $Z_1(t), Z_2(t), \dots, Z_h(t)$ , independent of each other, so that  $W_1(t), W_2(t), \dots, W_h(t)$  are found to be:

$$\begin{bmatrix} W_1(t) \\ W_2(t) \\ \vdots \\ W_h(t) \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & \dots & 0 \\ L_{21} & L_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{h1} & L_{h2} & \dots & L_{hh} \end{bmatrix} \begin{bmatrix} Z_1(t) \\ Z_2(t) \\ \vdots \\ Z_h(t) \end{bmatrix}$$

### 3.1.2 Geometric Brownian motion

The geometric Brownian motion  $Y(t)$  (also called generalized Wiener process or exponential Brownian motion) is a stochastic process that satisfies the

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<sup>2</sup>See Parker [25].

following stochastic differential equation, based on the standard Brownian motion  $W(t)$ :

$$dY(t) = \mu \cdot Y(t) \cdot dt + \sigma \cdot Y(t) \cdot dW(t)$$

where  $\mu$  and  $\sigma > 0$  are the constant drift and diffusion coefficients of the process respectively, i.e. the expected rate of return in a short period of time (annualized) and the volatility. The geometric Brownian motion is made of a predictable or expected part and an unpredictable or unexpected part, which reflects random changes. Moreover, the process remains always positive if it starts from a positive value.

The discrete-time version of the geometric Brownian motion, given by the Euler method, is found to be:

$$\Delta Y(t) = \mu \cdot Y(t) \cdot \Delta t + \sigma \cdot Y(t) \cdot \Delta W(t) \quad (3.2)$$

Using equation (3.1), it is found to be:

$$\Delta Y(t) = \mu \cdot Y(t) \cdot \Delta t + \sigma \cdot Y(t) \cdot \epsilon \cdot \sqrt{\Delta t}$$

The smaller the time increment  $\Delta t$ , the better the discretization.

Furthermore, the stochastic differential equation, satisfied by a geometric Brownian motion, admits an explicit solution. It can be shown that it is given by:<sup>3</sup>

$$Y(t) = Y(0) \cdot \exp\left[\left(\mu - \frac{1}{2} \cdot \sigma^2\right) \cdot t + \sigma \cdot W(t)\right] \quad (3.3)$$

Hence, since the standard Brownian motion has a Normal distribution, the variable  $Y$  is found to have a Lognormal distribution, such that:

$$E[Y(t) | Y(0)] = Y(0) \cdot \exp(\mu \cdot t) \quad (3.4)$$

and:

$$\text{Var}[Y(t) | Y(0)] = Y(0)^2 \cdot \exp(2 \cdot \mu \cdot t) \cdot [\exp(\sigma^2 \cdot t) - 1] \quad (3.5)$$

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<sup>3</sup>See Ballotta and Fusai [2].

### 3.1.3 Itô process

The Itô process  $Y(t)$  is a geometric Brownian motion in which the drift and diffusion coefficients are functions of the value of the underlying variable and time, so that it satisfies the following stochastic differential equation:

$$dY(t) = \mu(Y, t) \cdot dt + \sigma(Y, t) \cdot dW(t)$$

where  $\mu(Y, t)$  and  $\sigma(Y, t) > 0$  are the non-constant drift and diffusion coefficients of the process respectively. As above, the Itô process is made of a predictable or expected part and an unpredictable or unexpected part.

The discrete-time version of the Itô process, given by the Euler method, is found to be:

$$\Delta Y(t) = \mu(Y, t) \cdot \Delta t + \sigma(Y, t) \cdot \Delta W(t) \quad (3.6)$$

Using equation (3.1), it is found to be:

$$\Delta Y(t) = \mu(Y, t) \cdot \Delta t + \sigma(Y, t) \cdot \epsilon \cdot \sqrt{\Delta t}$$

This equation assumes that the drift and diffusion coefficients remain constant in the time interval between  $t$  and  $\Delta t$ . Moreover, the smaller the time increment  $\Delta t$ , the better the discretization.

## 3.2 Itô's lemma

A derivative contract is a financial instrument whose value depends on (or derives from) the values of other, more basic, stochastic underlying variables. The derivative price is a function of the underlying variables and time.<sup>4</sup>

The stochastic process for the price of the variable  $Y$  follows an Itô process that satisfies the following stochastic differential equation:

$$dY(t) = \mu(Y, t) \cdot dt + \sigma(Y, t) \cdot dW(t)$$

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<sup>4</sup>For this section see Hull [16], and Itô [18].



Let  $G$  be a function of  $Y$  and  $t$ . Then, Itô's lemma shows that the price of  $G$  (dropping the time index  $t$ ) follows an Itô process that satisfies the following stochastic differential equation:

$$dG = \left[ \frac{\partial G}{\partial t} + \mu(Y, t) \cdot \frac{\partial G}{\partial Y} + \frac{1}{2} \cdot \sigma^2(Y, t) \cdot \frac{\partial^2 G}{\partial Y^2} \right] \cdot dt + \sigma(Y, t) \cdot \frac{\partial G}{\partial Y} \cdot dW(t)$$

It can be expressed in the following form:

$$dG = \mu_G(Y, t) \cdot G \cdot dt + \sigma_G(Y, t) \cdot G \cdot dW(t)$$

then:

$$\mu_G(Y, t) = \frac{1}{G} \cdot \left[ \frac{\partial G}{\partial t} + \mu(Y, t) \cdot \frac{\partial G}{\partial Y} + \frac{1}{2} \cdot \sigma^2(Y, t) \cdot \frac{\partial^2 G}{\partial Y^2} \right]$$

and:

$$\sigma_G(Y, t) = \frac{1}{G} \cdot \sigma(Y, t) \cdot \frac{\partial G}{\partial Y}$$

We point out that both the variable  $Y$  and the function  $G$  are based on the same standard Brownian motion.

### 3.3 Market price of risk

The stochastic process for the price of the variable  $Y$  follows an Itô process that satisfies the following stochastic differential equation:<sup>5</sup>

$$dY(t) = \mu(Y, t) \cdot dt + \sigma(Y, t) \cdot dW(t)$$

We assume the prices of a pair of derivatives to be a function of  $Y$  and  $t$ . Itô's lemma shows that the dynamics of the derivative prices are given by:

$$\frac{dG_1(t)}{G_1(t)} = \mu_{G_1}(Y, t) \cdot dt + \sigma_{G_1}(Y, t) \cdot dW(t) \quad (3.7)$$

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<sup>5</sup>For this section see Etheridge [11], and Giordano and Siciliano [15].

and:

$$\frac{dG_2(t)}{G_2(t)} = \mu_{G_2}(Y, t) \cdot dt + \sigma_{G_2}(Y, t) \cdot dW(t) \quad (3.8)$$

We shall construct an instantaneously riskless portfolio, therefore we buy  $\sigma_{G_2}(Y, t) \cdot G_2(t)$  of the first derivative and we sell  $\sigma_{G_1}(Y, t) \cdot G_1(t)$  of the second derivative. As a result, the value of the portfolio at time  $t$  is given by:

$$\Pi(t) = [\sigma_{G_2}(Y, t) \cdot G_2(t)] \cdot G_1(t) - [\sigma_{G_1}(Y, t) \cdot G_1(t)] \cdot G_2(t) \quad (3.9)$$

and the change in the value of the portfolio in time  $dt$  is given by:

$$d\Pi(t) = [\sigma_{G_2}(Y, t) \cdot G_2(t)] \cdot dG_1(t) - [\sigma_{G_1}(Y, t) \cdot G_1(t)] \cdot dG_2(t)$$

Using equations (3.7) and (3.8), it is found to be:

$$\begin{aligned} d\Pi(t) &= [\mu_{G_1}(Y, t) \cdot \sigma_{G_2}(Y, t) \cdot G_1(t) \cdot G_2(t)] \cdot dt \\ &\quad - [\mu_{G_2}(Y, t) \cdot \sigma_{G_1}(Y, t) \cdot G_1(t) \cdot G_2(t)] \cdot dt \end{aligned} \quad (3.10)$$

Since the portfolio is instantaneously riskless, in order to avoid arbitrage opportunities, it must earn the risk-free short rate  $r(t)$ , so that:

$$d\Pi(t) = r(t) \cdot \Pi(t) \cdot dt$$

Using equations (3.9) and (3.10), we obtain:

$$\frac{\mu_{G_1}(Y, t) - r(t)}{\sigma_{G_1}(Y, t)} = \frac{\mu_{G_2}(Y, t) - r(t)}{\sigma_{G_2}(Y, t)}$$

Each side of the equation depends on the parameters of a single process. Hence, we drop numeric indices by equations (3.7) and (3.8), in order to obtain a generic dynamic of the derivative price. It is given by:

$$\frac{dG(t)}{G(t)} = \mu_G(Y, t) \cdot dt + \sigma_G(Y, t) \cdot dW(t)$$

The market price of risk of  $Y$  is given by:

$$\lambda(Y, t) = \frac{\mu_G(Y, t) - r(t)}{\sigma_G(Y, t)} \quad (3.11)$$

then:

$$\mu_G(Y, t) = r(t) + \lambda(Y, t) \cdot \sigma_G(Y, t) \quad (3.12)$$

The market price of risk (also called Sharpe Ratio, in the context of portfolio performance measurement) gives the extra increase in expected rate of return per an additional unit of risk. It depends on  $Y$  and  $t$ , but it does not depend on the nature of the derivative.

### 3.3.1 Risk-neutral world and real world

A risk-neutral world is a world where investors are risk-neutral. The real world is not a risk-neutral world, hence the higher the risks investors take, the higher the expected returns they require.

A risk-neutral measure (also called Q-measure) is a probability measure, which assumes a risk-neutral world. As a result, investors do not increase the expected return they require from an investment to compensate for increased risk and the expected return on all assets (and therefore the discount rate to use for all expected payoffs) is the risk-free rate. On the contrary, the real measure (also called P-measure) is a probability measure, which assumes the real world. As a result, the expected return on all assets is probably different from the risk-free rate.

We will deal with risk-neutral and real-world processes. The risk-neutral processes are characterized by a risk-neutral measure and a zero market price of risk. The real-world processes are characterized by a real measure and a non-zero market price of risk.

### Real-world process

The real-world process for the price of the variable  $Y$  satisfies the following stochastic differential equation:

$$dY(t) = \mu(Y, t) \cdot dt + \sigma(Y, t) \cdot dW(t) \quad (3.13)$$

where  $W(t)$  is a P-Brownian motion.

The real-world dynamic of the derivative price is given by:

$$\frac{dG(t)}{G(t)} = \mu_G(Y, t) \cdot dt + \sigma_G(Y, t) \cdot dW(t) \quad (3.14)$$

According to the Girsanov's theorem, as we move from the real world to the risk-neutral world, the expected rate of return changes, but the volatility remains the same.

### Risk-neutral process

The risk-neutral process for the price of the variable  $Y$ , that is equivalent to equation (3.13), is found to be:

$$dY(t) = [\mu(Y, t) - \lambda(Y, t) \cdot \sigma(Y, t)] \cdot dt + \sigma(Y, t) \cdot d\hat{W}(t) \quad (3.15)$$

where  $\hat{W}(t)$  is a Q-Brownian motion, that is given by:

$$d\hat{W}(t) = dW(t) + \lambda(Y, t) \cdot dt \quad (3.16)$$

The risk-neutral dynamic of the derivative price, that is equivalent to equation (3.14), is found to be:

$$\frac{dG(t)}{G(t)} = [\mu_G(Y, t) - \lambda(Y, t) \cdot \sigma_G(Y, t)] \cdot dt + \sigma_G(Y, t) \cdot d\hat{W}(t)$$

Using equation (3.12), it is found to be:

$$\frac{dG(t)}{G(t)} = r(t) \cdot dt + \sigma_G(Y, t) \cdot d\hat{W}(t) \quad (3.17)$$

The market price of risk of  $Y$  is used to move from the real-world process for the price of a single variable to the risk-neutral process, as well as from the real-world dynamic of the derivative price to the risk-neutral dynamic.

### 3.4 Stock price model

A stock price model is a mathematical model that describes the stochastic behaviour of the stock price at time  $t$ . In this section we deal with a non-dividend-paying stock price model described by the geometric Brownian motion.

#### Real-world process

The real-world process for the  $h$ -th stock price follows a geometric Brownian motion that satisfies the following stochastic differential equation:

$$dS_h(t) = \mu_h \cdot S_h(t) \cdot dt + \sigma_h \cdot S_h(t) \cdot dW_h(t) \quad (3.18)$$

where the drift and diffusion coefficients above are the expected rate of return on the  $h$ -th stock in a short period of time (annualized) and the volatility of the  $h$ -th stock price respectively.

Using equation (3.2), the discrete-time version of the real-world process for the  $h$ -th stock price is found to be:

$$\Delta S_h(t) = \mu_h \cdot S_h(t) \cdot \Delta t + \sigma_h \cdot S_h(t) \cdot \Delta W_h(t) \quad (3.19)$$

Furthermore, using equation (3.3), the solution of the stochastic differential equation above is found to be:

$$S_h(t) = S_h(0) \cdot \exp \left[ \left( \mu_h - \frac{1}{2} \cdot \sigma_h^2 \right) \cdot t + \sigma_h \cdot W_h(t) \right] \quad (3.20)$$

As shown in equations (3.4) and (3.5), the  $h$ -th stock price is found to have

a Lognormal distribution, such that:

$$\mathbb{E}[S_h(t) | S_h(0)] = S_h(0) \cdot \exp(\mu_h \cdot t) \quad (3.21)$$

and:

$$\text{Var}[S_h(t) | S_h(0)] = S_h(0)^2 \cdot \exp(2 \cdot \mu_h \cdot t) \cdot [\exp(\sigma_h^2 \cdot t) - 1]$$

In conclusion, the market price of risk of the  $h$ -th stock price is found to be:

$$\lambda(S_h, t) = \frac{\mu_h - r(t)}{\sigma_h}$$

then:

$$\mu_h = r(t) + \lambda(S_h, t) \cdot \sigma_h \quad (3.22)$$

Investors require a positive extra return over the risk-free rate for investing in stocks, hence the market price of risk is positive. In the stock model, we do not assume any form or value for the market price of risk.

### **Risk-neutral process**

The risk-neutral process for the  $h$ -th stock price, that is equivalent to equation (3.18), is found to be:

$$dS_h(t) = [\mu_h - \lambda(S_h, t) \cdot \sigma_h] \cdot S_h(t) \cdot dt + \sigma_h \cdot S_h(t) \cdot d\hat{W}_h(t)$$

where  $\hat{W}_h(t)$  is the  $h$ -th Q-Brownian motion, that is given by:

$$d\hat{W}_h(t) = dW_h(t) + \lambda(S_h, t) \cdot dt$$

Using equation (3.22), it is found to be:

$$dS_h(t) = r(t) \cdot S_h(t) \cdot dt + \sigma_h \cdot S_h(t) \cdot d\hat{W}_h(t)$$

Also in the risk-neutral case, we could find the discrete-time version of the process and the solution of the stochastic differential equation above.

### 3.5 One-factor short rate models

A one-factor short rate model is a mathematical model that describes a single stochastic factor, i.e. the short-term interest rate at time  $t$  (also called instantaneous short rate, since it applies to an infinitesimally short period of time at time  $t$ ). Once it is specified, we are able to compute the zero-coupon bond price and determine the initial zero curve and its future evolution.<sup>6</sup>

A zero-coupon bond is a long-term debt contract, where the face value (also called par value) is repaid at maturity date and coupon interest payments are absent. Consider a zero-coupon bond that provides a terminal payoff equal to 1 at maturity date  $T > t$ . Its price at time  $t$  is given by:

$$B(t, T) = e^{-R(t, T) \cdot (T-t)}$$

As a result, the zero-coupon interest rate (continuously compounded) at time  $t$  for a term of  $T - t$  is given by:

$$R(t, T) = -\frac{\ln B(t, T)}{T - t} \quad (3.23)$$

This equation enables the zero curve at any time.

#### Real-world process

In a one-factor short rate model, the real-world process for the short rate usually follows an Itô process that satisfies the following stochastic differential equation:

$$dr(t) = \mu(r, t) \cdot dt + \sigma(r, t) \cdot dZ(t) \quad (3.24)$$

Using equation (3.6), the discrete-time version of the real-world process for the short rate is found to be:

$$\Delta r(t) = \mu(r, t) \cdot \Delta t + \sigma(r, t) \cdot \Delta Z(t)$$

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<sup>6</sup>For this section see Hull [17], Kwok [22], and Svoboda [35].

We assume the zero-coupon bond price to be a function of  $r$  and  $t$ . Itô's lemma shows that the real-world dynamic of the bond price (dropping the time index  $t$  and the maturity date  $T$ ) is given by:

$$dB = \left[ \frac{\partial B}{\partial t} + \mu(r, t) \cdot \frac{\partial B}{\partial r} + \frac{1}{2} \cdot \sigma^2(r, t) \cdot \frac{\partial^2 B}{\partial r^2} \right] \cdot dt + \sigma(r, t) \cdot \frac{\partial B}{\partial r} \cdot dZ(t)$$

It can be expressed in the following form:

$$dB = \mu_B(r, t) \cdot B \cdot dt + \sigma_B(r, t) \cdot B \cdot dZ(t) \quad (3.25)$$

then:

$$\mu_B(r, t) = \frac{1}{B} \cdot \left[ \frac{\partial B}{\partial t} + \mu(r, t) \cdot \frac{\partial B}{\partial r} + \frac{1}{2} \cdot \sigma^2(r, t) \cdot \frac{\partial^2 B}{\partial r^2} \right] \quad (3.26)$$

and:

$$\sigma_B(r, t) = \frac{1}{B} \cdot \sigma(r, t) \cdot \frac{\partial B}{\partial r} \quad (3.27)$$

where the drift and diffusion coefficients above are the expected rate of return on the bond in a short period of time (annualized) and the (negative) bond's risk exposure respectively.

In conclusion, using equation (3.11), the market price of risk of the short rate is found to be:

$$\lambda(r, t) = \frac{\mu_B(r, t) - r(t)}{\sigma_B(r, t)}$$

then:

$$\mu_B(r, t) = r(t) + \lambda(r, t) \cdot \sigma_B(r, t) \quad (3.28)$$

Investors require a positive extra return over the risk-free rate for investing in bonds, hence the market price of risk is negative, because the bond's risk exposure is negative as well.

### Risk-neutral process

Using equation (3.15), the risk-neutral process for the short rate, that is equivalent to equation (3.24), is found to satisfy the following stochastic



differential equation:

$$dr(t) = [\mu(r, t) - \lambda(r, t) \cdot \sigma(r, t)] \cdot dt + \sigma(r, t) \cdot d\hat{Z}(t) \quad (3.29)$$

As previously shown in equation (3.16), the Q-Brownian motion is found to be:

$$d\hat{Z}(t) = dZ(t) + \lambda(r, t) \cdot dt$$

Furthermore, using equation (3.17), the risk-neutral dynamic of the bond price, that is equivalent to equation (3.25), is found to be:

$$dB = r(t) \cdot B \cdot dt + \sigma_B(r, t) \cdot B \cdot d\hat{Z}(t)$$

Also in the risk-neutral case, we could find the discrete-time version of the process.

### 3.5.1 Vasicek model

The Vasicek model was introduced in 1977 by Oldřich Alfons Vašíček and it is one of the earliest stochastic models of the term structure of interest rates.<sup>7</sup>

#### Real-world process

According to the Vasicek model, the real-world process for the short rate satisfies the following stochastic differential equation:

$$dr(t) = \kappa \cdot (\theta - r(t)) \cdot dt + v \cdot dZ(t) \quad (3.30)$$

where  $\kappa > 0$  is the speed of mean-reversion,  $\theta$  is the long-run mean interest rate level and  $v > 0$  is the interest rate risk exposure. The short rate tends to revert to  $\theta$ , because  $\kappa$  steadies its fluctuations. As a result, it is stationary.

Using equation (3.6), the discrete-time version of the real-world process

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<sup>7</sup>See Vasicek [36].

for the short rate is found to be:

$$\Delta r(t) = \kappa \cdot (\theta - r(t)) \cdot \Delta t + v \cdot \Delta Z(t) \quad (3.31)$$

Furthermore, the stochastic differential equation of the Vasicek model admits an explicit solution. It can be shown that it is given by:<sup>8</sup>

$$r(t) = e^{-\kappa \cdot t} \cdot r(0) + \theta \cdot (1 - e^{-\kappa \cdot t}) + v \cdot \int_0^t e^{-\kappa \cdot (t-s)} \cdot dZ(s) \quad (3.32)$$

Hence, since the standard Brownian motion has Normal distribution, the short rate is found to have a Normal distribution as well, such that:

$$\mathbb{E}[r(t) | r(0)] = e^{-\kappa \cdot t} \cdot r(0) + \theta \cdot (1 - e^{-\kappa \cdot t}) \quad (3.33)$$

and:

$$\text{Var}[r(t) | r(0)] = \frac{v^2}{2 \cdot \kappa} \cdot (1 - e^{-2 \cdot \kappa \cdot t})$$

The real-world dynamic of the bond price is given by equation (3.25). Using equation (3.26), the expected rate of return on the bond is found to be:

$$\mu_B(r, t) = \frac{1}{B} \cdot \left[ \frac{\partial B}{\partial t} + \kappa \cdot (\theta - r(t)) \cdot \frac{\partial B}{\partial r} + \frac{1}{2} \cdot v^2 \cdot \frac{\partial^2 B}{\partial r^2} \right] \quad (3.34)$$

Using equation (3.27), the bond's risk exposure is found to be:

$$\sigma_B(r, t) = \frac{1}{B} \cdot v \cdot \frac{\partial B}{\partial r}$$

We assume the market price of risk to be the (negative) constant  $\lambda$ . As a result, using equation (3.28), the expected rate of return on the bond is also found to be:

$$\mu_B(r, t) = r(t) + \lambda \cdot v \cdot \frac{1}{B} \cdot \frac{\partial B}{\partial r} \quad (3.35)$$

Hence, combining equations (3.34) and (3.35), the Vasicek partial differential

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<sup>8</sup>See Ballotta and Fusai [2].

equation is found to be:

$$\frac{\partial B}{\partial t} + \kappa \cdot (\theta - r(t)) \cdot \frac{\partial B}{\partial r} + \frac{1}{2} \cdot v^2 \cdot \frac{\partial^2 B}{\partial r^2} - \lambda \cdot v \cdot \frac{\partial B}{\partial r} = B \cdot r(t)$$

We solve the equation above, with the following boundary conditions:

$$\lim_{t \rightarrow T} B(t, T) = 1 \quad \text{terminal payoff condition}$$

$$\lim_{r(t) \rightarrow +\infty} B(t, T) = 0 \quad \text{present value infimum condition}$$

then the zero-coupon bond price is found to be:

$$B(t, T) = a(t, T) \cdot e^{-b(t, T) \cdot r(t)} \quad (3.36)$$

where:

$$b(t, T) = \frac{1 - e^{-\kappa \cdot (T-t)}}{\kappa}$$

and:

$$a(t, T) = \exp \left[ \gamma \cdot (b(t, T) - (T - t)) - \frac{v^2}{4 \cdot \kappa} \cdot b(t, T)^2 \right] \quad (3.37)$$

with:

$$\gamma = \theta - \frac{\lambda \cdot v}{\kappa} - \frac{v^2}{2 \cdot \kappa^2}$$

### Risk-neutral process

Using equation (3.29), the risk-neutral process for the short rate, that is equivalent to equation (3.30), is found to satisfy the following stochastic differential equation:

$$dr(t) = [\kappa \cdot (\theta - r(t)) - \lambda \cdot v] \cdot dt + v \cdot d\hat{Z}(t)$$

It can be written as follows:

$$dr(t) = \kappa \cdot (\theta^* - r(t)) \cdot dt + v \cdot d\hat{Z}(t) \quad (3.38)$$

where:

$$\theta^* = \theta - \frac{\lambda \cdot v}{\kappa}$$

The risk-neutral process is the same as the real-world process, except that the long-run mean interest rate level is higher (because  $\lambda$  is negative).

Also in the risk-neutral case, we could find the discrete-time version of the process and the solution of the stochastic differential equation above.

### 3.5.2 Cox-Ingersoll-Ross model

The Cox-Ingersoll-Ross model was introduced in 1985 by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross as an extension of the Vasicek model.<sup>9</sup>

#### Real-world process

According to the Cox-Ingersoll-Ross model, the real-world process for the short rate satisfies the following stochastic differential equation:

$$dr(t) = \kappa \cdot (\theta - r(t)) \cdot dt + v \cdot \sqrt{r(t)} \cdot dZ(t) \quad (3.39)$$

The drift coefficient is the same as that of Vasicek, but the interest rate risk exposure is proportional to  $\sqrt{r(t)}$ . As a result, the diffusion coefficient increases as the short rate increases. Moreover, the short rate can never be negative and, in addition, when  $2 \cdot \kappa \cdot \theta \geq v^2$  it can never be zero. As above, the short rate is stationary.

Using equation (3.6), the discrete-time version of the real-world process for the short rate is found to be:

$$\Delta r(t) = \kappa \cdot (\theta - r(t)) \cdot \Delta t + v \cdot \sqrt{r(t)} \cdot \Delta Z(t) \quad (3.40)$$

Furthermore, the stochastic differential equation of the Cox-Ingersoll-Ross model does not admit an explicit solution. However, it can be shown that

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<sup>9</sup>See Cox, Ingersoll and Ross [5].

the short rate has a Non-Central Chi-Square distribution, such that:<sup>10</sup>

$$\mathbb{E}[r(t) | r(0)] = e^{-\kappa \cdot t} \cdot r(0) + \theta \cdot (1 - e^{-\kappa \cdot t}) \quad (3.41)$$

and:

$$\text{Var}[r(t) | r(0)] = r(0) \cdot \frac{v^2}{\kappa} \cdot (e^{-\kappa \cdot t} - e^{-2 \cdot \kappa \cdot t}) + \theta \cdot \frac{v^2}{2 \cdot \kappa} \cdot (1 - e^{-\kappa \cdot t})^2$$

The real-world dynamic of the bond price is given by equation (3.25). Using equation (3.26), the expected rate of return on the bond is found to be:

$$\mu_B(r, t) = \frac{1}{B} \cdot \left[ \frac{\partial B}{\partial t} + \kappa \cdot (\theta - r(t)) \cdot \frac{\partial B}{\partial r} + \frac{1}{2} \cdot v^2 \cdot r(t) \cdot \frac{\partial^2 B}{\partial r^2} \right] \quad (3.42)$$

Using equation (3.27), the bond's risk exposure is found to be:

$$\sigma_B(r, t) = \frac{1}{B} \cdot v \cdot \sqrt{r(t)} \cdot \frac{\partial B}{\partial r}$$

We assume the market price of risk to be  $\eta \cdot \sqrt{r(t)} / v$  and  $\eta$  to be a (negative) constant. As a result, using equation (3.28), the expected rate of return on the bond is also found to be:

$$\mu_B(r, t) = r(t) + \eta \cdot r(t) \cdot \frac{1}{B} \cdot \frac{\partial B}{\partial r} \quad (3.43)$$

Hence, combining equations (3.42) and (3.43), the Cox-Ingersoll-Ross partial differential equation is found to be:

$$\frac{\partial B}{\partial t} + \kappa \cdot (\theta - r(t)) \cdot \frac{\partial B}{\partial r} + \frac{1}{2} \cdot v^2 \cdot r(t) \cdot \frac{\partial^2 B}{\partial r^2} - \eta \cdot r(t) \cdot \frac{\partial B}{\partial r} = B \cdot r(t)$$

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<sup>10</sup>See Ballotta and Fusai [2].

We solve the equation above, with the following boundary conditions:

$$\lim_{t \rightarrow T} B(t, T) = 1 \quad \text{terminal payoff condition}$$

$$\lim_{r(t) \rightarrow +\infty} B(t, T) = 0 \quad \text{present value infimum condition}$$

then the zero-coupon bond price is found to be:

$$B(t, T) = a(t, T) \cdot e^{-b(t, T) \cdot r(t)} \quad (3.44)$$

where:

$$b(t, T) = \frac{2 \cdot (e^{\gamma \cdot (T-t)} - 1)}{(\gamma + \kappa + \eta) \cdot (e^{\gamma \cdot (T-t)} - 1) + 2 \cdot \gamma}$$

with:

$$\gamma = \sqrt{(\kappa + \eta)^2 + 2 \cdot v^2}$$

and:

$$a(t, T) = \left[ \frac{2 \cdot \gamma \cdot e^{(\gamma + \kappa + \eta) \cdot (T-t)/2}}{(\gamma + \kappa + \eta) \cdot (e^{\gamma \cdot (T-t)} - 1) + 2 \cdot \gamma} \right]^{2 \cdot \kappa \cdot \theta / v^2}$$

### Risk-neutral process

Using equation (3.29), the risk-neutral process for the short rate, that is equivalent to equation (3.39), is found to satisfy the following stochastic differential equation:

$$dr(t) = [\kappa \cdot (\theta - r(t)) - \eta \cdot r(t)] \cdot dt + v \cdot \sqrt{r(t)} \cdot d\hat{Z}(t)$$

It can be written as follows:

$$dr(t) = \kappa^* \cdot (\theta^* - r(t)) \cdot dt + v \cdot \sqrt{r(t)} \cdot d\hat{Z}(t) \quad (3.45)$$

where:

$$\kappa^* = \kappa + \eta$$

and:

$$\theta^* = \frac{\kappa \cdot \theta}{\kappa^*}$$

The risk-neutral process is the same as the real-world process, except that the speed of mean-reversion is lower (because  $\eta$  is negative) and the long-run mean interest rate level is higher.

Also in the risk-neutral case, we could find the discrete-time version of the process.

### 3.5.3 Comparison

The Vasicek model has some disadvantages. First of all, its parameters do not change over time, therefore there is no term structure of interest rate exposure. Moreover, it is a one-factor model, hence only parallel shifts in the yield curve are possible. Some years ago, it was believed that interest rates could not become negative. As a result, the possibility that the model could produce negative short rates was considered an important drawback. Nowadays, it could be an advantage.

The Cox-Ingersoll-Ross model is a bit more sophisticated than the Vasicek model, indeed there is term structure of interest rate exposure, since it is proportional to the root square of the short rate level. Actually also the Cox-Ingersoll-Ross model has some disadvantages. It is a one-factor model, hence only parallel shifts in the yield curve are possible. Moreover, it can never produce negative short rates.

# Chapter 4

## Aggregate claim model

The inversion of the production cycle also implies that insurance companies have to measure and manage the future aggregate claim amount, in order to control losses and determine insurance premiums.

In this chapter we deal with the collective risk model, in order to describe over time the distribution of the aggregate claim amount.

### 4.1 Collective risk model

The collective risk model is used to determine the aggregate claim amount. It is very popular in non-life insurance, because each risk can produce claims of different dimensions. We consider the entire portfolio or a sub-portfolio of homogeneous risks and we separately analyze the number of claims and the single claim amount, that is assumed to be independent of the contract that generated it.<sup>1</sup>

The stochastic aggregate claim amount at the end of time  $t$  is given by:

$$\tilde{X}_t = \sum_{i=1}^{\tilde{K}_t} \tilde{Z}_{i,t} \quad (4.1)$$

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<sup>1</sup>For this section see Savelli [29], Savelli and Clemente [30], and Zappa and Facchinetti [38].



where  $\tilde{K}_t$  is the stochastic number of claims and  $\tilde{Z}_{i,t}$  is the stochastic amount for the  $i$ -th claim. We assume that:

1. The random variables  $\tilde{Z}_{i,t}$  are independent.
2. The random variables  $\tilde{Z}_{i,t}$  are identically distributed.
3. The random variables  $\tilde{Z}_{i,t}$  and  $\tilde{K}_t$  are independent.

The first and second assumption are satisfied, in particular, for limited time periods. The third assumption is usually satisfied, but it could be refuted in some situation. In case of storm, for example, the number of claims and single claim amount increase significantly at the same time.

#### 4.1.1 Number of claims

We now define different distributions associated with the number of claims, such as the Poisson and Negative Binomial. We point out that the aggregate claim amount is found to be zero, in case that the number of claims is zero as well. Moreover, since the insurance portfolio is dynamic, we assume that the expected number of claims increases every year:

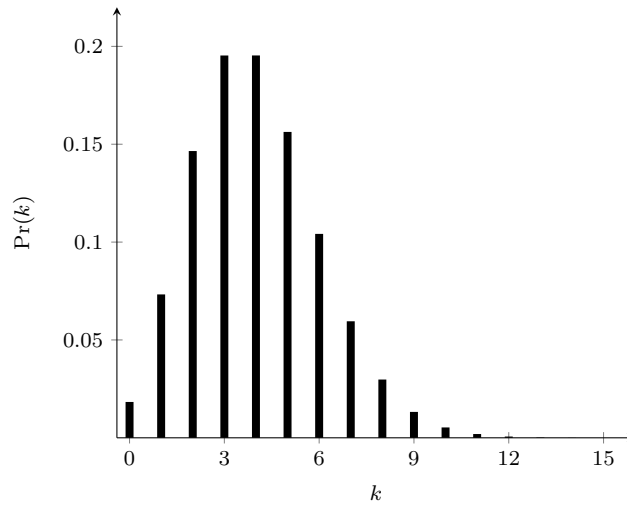
$$n_t = n_{t-1} \cdot (1 + g) = n_0 \cdot (1 + g)^t \quad \text{with } n_0 > 0 \quad (4.2)$$

The expected number of claims increases every year in the same way as the insurance portfolio. As a result, the claims frequency of the portfolio remains the same over time. This assumption could be refuted in some situation, such as in case of consistent modification of the portfolio. Moreover, we point out that not only does the initial expected number of claims depend on the dimension of the insurance portfolio, but also on the individual claims frequency of the people insured.

## Poisson distribution

The Poisson distribution with parameter  $n_t$  (see Figure 4.1) is defined by the following probability mass function:

$$\Pr(\tilde{K}_t = k) = \frac{n_t^k}{k!} \cdot e^{-n_t} \quad \text{with } k = 0, 1, \dots \text{ and } n_t > 0$$



**Figure 4.1:** Poisson distribution with parameter 4

The moment generating function of the Poisson distribution is given by:

$$M_{\tilde{K}_t}(s) = \exp(n_t \cdot (e^s - 1))$$

and the cumulant generating function is given by:

$$\Psi_{\tilde{K}_t}(s) = \ln[\exp(n_t \cdot (e^s - 1))] = n_t \cdot (e^s - 1) \quad (4.3)$$

As a result, the mean of the Poisson distribution is found to be:

$$E(\tilde{K}_t) = n_t$$

the variance is found to be:

$$\text{Var}(\tilde{K}_t) = n_t$$

the skewness is found to be:

$$\text{Sk}(\tilde{K}_t) = \frac{1}{\sqrt{n_t}}$$

and the coefficient of variation is found to be:

$$\text{CV}(\tilde{K}_t) = \frac{1}{\sqrt{n_t}}$$

We point out that, as  $n_t$  increases, the expected value and variance increase proportionally, then the standard deviation increases less than proportionally. Moreover, the skewness is always positive and approaches zero, because of the central limit theorem:

$$\lim_{n_t \rightarrow \infty} \text{Sk}(\tilde{K}_t) = 0$$

and the coefficient of variation approaches zero, because of the dimension diversification effect (also called pooling of the risk):

$$\lim_{n_t \rightarrow \infty} \text{CV}(\tilde{K}_t) = 0$$

### **Mixed Poisson distribution**

The Mixed Poisson distribution is found to be a Poisson with stochastic parameter  $\tilde{n}_t > 0$ , such that:

$$\tilde{n}_t = n_t \cdot \tilde{q}$$

where  $\tilde{q}$  is the stochastic structure variable. In order to describe the number of claims, considering the short-term fluctuations only, we assume that:

1. It is defined for  $q > 0$ .
2. It has mean equal to  $E(\tilde{q}) = 1$ .

The moment generating function of the Mixed Poisson distribution is given by:

$$M_{\tilde{K}_t}(s) = \mathbb{E}[M_{\tilde{K}_t}(s | \tilde{q} = q)] = \mathbb{E}[\exp(n_t \cdot (e^s - 1) \cdot \tilde{q})] = M_{\tilde{q}}[n_t \cdot (e^s - 1)]$$

and the cumulant generating function is given by:

$$\Psi_{\tilde{K}_t}(s) = \Psi_{\tilde{q}}[n_t \cdot (e^s - 1)] \quad (4.4)$$

Let  $\sigma_{\tilde{q}}$  and  $\gamma_{\tilde{q}}$  the standard deviation and skewness of the structure variable. Then, the mean of the Mixed Poisson distribution is found to be:

$$\mathbb{E}(\tilde{K}_t) = n_t$$

the variance is found to be:

$$\text{Var}(\tilde{K}_t) = n_t + n_t^2 \cdot \sigma_{\tilde{q}}^2$$

the skewness is found to be:

$$\text{Sk}(\tilde{K}_t) = \frac{n_t + 3 \cdot n_t^2 \cdot \sigma_{\tilde{q}}^2 + n_t^3 \cdot \gamma_{\tilde{q}} \cdot \sigma_{\tilde{q}}^3}{(n_t + n_t^2 \cdot \sigma_{\tilde{q}}^2)^{3/2}}$$

and the coefficient of variation is found to be:

$$\text{CV}(\tilde{K}_t) = \sqrt{\frac{1}{n_t} + \sigma_{\tilde{q}}^2}$$

We point out that the expected value is the same as that of the Poisson distribution and the variance is bigger, because depends on  $\sigma_{\tilde{q}}^2$ . Moreover, the skewness is positive or negative depending on  $\gamma_{\tilde{q}}$  and approaches  $\gamma_{\tilde{q}}$  as  $n_t$  increases:

$$\lim_{n_t \rightarrow \infty} \text{Sk}(\tilde{K}_t) = \gamma_{\tilde{q}}$$

and the coefficient of variation approaches  $\sigma_{\tilde{q}}$  (also called non-diversifiable systematic risk):

$$\lim_{n_t \rightarrow \infty} \text{CV}(\tilde{K}_t) = \sigma_{\tilde{q}}$$

The calibration of the non-diversifiable systematic risk is crucial for large insurance companies, because the diversifiable risk is close to be null.

### Negative Binomial distribution

The Negative Binomial distribution is found to be a Mixed Poisson with structure variable  $\tilde{q}$  distributed as a Gamma. In order to describe the number of claims, considering the short-term fluctuations only, we assume that the structure variable is distributed as a Gamma with equal parameters  $(h, h)$ , that is defined by the following probability density function:

$$f_{\tilde{q}}(q) = \frac{h^h \cdot q^{h-1}}{\Gamma(h)} \cdot e^{-h \cdot q} \quad \text{with } q > 0 \text{ and } h > 0$$

The mean of the Gamma distribution with equal parameters is given by:

$$E(\tilde{q}) = 1$$

the variance is given by:

$$\text{Var}(\tilde{q}) = \frac{1}{h}$$

the skewness is given by:

$$\text{Sk}(\tilde{q}) = \frac{2}{\sqrt{h}} = 2 \cdot \sigma_{\tilde{q}}$$

and the coefficient of variation is given by:

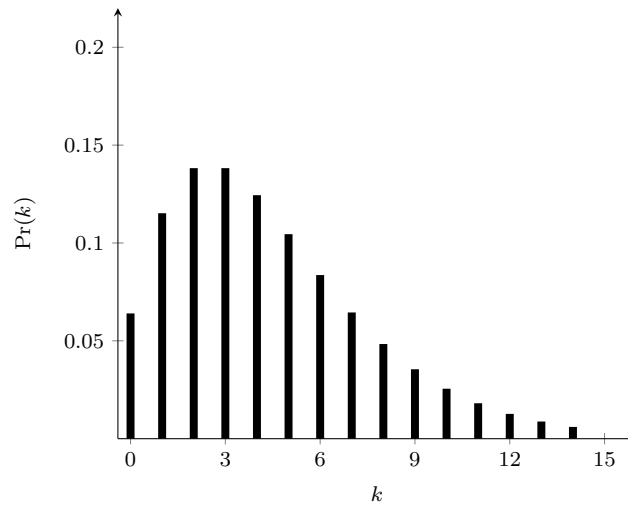
$$\text{CV}(\tilde{q}) = \frac{1}{\sqrt{h}}$$

We point out that, as  $h$  increases, the variance and coefficient of variation of the structure variable approach zero, and the skewness, that is always positive, approaches zero as well.

Hence, the Negative Binomial distribution is found to have parameters  $(h, p_t)$  with  $p_t = h/(h+n_t)$  (see Figure 4.2) and to be defined by the following probability mass function:

$$\Pr(\tilde{K}_t = k) = \binom{k+h-1}{h-1} \cdot p_t^h \cdot (1-p_t)^k$$

with  $k = 0, 1, \dots$  and  $h > 0$  and  $p_t \in (0, 1)$



**Figure 4.2:** Negative Binomial distribution with parameters 3 and 0.4

The mean, variance, skewness and coefficient of variation of the Negative Binomial distribution can be obtained from the Mixed Poisson distribution.

### 4.1.2 Single claim amount

We now define different distributions associated with the single claim amount, i.e. the Lognormal, Gamma, Weibull, Inverse Normal and Pareto. The first four distributions fit better the attritional claims, i.e. the most frequent and least expensive claims. The last distribution fits better the large claims, i.e. the least frequent and most expensive claims. A popular alternative is to use compound or mixed distributions. Moreover, since the insurance portfolio is dynamic, we assume that the single claim amount distribution (dropping the

index  $i$  because of the assumption of identical distribution) is scaled for the claims inflation every year:

$$\tilde{Z}_t = \tilde{Z}_{t-1} \cdot (1 + i) = \tilde{Z}_0 \cdot (1 + i)^t$$

Hence, the  $j$ -th raw moment is found to be:

$$\alpha_{j, \tilde{Z}_t} = \alpha_{j, \tilde{Z}_0} \cdot (1 + i)^{j \cdot t} \quad (4.5)$$

As a result, we start by describing the initial single claim distribution and, by changing the parameters, we scale it to obtain the subsequent ones. It follows that the relative indicators, such as the skewness and coefficient of variation, remain the same over time.

### Lognormal distribution

The logarithm of a Lognormal distribution is found to be a Normal. As a result, the Lognormal distribution with parameters  $(\mu, \sigma)$  (see Figure 4.3) is defined by the following probability density function:

$$f_{\tilde{Z}}(z) = \frac{1}{z \cdot \sqrt{2 \cdot \pi} \cdot \sigma} \cdot \exp \left[ -\frac{1}{2} \cdot \left( \frac{\ln(z) - \mu}{\sigma} \right)^2 \right]$$

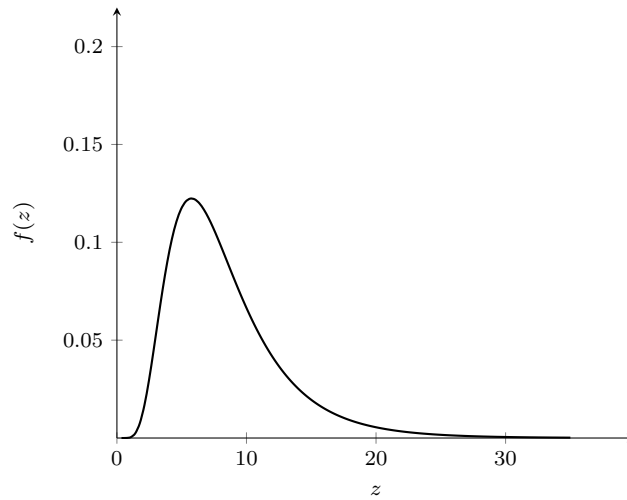
with  $z > 0$  and  $-\infty < \mu < +\infty$  and  $\sigma > 0$

The  $j$ -th raw moment of the Lognormal distribution is given by:

$$\alpha_{j, \tilde{Z}} = \exp \left( \frac{j^2 \cdot \sigma^2}{2} + \mu \cdot j \right)$$

As a result, the mean of the Lognormal distribution is found to be:

$$E(\tilde{Z}) = \exp \left( \frac{\sigma^2}{2} + \mu \right)$$



**Figure 4.3:** Lognormal distribution with parameters 2 and 0.5

the variance is found to be:

$$\text{Var}(\tilde{Z}) = \exp(\sigma^2 + 2 \cdot \mu) \cdot (\exp \sigma^2 - 1)$$

the skewness is found to be:

$$\text{Sk}(\tilde{Z}) = (\exp \sigma^2 + 2) \cdot \sqrt{\exp \sigma^2 - 1} = \text{CV}(\tilde{Z}) \cdot [3 + \text{CV}(\tilde{Z})^2]$$

and the coefficient of variation is found to be:

$$\text{CV}(\tilde{Z}) = \sqrt{\exp \sigma^2 - 1}$$

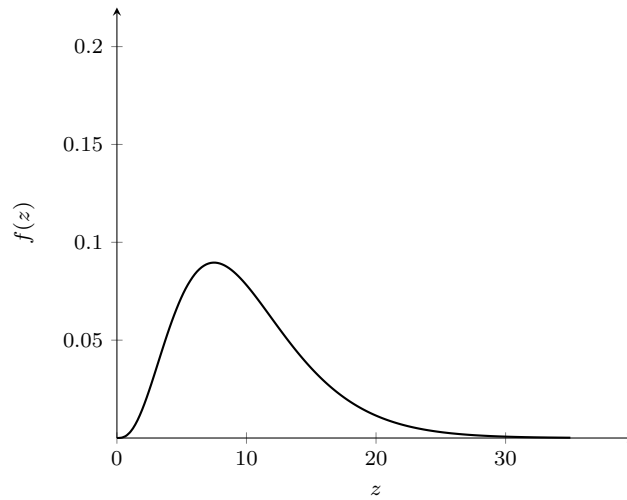
We point out that the skewness, that is always positive, increases as the coefficient of variation increases.

### Gamma distribution

The Gamma distribution with parameters  $(\alpha, \beta)$  (see Figure 4.4) is defined by the following probability density function:

$$f_{\tilde{Z}}(z) = \frac{\beta^\alpha \cdot z^{\alpha-1}}{\Gamma(\alpha)} \cdot e^{-\beta \cdot z} \quad \text{with } z > 0 \text{ and } \alpha > 0 \text{ and } \beta > 0$$





**Figure 4.4:** Gamma distribution with parameters 4 and 0.4

The mean of the Gamma distribution is given by:

$$E(\tilde{Z}) = \frac{\alpha}{\beta}$$

the variance is given by:

$$\text{Var}(\tilde{Z}) = \frac{\alpha}{\beta^2}$$

the skewness is given by:

$$\text{Sk}(\tilde{Z}) = \frac{2}{\sqrt{\alpha}}$$

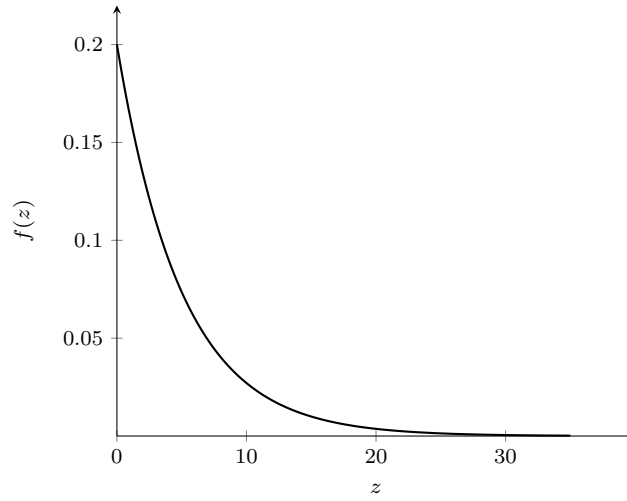
and the coefficient of variation is given by:

$$\text{CV}(\tilde{Z}) = \frac{1}{\sqrt{\alpha}}$$

### Weibull distribution

The Weibull distribution with parameters  $(\lambda, k)$  (see Figure 4.5) is defined by the following probability density function:

$$f_{\tilde{Z}}(z) = \frac{k \cdot z^{k-1}}{\lambda^k} \cdot \exp\left(-\frac{z}{\lambda}\right)^k \quad \text{with } z > 0 \text{ and } \lambda > 0 \text{ and } k > 0$$



**Figure 4.5:** Weibull distribution with parameters 5 and 1

The mean of the Weibull distribution is given by:

$$E(\tilde{Z}) = \lambda \cdot \Gamma\left(1 + \frac{1}{k}\right)$$

the variance is given by:

$$\text{Var}(\tilde{Z}) = \lambda^2 \cdot \left[ \Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right) \right]$$

the skewness is given by:

$$\text{Sk}(\tilde{Z}) = \frac{\Gamma\left(1 + \frac{3}{k}\right) - 3 \cdot \Gamma\left(1 + \frac{2}{k}\right) \cdot \Gamma\left(1 + \frac{1}{k}\right) + 2 \cdot \Gamma^3\left(1 + \frac{1}{k}\right)}{\left[ \Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right) \right]^{3/2}}$$

and the coefficient of variation is given by:

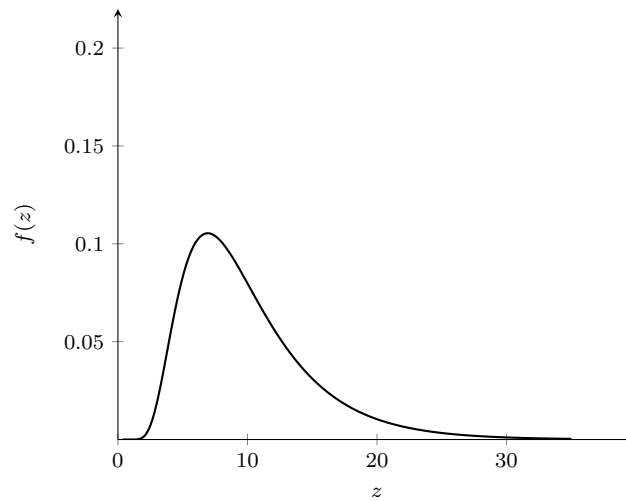
$$\text{CV}(\tilde{Z}) = \frac{\sqrt{\Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right)}}{\Gamma\left(1 + \frac{1}{k}\right)}$$

### Inverse Normal distribution

The Inverse Normal distribution with parameters  $(\mu, \lambda)$  (see Figure 4.6) is defined by the following probability density function:

$$f_{\tilde{Z}}(z) = \sqrt{\frac{\lambda}{2 \cdot \pi \cdot z^3}} \cdot \exp\left[-\frac{\lambda \cdot (z - \mu)^2}{2 \cdot \mu^2 \cdot z}\right]$$

with  $z > 0$  and  $\mu > 0$  and  $\lambda > 0$



**Figure 4.6:** Inverse Normal distribution with parameters 10 and 40

The mean of the Inverse Normal distribution is given by:

$$E(\tilde{Z}) = \mu$$

the variance is given by:

$$\text{Var}(\tilde{Z}) = \frac{\mu^3}{\lambda}$$

the skewness is given by:

$$\text{Sk}(\tilde{Z}) = 3 \cdot \sqrt{\frac{\mu}{\lambda}}$$

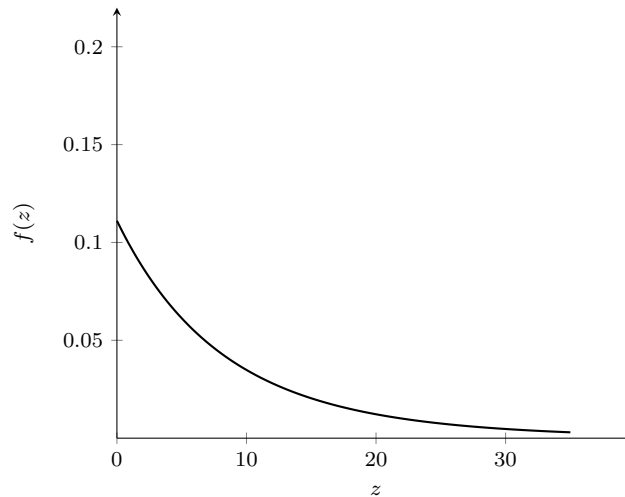
and the coefficient of variation is given by:

$$\text{CV}(\tilde{Z}) = \sqrt{\frac{\mu}{\lambda}}$$

### Pareto distribution

The Pareto (type II) distribution with parameters  $(\alpha, \lambda)$  (see Figure 4.7) is defined by the following probability density function:

$$f_{\tilde{Z}}(z) = \frac{\alpha}{\lambda} \cdot \left(1 + \frac{z}{\lambda}\right)^{-(\alpha+1)} \quad \text{with } z > 0 \text{ and } \alpha > 0 \text{ and } \lambda > 0$$



**Figure 4.7:** Pareto distribution with parameters 10 and 90

The mean of the Pareto distribution is given by:

$$\text{E}(\tilde{Z}) = \frac{\lambda}{\alpha - 1} \quad \text{with } \alpha > 1$$

the variance is given by:

$$\text{Var}(\tilde{Z}) = \frac{\lambda^2 \cdot \alpha}{(\alpha - 1)^2 \cdot (\alpha - 2)} \quad \text{with } \alpha > 2$$

the skewness is given by:

$$\text{Sk}(\tilde{Z}) = \frac{2 \cdot (1 + \alpha)}{\alpha - 3} \cdot \sqrt{\frac{\alpha - 2}{\alpha}} \quad \text{with } \alpha > 3$$

and the coefficient of variation is given by:

$$\text{CV}(\tilde{Z}) = \sqrt{\frac{\alpha}{\alpha - 2}} \quad \text{with } \alpha > 2$$

### 4.1.3 Aggregate claim amount

Using the definitions and assumptions above, we are able to determine the aggregate claim amount distribution. The cumulant generating function of the aggregate claim amount distribution is found to be:<sup>2</sup>

$$\Psi_{\tilde{X}_t}(s) = \Psi_{\tilde{K}_t}[\Psi_{\tilde{Z}_t}(s)]$$

#### Simple compound Poisson process

The simple compound Poisson process is a compound process that produces the aggregate claim amount distribution, assuming that the distribution of the number of claims is a simple Poisson. Hence, using equation (4.3), the cumulant generating function of the aggregate claim amount distribution is found to be:

$$\Psi_{\tilde{X}_t}(s) = \Psi_{\tilde{K}_t}[\Psi_{\tilde{Z}_t}(s)] = n_t \cdot (e^{\Psi_{\tilde{Z}_t}(s)} - 1) = n_t \cdot (M_{\tilde{Z}_t}(s) - 1)$$

Let  $m_t$  and  $c_{\tilde{Z}}$  be the mean at time  $t$  and time-independent coefficient of variation of the single claim amount distribution. Then, using equations (4.2) and (4.5), the mean of the aggregate claim amount distribution is found to be:

$$\text{E}(\tilde{X}_t) = n_t \cdot m_t = \text{E}(\tilde{X}_0) \cdot (1 + g)^t \cdot (1 + i)^t$$

---

<sup>2</sup>See Savelli and Clemente [30].

the variance is found to be:

$$\text{Var}(\tilde{X}_t) = n_t \cdot \alpha_{2, \tilde{Z}_t} = \text{Var}(\tilde{X}_0) \cdot (1 + g)^t \cdot (1 + i)^{2 \cdot t}$$

the skewness is found to be:

$$\text{Sk}(\tilde{X}_t) = \frac{n_t \cdot \alpha_{3, \tilde{Z}_t}}{(n_t \cdot \alpha_{2, \tilde{Z}_t})^{3/2}} = \text{Sk}(\tilde{X}_0) \cdot \frac{1}{\sqrt{(1 + g)^t}}$$

and the coefficient of variation is found to be:

$$\text{CV}(\tilde{X}_t) = \sqrt{\frac{1}{n_t} \cdot r_{2, \tilde{Z}_t}} = \text{CV}(\tilde{X}_0) \cdot \frac{1}{\sqrt{(1 + g)^t}}$$

where:

$$r_{2, \tilde{Z}_t} = \frac{\alpha_{2, \tilde{Z}_t}}{m_t^2} = 1 + c_{\tilde{Z}}^2$$

We point out that, when  $g$  is positive and increases, the expected value and variance increase as a function of the power  $t$ , then the standard deviation increases as a function of the power  $t/2$ . When  $i$  is positive and increases, the expected value increases as a function of the power  $t$  and the variance increases as a function of the power  $2 \cdot t$ , then the standard deviation increases as a function of the power  $t$ . Moreover, we point out that the skewness is always positive and, when  $g$  is positive, it approaches zero over time, because of the central limit theorem:

$$\lim_{t \rightarrow \infty} \text{Sk}(\tilde{X}_t) = 0 \quad \text{with } g > 0$$

and the coefficient of variation approaches zero over time, because of the dimension diversification effect:

$$\lim_{t \rightarrow \infty} \text{CV}(\tilde{X}_t) = 0 \quad \text{with } g > 0$$

Moreover, because of the central limit theorem, the skewness approaches zero as  $n_0$  or  $g$  increase, i.e. the dimension of the insurance company increases:

$$\lim_{n_0 \rightarrow \infty} \text{Sk}(\tilde{X}_t) = \lim_{g \rightarrow \infty} \text{Sk}(\tilde{X}_t) = 0$$

and, because of the dimension diversification effect, the coefficient of variation approaches zero:

$$\lim_{n_0 \rightarrow \infty} \text{CV}(\tilde{X}_t) = \lim_{g \rightarrow \infty} \text{CV}(\tilde{X}_t) = 0$$

In conclusion, a variation of  $i$  does not affect the skewness and coefficient of variation, because they are relative indicators.

### Mixed compound Poisson process

The mixed compound Poisson process is a compound process that produces the aggregate claim amount distribution, assuming that the distribution of the number of claims is a Mixed Poisson. Hence, using equation (4.4), the cumulant generating function of the aggregate claim amount distribution is found to be:

$$\Psi_{\tilde{X}_t}(s) = \Psi_{\tilde{K}_t}[\Psi_{\tilde{Z}_t}(s)] = \Psi_{\tilde{q}}[n_t \cdot (e^{\Psi_{\tilde{Z}_t}(s)} - 1)] = \Psi_{\tilde{q}}[n_t \cdot (M_{\tilde{Z}_t}(s) - 1)]$$

Hence, the mean of the aggregate claim amount distribution is found to be:

$$\text{E}(\tilde{X}_t) = n_t \cdot m_t = \text{E}(\tilde{X}_0) \cdot (1 + g)^t \cdot (1 + i)^t$$

the variance is found to be:

$$\text{Var}(\tilde{X}_t) = n_t \cdot \alpha_{2, \tilde{Z}_t} + n_t^2 \cdot m_t^2 \cdot \sigma_{\tilde{q}}^2 \quad (4.6)$$

the skewness is found to be:

$$\text{Sk}(\tilde{X}_t) = \frac{n_t \cdot \alpha_{3, \tilde{Z}_t} + 3 \cdot n_t^2 \cdot m_t \cdot \alpha_{2, \tilde{Z}_t} \cdot \sigma_{\tilde{q}}^2 + n_t^3 \cdot m_t^3 \cdot \gamma_{\tilde{q}} \cdot \sigma_{\tilde{q}}^3}{(n_t \cdot \alpha_{2, \tilde{Z}_t} + n_t^2 \cdot m_t^2 \cdot \sigma_{\tilde{q}}^2)^{3/2}}$$

and the coefficient of variation is found to be:

$$\text{CV}(\tilde{X}_t) = \sqrt{\frac{1}{n_t} \cdot r_{2, \tilde{Z}_t} + \sigma_{\tilde{q}}^2}$$

We point out that the expected value is the same as that of the simple compound Poisson process and the variance is bigger, because it depends on  $\sigma_{\tilde{q}}^2$ . Moreover, we point out that the skewness is positive or negative depending on  $\gamma_{\tilde{q}}$ . In conclusion, because of equation (4.2), when  $g$  is positive the skewness approaches  $\gamma_{\tilde{q}}$  over time:

$$\lim_{t \rightarrow \infty} \text{Sk}(\tilde{X}_t) = \gamma_{\tilde{q}} \quad \text{with } g > 0$$

and the coefficient of variation approaches  $\sigma_{\tilde{q}}$  over time:

$$\lim_{t \rightarrow \infty} \text{CV}(\tilde{X}_t) = \sigma_{\tilde{q}} \quad \text{with } g > 0$$

Moreover, the skewness approaches  $\gamma_{\tilde{q}}$  as  $n_0$  or  $g$  increase, i.e. the dimension of the insurance company increases:

$$\lim_{n_0 \rightarrow \infty} \text{Sk}(\tilde{X}_t) = \lim_{g \rightarrow \infty} \text{Sk}(\tilde{X}_t) = \gamma_{\tilde{q}}$$

and the coefficient of variation approaches  $\sigma_{\tilde{q}}$ :

$$\lim_{n_0 \rightarrow \infty} \text{CV}(\tilde{X}_t) = \lim_{g \rightarrow \infty} \text{CV}(\tilde{X}_t) = \sigma_{\tilde{q}}$$

As in the simple compound Poisson process, a variation of  $i$  does not affect the skewness and coefficient of variation.



# Chapter 5

## Capital requirements for a single-line insurance company

In this chapter we calculate the capital requirements for market and non-life premium risk of a single-line insurance company.<sup>1</sup> We lay down the following assumptions:

- the insurance company is single-line, since it only works in the Motor Third-Party Liability (MTPL) line of business;
- the non-life premium risk, interest rate risk and equity risk are the only sources of risk;
- the insurance contracts are not multi-annual and the geographical diversification is absent;
- the security trading is continuous, all securities are perfectly divisible, no short-sale restrictions, transaction costs, or taxes are present and there are no riskless arbitrage opportunities;
- the market zero curve is used instead of the EIOPA basic risk-free interest rates and no volatility or matching adjustment is considered;
- the bond investments are government bonds without credit risk;

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<sup>1</sup>The case study in this chapter is inspired by Ballotta and Savelli [3].

- the stock investments are non-dividend-paying stocks without strategic nature, because of the absence of a clear decisive strategy to continue holding them for long period;
- the stock investments can be considered as type 1 equities;
- the symmetric adjustment is equal to zero;
- the interest rate risk only affects the bond investments and the equity risk only affects the stock investments.

Furthermore, we assume the relevant parameters shown in Table 5.1. The initial risk reserve ratio is roughly 1.5 times the Required Solvency Margin for non-life insurance, required by Solvency 0 and Solvency I.

**Table 5.1:** Relevant parameters

$u_0$	$\pi_0$	$F_0$	$i$	$g$
25%	100,000,000	0	3%	2%

The other relevant parameters, for reasons of simplicity, are estimated by the Italian market data, provided by the National Association of Insurance Companies (ANIA) and Institute for the Supervision of Insurance (IVASS).<sup>2</sup> Table 5.2 shows the combined ratio net of run-off of the MTPL line of business, Table 5.3 shows the expense ratio, and Table 5.4 shows the ratio of claims reserve and written premium amount.

**Table 5.2:** Combined ratio net of run-off of the MTPL line of business

2014	2015	2016	2017	2018
92.8%	97.9%	102.0%	102.5%	101.3%

The safety loading coefficient, as a multiplier of the gross premium amount, is estimated by the complement of 100% of the average of the last five observations of the combined ratio net of run-off:

$$\varphi_\pi = 0.70\%$$

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<sup>2</sup>See ANIA [1], and IVASS [20].

**Table 5.3:** Expense ratio of the MTPL line of business

2014	2015	2016	2017	2018
21.0%	21.5%	21.4%	21.2%	21.1%

**Table 5.4:** Ratio of claims reserve and written premium amount of the MTPL line of business

2014	2015	2016	2017	2018
153.5%	158.1%	161.1%	157.5%	150.3%

The expense loading coefficient is estimated by the average of the last five observations of the expense ratio:

$$c = 21.24\%$$

The gross premium amount can be written as follows:

$$\pi_t = P_t + \varphi_\pi \cdot \pi_t + c \cdot \pi_t$$

then:

$$\frac{\pi_t}{P_t} = \frac{1}{1 - \varphi_\pi - c} = 128.11\%$$

Hence, the safety loading coefficient, as a multiplier of the risk premium amount, is found to be:

$$\varphi = \varphi_\pi \cdot \frac{\pi_t}{P_t} = 0.90\%$$

The ratio of claims reserve and gross premium amount is estimated by the average of the last five observations of the ratio of claims reserve and written premium amount:

$$\delta = 156.10\%$$

As a result, the initial claims reserve is found to be:

$$L_0 = 156,100,000$$

and the initial asset value of the portfolio is found to be:

$$A_0 = U_0 + L_0 = 181,100,000$$

We now assume to allocate 15% to the stock portfolio and 85% to the bond portfolio. The composition of the stock portfolio is determined through an optimization procedure. Table 5.5 shows the composition of the bond portfolio.

**Table 5.5:** Composition of the bond portfolio

$\gamma_1$	$\gamma_2$	$\gamma_3$	$\gamma_5$	$\gamma_{10}$
40%	25%	15%	10%	10%

Therefore, the initial value of the stock portfolio is found to be:

$$A_0^S = 15\% \cdot A_0 = 27,165,000$$

and the initial value of the bond portfolio is found to be:

$$A_0^B = 85\% \cdot A_0 = 153,935,000$$

In conclusion, Table 5.6 shows the initial value of each bond investment.

**Table 5.6:** Initial values of the bond investments

$A_0^{B_1}$	$A_0^{B_2}$	$A_0^{B_3}$	$A_0^{B_5}$	$A_0^{B_{10}}$
61,574,000	38,483,750	23,090,250	15,393,500	15,393,500

## 5.1 Annual rate of return

In this section we describe the distributions of the three stocks and five zero-coupon bonds over time. Therefore, once we have fixed the percentage invested in each category, we obtain the distribution of the annual rate of return over time.

In order to provide correlated prices of stocks and zero-coupon bonds, we make use of the Monte Carlo simulation to generate the P-Brownian motions of each stock price and of the short rate (since zero-coupon bond prices depend on it), and we take advantage of the Cholesky decomposition to inject correlation. Table 5.7 shows the correlation matrix and Figure 5.1 shows the simulated standard Brownian motions on weekly basis over a period of three years. We point out that the standard Brownian motions of the three stock prices are positively, but not highly correlated between themselves, while the standard Brownian motion of the short rate is negatively correlated with the others. Indeed, when the short rate decreases (increases), stock prices increase (decrease), because investments are cheaper.

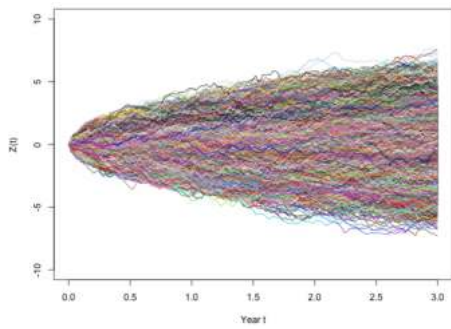
**Table 5.7:** Matrix of correlations between the P-Brownian motion of the short rate  $Z(t)$  and the P-Brownian motions of the stock prices  $W_h(t)$

	$Z(t)$	$W_1(t)$	$W_2(t)$	$W_3(t)$
$Z(t)$	1	-0.2	-0.2	-0.2
$W_1(t)$	-0.2	1	0.25	0.25
$W_2(t)$	-0.2	0.25	1	0.25
$W_3(t)$	-0.2	0.25	0.25	1

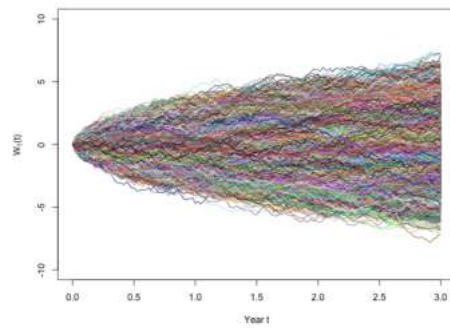
### 5.1.1 Stock price distributions

We can now describe the distributions of the three stocks over time, using the model presented in section 3.4 and making use of the Monte Carlo simulation. We base our analysis on the P-Brownian motions  $W_1(t)$ ,  $W_2(t)$  and  $W_3(t)$  above. Table 5.8 shows the real-world drift and diffusion coefficients. We point out that the stock 1 has a small expected return and a small risk, the stock 2 has a medium expected return and a medium risk, and the stock 3 has a big expected return and a big risk. Therefore, each stock investment is efficient from a risk-return perspective and the choice to invest in the stock 1, 2 or 3 depends on the risk-return preferences of the insurance company.

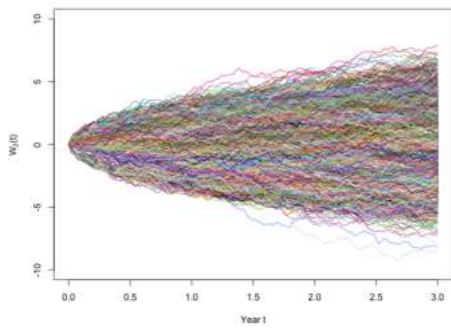
We are able to simulate the stock price over time through the explicit solution of the stochastic differential equation, shown in equation (3.20).



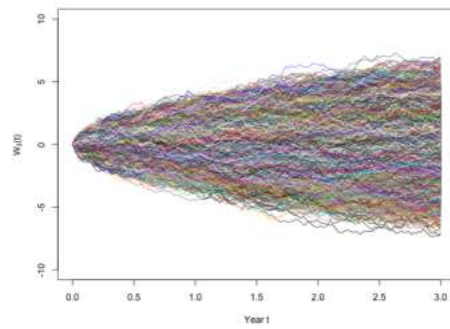
**(a)** Short rate



**(b)** Stock 1



**(c)** Stock 2



**(d)** Stock 3

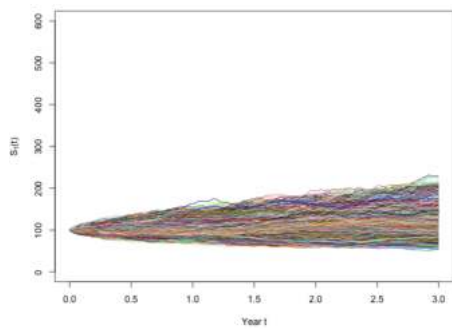
**Figure 5.1:** Samples of 100,000 possible trajectories of the correlated P-Brownian motions on weekly basis over a period of three years

**Table 5.8:** Real-world drift and diffusion coefficients of the stock price models

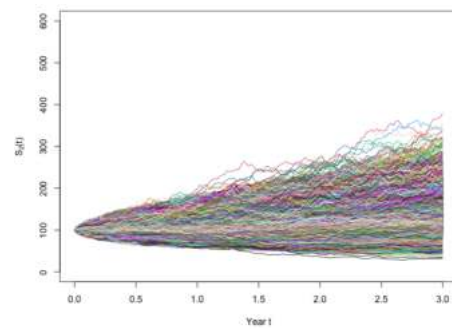
$S_h(t)$	$\mu_h$	$\sigma_h$
$h = 1$	4%	10%
$h = 2$	6%	15%
$h = 3$	8%	20%

However, in order to be consistent with the Cox-Ingersoll-Ross model, used in next few pages, we prefer to use the Euler method, shown in equation (3.19). Figure 5.2 shows the simulated stock prices on weekly basis over a period of three years, assuming initial prices equal to 100. Figure 5.3 shows the resulting distributions of the stock prices after one, two and three years, and Table 5.9 shows some elements of descriptive statistics. We point out that, because of our assumptions, the mean and standard deviation increase as the riskiness of the stock raises. The bigger the expected value of the stock price, the bigger its volatility. Similarly, the mean and standard deviation increase over time, because of a higher level of uncertainty. The skewness is always positive, i.e. the tail on the right side of the distribution is heavier than the tail on the left side, resulting in a higher likelihood of great profits rather than great losses. The distribution becomes more skewed over time and also as the riskiness of the stock increases. On the contrary, the first quartile, i.e. the minimum value that cannot be exceeded in a quarter of all cases, decreases as the riskiness of the stock increases. In conclusion, we can observe that stock price models described by geometric Brownian motions produce stocks with Lognormal distributions.

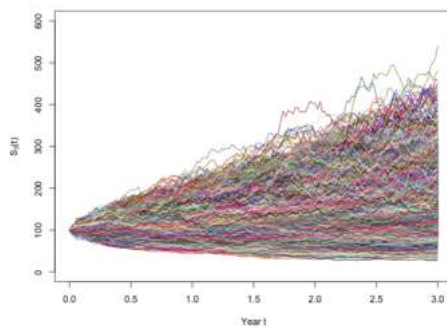
Figure 5.4 shows the resulting distributions of the annual stock capitalization factors for the first, second and third year, shown in equation (1.8), and Table 5.10 shows some elements of descriptive statistics. We point out that we can make the same comments as in the case of the simulated distributions of the stock prices. However, in this case the distributions are quite stable over time. Indeed, when a variable is described by a geometric Brownian motion, its distribution changes over time, but the distribution of the capitalization factor remains constant for a fixed time increment, such as one year.



(a) Stock 1



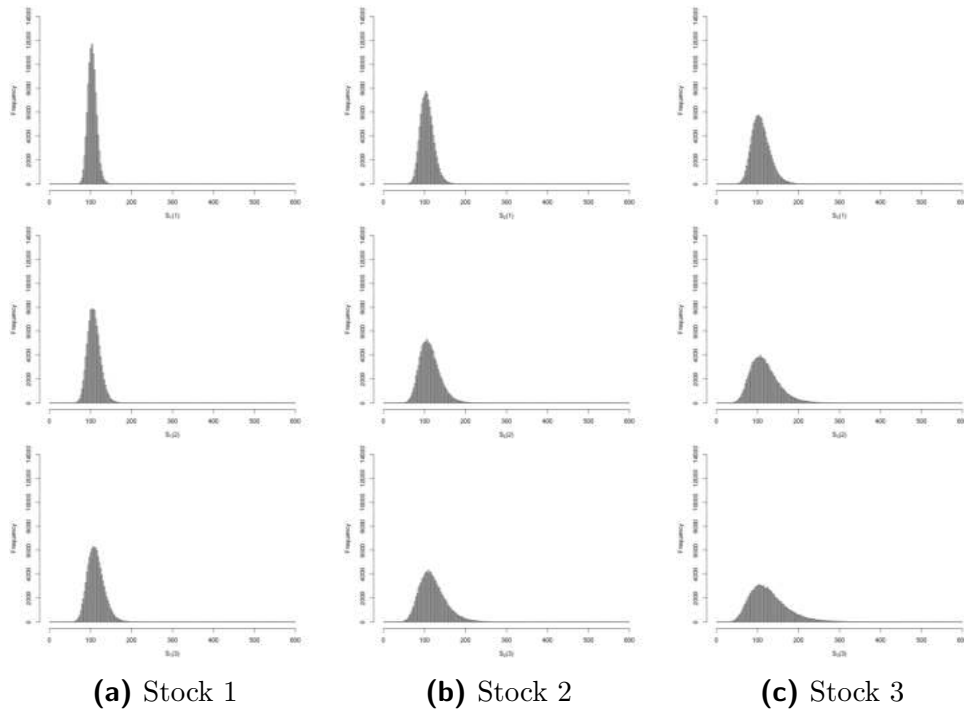
(b) Stock 2



(c) Stock 3

**Figure 5.2:** Samples of 100,000 possible trajectories of the stock prices on weekly basis over a period of three years





**Figure 5.3:** Simulated distributions of the stock prices after one, two and three years

**Table 5.9:** Descriptive statistics of the simulated stock prices after one, two and three years

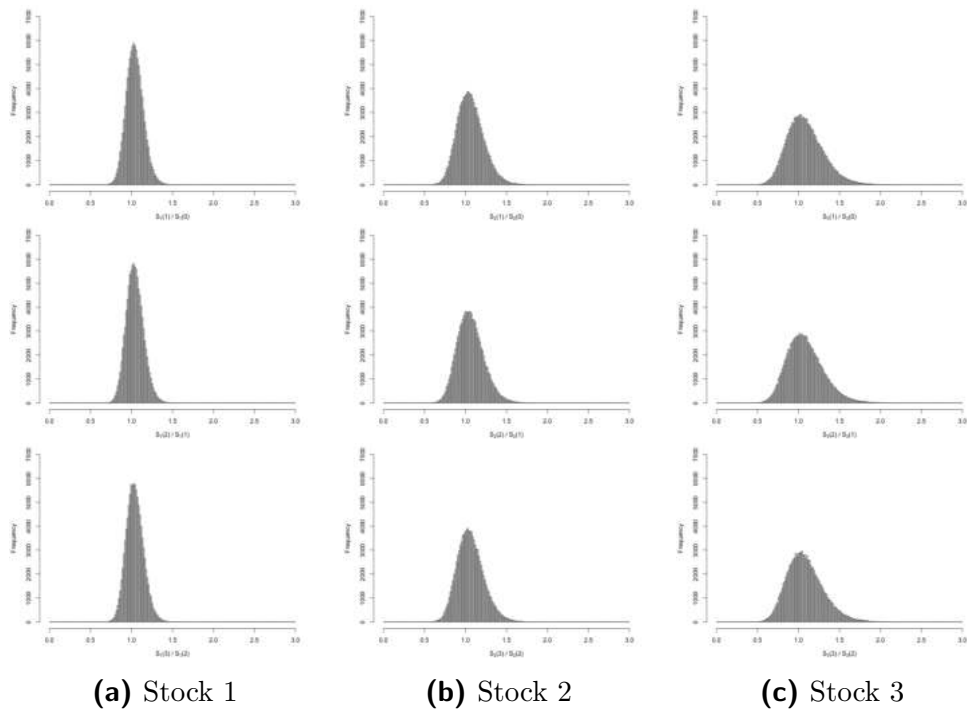
$S_h(1)$	Mean	St.Dev.	Skew.	Min.	1 <sup>st</sup> Qu.	Median	3 <sup>rd</sup> Qu.	Max.
$h = 1$	104.04	10.38	0.2753	64.50	96.84	103.58	110.71	163.03
$h = 2$	106.19	15.96	0.4489	56.14	94.86	105.02	116.15	198.26
$h = 3$	108.38	21.83	0.5890	43.37	92.78	106.25	121.64	242.21

$S_h(2)$	Mean	St.Dev.	Skew.	Min.	1 <sup>st</sup> Qu.	Median	3 <sup>rd</sup> Qu.	Max.
$h = 1$	108.31	15.38	0.4210	57.71	97.50	107.26	117.96	194.14
$h = 2$	112.70	24.17	0.6574	36.17	95.40	110.15	127.12	274.81
$h = 3$	117.33	33.86	0.8664	29.46	93.02	112.89	136.67	403.24

$S_h(3)$	Mean	St.Dev.	Skew.	Min.	1 <sup>st</sup> Qu.	Median	3 <sup>rd</sup> Qu.	Max.
$h = 1$	112.67	19.63	0.5165	53.55	98.73	111.09	124.75	230.11
$h = 2$	119.57	31.56	0.8209	32.51	97.18	115.53	137.54	379.65
$h = 3$	126.96	45.17	1.0845	28.17	94.70	119.90	151.14	541.61



**Figure 5.4:** Simulated distributions of the annual stock capitalization factors for the first, second and third year

**Table 5.10:** Descriptive statistics of the simulated annual stock capitalization factors for the first, second and third year

$\frac{S_h(1)}{S_h(0)}$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$h = 1$	1.0404	0.1038	0.2753	0.6450	0.9684	1.0358	1.1071	1.6303
$h = 2$	1.0619	0.1596	0.4489	0.5614	0.9486	1.0502	1.1615	1.9826
$h = 3$	1.0838	0.2183	0.5890	0.4337	0.9278	1.0625	1.2164	2.4221

$\frac{S_h(2)}{S_h(1)}$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$h = 1$	1.0411	0.1047	0.2943	0.6454	0.9685	1.0359	1.1085	1.6289
$h = 2$	1.0612	0.1598	0.4434	0.5448	0.9479	1.0501	1.1609	2.1111
$h = 3$	1.0824	0.2184	0.5944	0.4302	0.9265	1.0616	1.2153	2.2938

$\frac{S_h(3)}{S_h(2)}$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$h = 1$	1.0403	0.1045	0.3045	0.6688	0.9677	1.0352	1.1075	1.6323
$h = 2$	1.0610	0.1599	0.4563	0.5147	0.9484	1.0492	1.1603	1.9338
$h = 3$	1.0822	0.2183	0.5992	0.4113	0.9268	1.0611	1.2139	2.6456

## 5.1.2 Zero-coupon bond price distributions

We can now describe the distributions of the five zero-coupon bonds over time, using the one-factor short rate models presented in section 3.5 and making use of the Monte Carlo simulation. We base our analysis on the P-Brownian motion  $Z(t)$  above.

### Vasicek model

We now estimate the real-world parameters of the Vasicek model, using linear regression.<sup>3</sup> Daily data on 1-month Treasury rates in the United States between July 31, 2001, and September 11, 2019, is used as a proxy of short rates. Let  $r(t)$  be the 1-month rate at time  $t$  and  $\Delta t$  be the daily time interval. Then, regressing  $\Delta r(t)$  on  $r(t)$ , we obtain:

$$\Delta r(t) = 0.0000114 - 0.00117 \cdot r(t) + 0.000640 \cdot \epsilon$$

$$\text{with } \epsilon \sim \mathcal{N}(0, 1)$$

There are about 250 observations per year, so that  $\Delta t = 1/250$ . Therefore, we can write:

$$\Delta r(t) = 0.292 \cdot (0.0097 - r(t)) \cdot \Delta t + 0.0101 \cdot \epsilon \cdot \sqrt{\Delta t}$$

Since we use historical data, we do not deal with a risk-neutral world, but with the real world. Using equations (3.1) and (3.31), the discrete-time version of the real-world Vasicek model is found to be:

$$\Delta r(t) = 0.292 \cdot (0.0097 - r(t)) \cdot \Delta t + 0.0101 \cdot \Delta Z(t)$$

As a result, the real-world parameters are found to be:

$$\kappa = 0.292$$

---

<sup>3</sup>See Hull [16].

and:

$$\theta = 0.0097 = 0.97\%$$

and:

$$v = 0.0101 = 1.01\%$$

The market price of risk is obtained by minimizing the sum of squared errors between the initial zero-coupon interest rates given by the Vasicek model and those in the United States market on September 11, 2019, (see Table 5.11) namely:

$$\lambda = \arg \min_{\lambda} \sum_{\{t=0.5,1,2,3,5,7,10,20,30\}} [R_V(0,t) - R_M(0,t)]^2$$

where, using equations (3.23) and (3.36), we have:

$$R_V(0,t) = \frac{-\ln a(0,t) + b(0,t) \cdot r(0)}{t}$$

where  $R_V(0,t)$  and  $R_M(0,t)$  are the spot interest rate with maturity  $t$  according to the Vasicek model and according to the market. As shown in equation (3.37), we remember that  $a(0,t)$  is a function of  $\lambda$ .

**Table 5.11:** Best interest rates given by the Vasicek model and market interest rates (continuously compounded)

Maturity (years)	Model rate	Market rate
0.5	2.00%	1.88%
1	1.98%	1.79%
2	1.96%	1.68%
3	1.94%	1.62%
5	1.90%	1.60%
7	1.88%	1.68%
10	1.85%	1.75%
20	1.81%	2.02%
30	1.79%	2.22%

As a result, the market price of risk is found to be:

$$\lambda = -0.246 \quad (5.1)$$

Alternatively, daily data on 3-month Treasury rates can be used as a proxy of short rates. In this case we obtain similar real-world parameters, except for a lower speed of mean-reversion and for a lower market price of risk.

Using equation (3.38), the risk-neutral parameters are found to be:

$$\kappa = 0.292 \quad (5.2)$$

and:

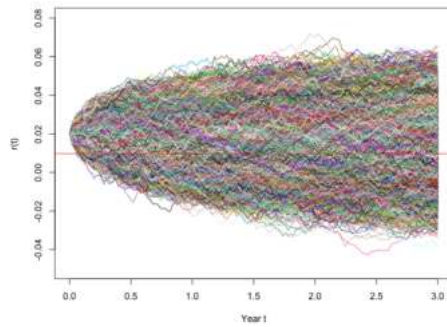
$$\theta^* = 0.97\% - \frac{-0.246 \cdot 1.01\%}{0.292} = 1.82\% \quad (5.3)$$

and:

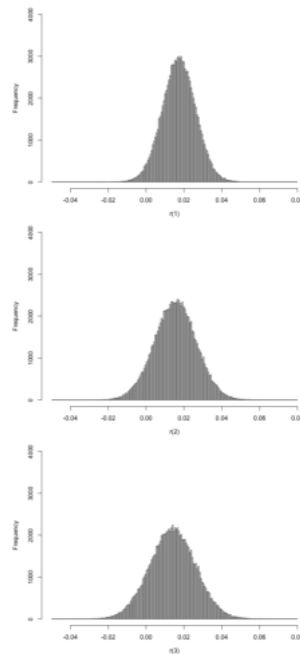
$$v = 1.01\% \quad (5.4)$$

We are able to simulate the short rate over time through the explicit solution of the Vasicek stochastic differential equation, shown in equation (3.32). However, in order to be consistent with the Cox-Ingersoll-Ross model, used in next few pages, we prefer to use the Euler method, shown in equation (3.31). Figure 5.5 shows the simulated short rate on weekly basis over a period of three years and the long-run mean interest rate level, assuming the initial rate equal to 2.01% (1-month Treasury rate in the United States on September 11, 2019). Figure 5.6 shows the resulting distribution of the short rate after one, two and three years, and Table 5.12 shows some elements of descriptive statistics. We point out that, because of our assumptions, the mean decreases over time, since the long-run mean interest rate level is smaller than the initial short rate. The standard deviation increases over time, because of a higher level of uncertainty. Moreover, the skewness is always close to zero. In conclusion, we can observe that the Vasicek model produces a short rate with Normal distribution.

Using equation (3.36), we can obtain the simulated zero-coupon bond prices over time, through the simulated short rate. Figure 5.7 shows the simulated zero-coupon bond prices on weekly basis over a period of three



**Figure 5.5:** Samples of 100,000 possible trajectories of the Vasicek short rate on weekly basis over a period of three years and, superimposed in red, the long-run mean interest rate level



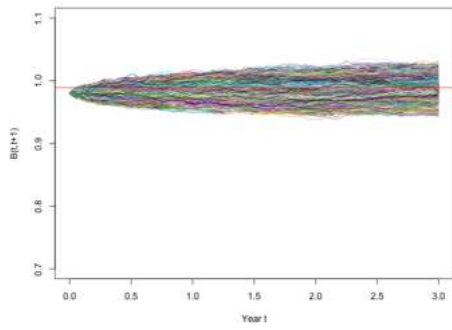
**Figure 5.6:** Simulated distribution of the Vasicek short rate after one, two and three years

**Table 5.12:** Descriptive statistics of the simulated Vasicek short rate after one, two and three years

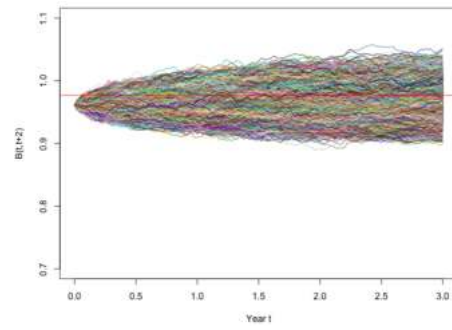
$r(t)$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	1.74%	0.88%	-0.0003	-2.28%	1.15%	1.74%	2.33%	5.74%
$t = 2$	1.54%	1.10%	-0.0008	-2.94%	0.80%	1.55%	2.28%	6.98%
$t = 3$	1.40%	1.20%	0.0067	-3.89%	0.59%	1.40%	2.21%	6.51%

years and the long-run mean bond price levels, assuming terminal payoffs equal to 1. Figure 5.8 shows the resulting distributions of the zero-coupon bond prices after one, two and three years, and Table 5.13 shows some elements of descriptive statistics. We point out that, as we could expect, the mean decreases as the time to maturity increases, because the discounting effect is bigger. The standard deviation increases as the time to maturity raises, since a higher time to maturity implies a bigger level of uncertainty, and the skewness, that is always positive, increases as well. Moreover, according to the Vasicek model, the short rate can be negative, and thus zero-coupon bond prices can be higher than one. However, the bonds with the highest time to maturity have maxima lower than one. This is because the short rate is not expected to remain negative so long that also the long interest rates become negative. Indeed, the long-run mean interest rate level is positive and the negative short rate is not expected to become too large in absolute value. This situation is more likely to be refuted over time, because the standard deviation increases, implying a bigger level of uncertainty, that can produce situations rather remote from expectations. In conclusion, the mean increases over time, because the short rate decreases over time. This is because the long-run mean interest rate level is smaller than the initial short rate. The standard deviation and skewness increase over time.

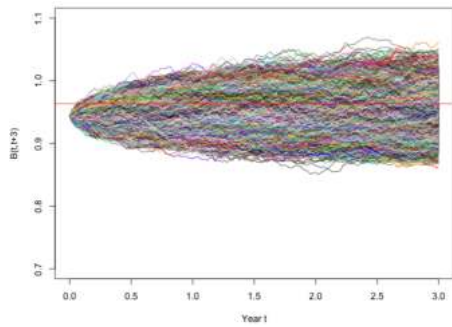
Figure 5.9 shows the resulting distributions of the annual zero-coupon bond capitalization factors for the first, second and third year, shown in equation (1.10), and Table 5.14 shows some elements of descriptive statistics. We point out that, for the first year, the 1 year zero-coupon bond capitalization factor is not stochastic. Indeed, the initial zero-coupon bond prices are deterministic, because the initial short rate is given and, after one year, the 1 year zero-coupon bond price is equal to one for sure. Moreover, the mean and standard deviation increase as the time to maturity raises. The bigger the expected value of the return on the bond investment, the bigger its volatility. The skewness, that is always positive, increases as the time to maturity raises. In conclusion, the mean decreases over time, because the short rate decreases as well. The standard deviation increases over time, because of a higher level of uncertainty, and the skewness increases as well.



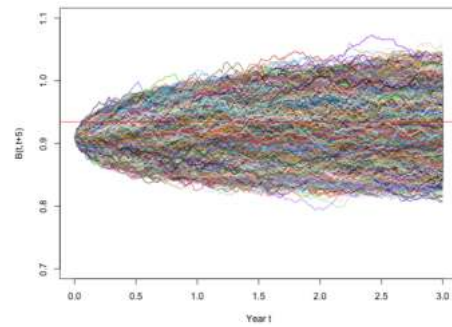
**(a)** 1 year bond



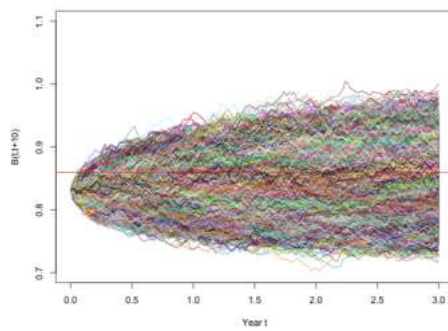
**(b)** 2 years bond



**(c)** 3 years bond



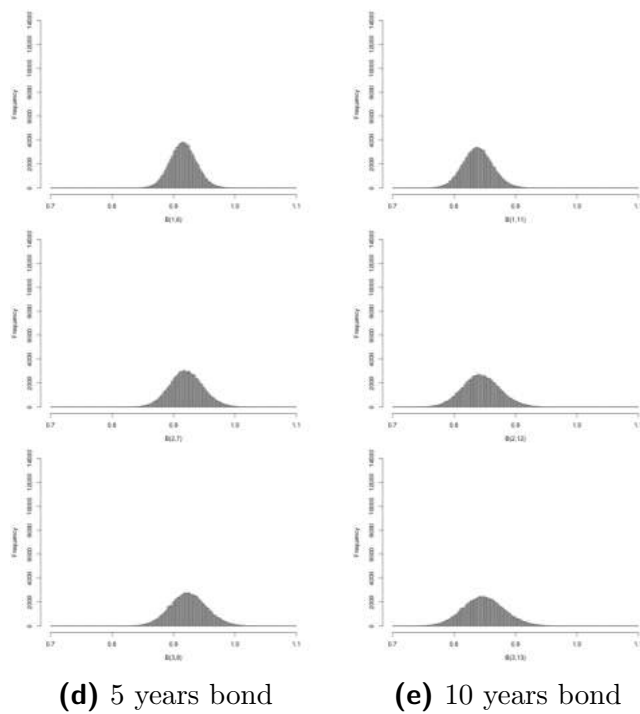
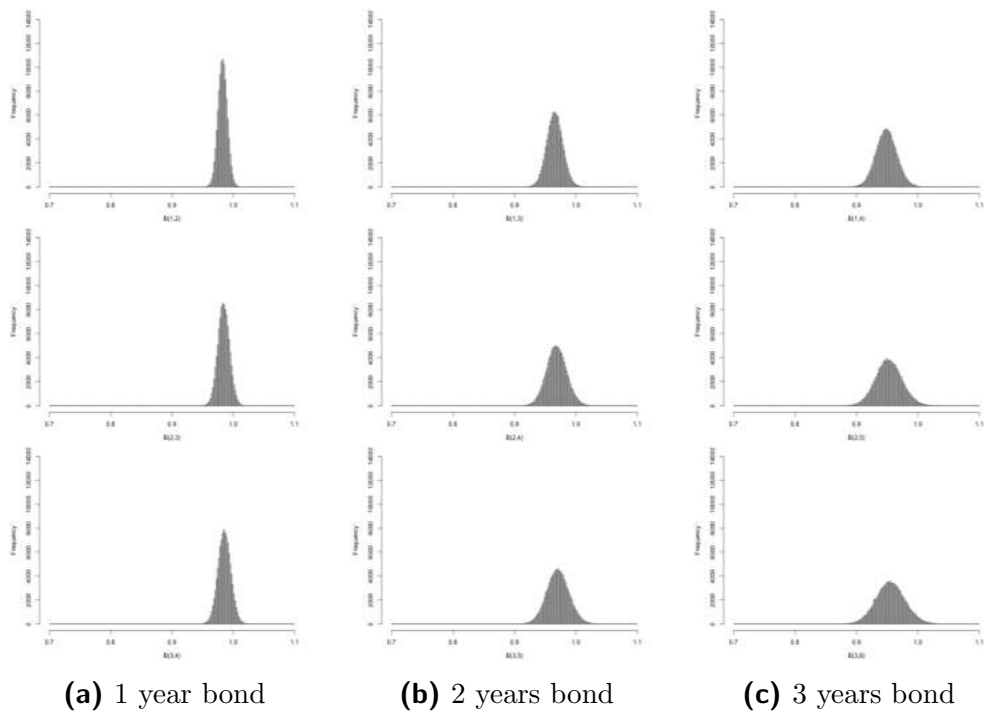
**(d)** 5 years bond



**(e)** 10 years bond

**Figure 5.7:** Samples of 100,000 possible trajectories of the zero-coupon bond prices on weekly basis given by the Vasicek model over a period of three years and, superimposed in red, the long-run mean bond price levels





**Figure 5.8:** Simulated distributions of the zero-coupon bond prices given by the Vasicek model after one, two and three years

**Table 5.13:** Descriptive statistics of the simulated zero-coupon bond prices given by the Vasicek model after one, two and three years

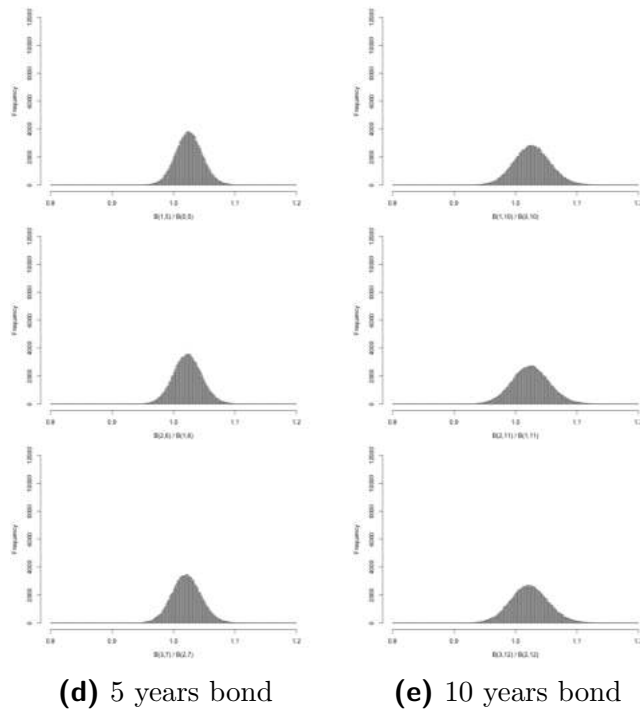
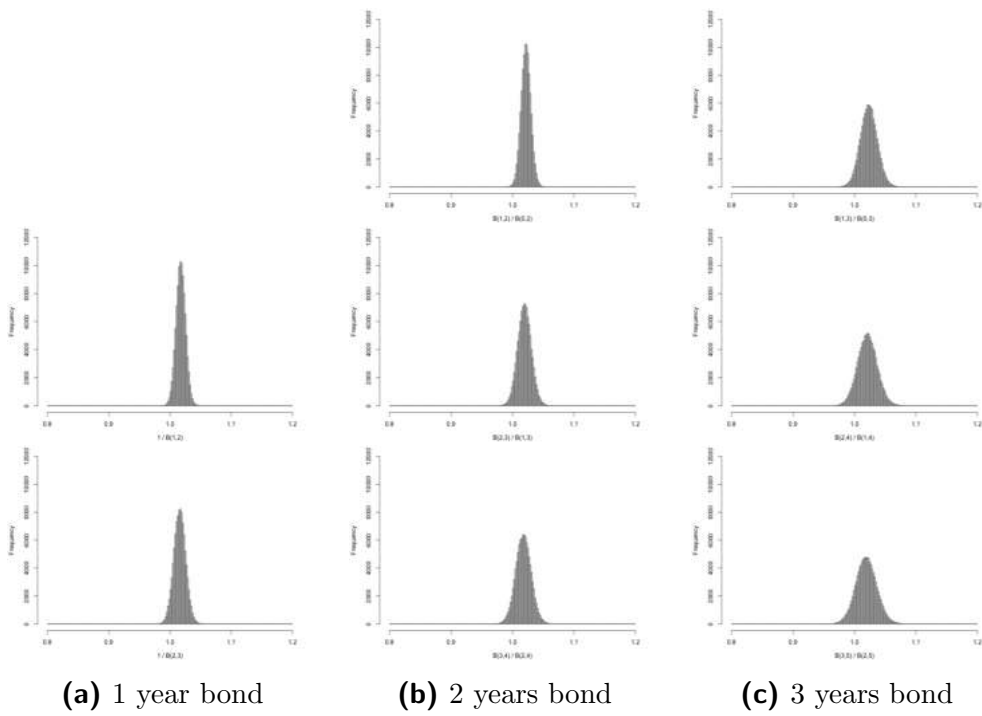
$B(1, 1+i)$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$i = 1$	0.9827	0.0075	0.0235	0.9492	0.9776	0.9827	0.9877	1.0175
$i = 2$	0.9656	0.0129	0.0408	0.9087	0.9569	0.9655	0.9742	1.0262
$i = 3$	0.9487	0.0167	0.0537	0.8757	0.9375	0.9486	0.9598	1.0280
$i = 5$	0.9159	0.0212	0.0706	0.8243	0.9015	0.9157	0.9300	1.0178
$i = 10$	0.8387	0.0239	0.0869	0.7364	0.8224	0.8383	0.8545	0.9550

$B(2, 2+i)$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$i = 1$	0.9844	0.0094	0.0293	0.9390	0.9781	0.9843	0.9907	1.0233
$i = 2$	0.9685	0.0161	0.0506	0.8918	0.9576	0.9683	0.9793	1.0364
$i = 3$	0.9526	0.0209	0.0665	0.8542	0.9384	0.9522	0.9665	1.0415
$i = 5$	0.9208	0.0265	0.0872	0.7977	0.9028	0.9203	0.9385	1.0354
$i = 10$	0.8442	0.0300	0.1072	0.7074	0.8238	0.8435	0.8641	0.9754

$B(3, 3+i)$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$i = 1$	0.9856	0.0103	0.0245	0.9428	0.9786	0.9856	0.9925	1.0318
$i = 2$	0.9706	0.0177	0.0477	0.8982	0.9586	0.9705	0.9824	1.0515
$i = 3$	0.9553	0.0230	0.0651	0.8623	0.9397	0.9551	0.9706	1.0616
$i = 5$	0.9243	0.0292	0.0877	0.8077	0.9044	0.9239	0.9437	1.0617
$i = 10$	0.8482	0.0331	0.1097	0.7183	0.8256	0.8476	0.8701	1.0061



**Figure 5.9:** Simulated distributions of the annual zero-coupon bond capitalization factors given by the Vasicek model for the first, second and third year

**Table 5.14:** Descriptive statistics of the simulated annual zero-coupon bond capitalization factors given by the Vasicek model for the first, second and third year

$\frac{B(1,0+i)}{B(0,0+i)}$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$i = 1$	1.0200	-	-	-	-	-	-	-
$i = 2$	1.0220	0.0078	0.0235	0.9871	1.0167	1.0219	1.0272	1.0582
$i = 3$	1.0234	0.0136	0.0408	0.9632	1.0142	1.0233	1.0325	1.0876
$i = 5$	1.0253	0.0213	0.0634	0.9328	1.0109	1.0251	1.0394	1.1272
$i = 10$	1.0271	0.0287	0.0853	0.9042	1.0076	1.0267	1.0461	1.1667

$\frac{B(2,1+i)}{B(1,1+i)}$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$i = 1$	1.0177	0.0078	0.0229	0.9828	1.0125	1.0176	1.0229	1.0536
$i = 2$	1.0196	0.0110	0.0289	0.9743	1.0121	1.0195	1.0269	1.0692
$i = 3$	1.0210	0.0157	0.0439	0.9552	1.0103	1.0209	1.0315	1.0921
$i = 5$	1.0228	0.0226	0.0657	0.9265	1.0074	1.0227	1.0379	1.1227
$i = 10$	1.0246	0.0297	0.0873	0.8995	1.0044	1.0243	1.0443	1.1565

$\frac{B(3,2+i)}{B(2,2+i)}$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$i = 1$	1.0160	0.0097	0.0277	0.9772	1.0094	1.0160	1.0224	1.0650
$i = 2$	1.0178	0.0124	0.0361	0.9668	1.0094	1.0177	1.0261	1.0725
$i = 3$	1.0192	0.0167	0.0536	0.9443	1.0079	1.0190	1.0303	1.0914
$i = 5$	1.0210	0.0233	0.0760	0.9154	1.0052	1.0207	1.0365	1.1209
$i = 10$	1.0227	0.0302	0.0971	0.8882	1.0022	1.0221	1.0427	1.1567

### Cox-Ingersoll-Ross model

We now estimate the real-world parameters of the Cox-Ingersoll-Ross model, by starting from the parameters of the Vasicek model. We assume that the risk-neutral speed of mean-reversion and long-run mean interest rate level of the Cox-Ingersoll-Ross model are the same as those used in the Vasicek model. Using equations (5.2) and (5.3), they are found to be:

$$\kappa^* = 0.292$$

and:

$$\theta^* = 1.82\%$$

Furthermore, we assume that the initial risk-neutral interest rate exposure of the Cox-Ingersoll-Ross model is the same as that of the Vasicek model. Using equation (5.4), we obtain:

$$v \cdot \sqrt{r(0)} = 1.01\%$$

then:

$$v = \frac{1.01\%}{\sqrt{2.01\%}} = 7.12\%$$

Using equation (3.45), the real-world interest rate exposure is found to be the same as the risk-neutral interest rate exposure.

In order to compute the other real-world parameters, we assume that the initial market price of risk of the Cox-Ingersoll-Ross model is the same as that of the Vasicek model. Using equation (5.1), we obtain:

$$\frac{\eta \cdot \sqrt{r(0)}}{v} = -0.246$$

then:

$$\eta = \frac{-0.246 \cdot 7.12\%}{\sqrt{2.01\%}} = -0.124$$

As a result, using equation (3.45), the real-world speed of mean-reversion

and long-run mean interest rate level are found to be:

$$\kappa = 0.292 + 0.124 = 0.416$$

and:

$$\theta = \frac{1.82\% \cdot 0.292}{0.416} = 1.28\%$$

Table 5.15 shows the initial zero-coupon interest rates given by the Cox-Ingersoll-Ross model. We point out that the initial zero curve given by the Cox-Ingersoll-Ross model lies above the initial zero curve given by the Vasicek model, therefore the initial zero-coupon bond prices given by the Cox-Ingersoll-Ross model are smaller.

**Table 5.15:** Interest rates given by the Cox-Ingersoll-Ross model (continuously compounded)

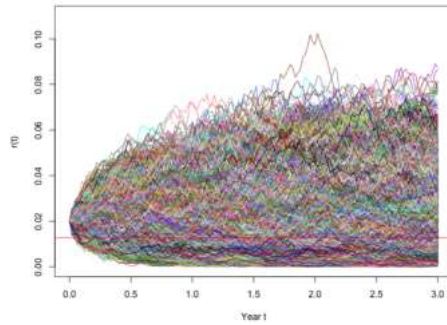
Maturity (years)	Model rate
0.5	2.00%
1	1.98%
2	1.96%
3	1.94%
5	1.90%
7	1.88%
10	1.85%
20	1.81%
30	1.80%

Using equation (3.40), we are able to simulate the short rate over time. Figure 5.10 shows the simulated short rate on weekly basis over a period of three years and the long-run mean interest rate level, assuming the initial rate equal to 2.01% as seen in the Vasicek model. Figure 5.11 shows the resulting distribution of the short rate after one, two and three years, and Table 5.16 shows some elements of descriptive statistics. We point out that we can make the same comments about the mean and standard deviation as in the case of the Vasicek model but, on the contrary, the skewness is always positive and increases over time. Moreover, the mean is bigger than

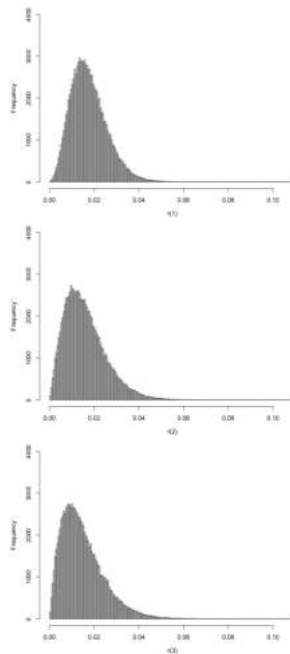
in the Vasicek model and the standard deviation is smaller. In conclusion, we can observe that the Cox-Ingersoll-Ross model produces a short rate with Lognormal distribution.

Using equation (3.44), we can obtain the simulated zero-coupon bond prices over time, through the simulated short rate. Figure 5.12 shows the simulated zero-coupon bond prices on weekly basis over a period of three years and the long-run mean bond price levels, assuming terminal payoffs equal to 1. Figure 5.13 shows the resulting distributions of the zero-coupon bond prices after one, two and three years, and Table 5.17 shows some elements of descriptive statistics. We point out that we can make the same comments about the mean and standard deviation as in the case of the Vasicek model but, on the contrary, the skewness, that is always negative, decreases in absolute value as the time to maturity increases and it increases in absolute value over time. Moreover, according to the Cox-Ingersoll-Ross model, the short rate cannot be negative, and thus zero-coupon bond prices are always lower than one. In conclusion, the mean is smaller than in the Vasicek model, and the standard deviation is smaller as well, therefore the zero curve lies above the zero curve given by the Vasicek model.

Figure 5.14 shows the resulting distributions of the annual zero-coupon bond capitalization factors for the first, second and third year, and Table 5.18 shows some elements of descriptive statistics. We point out that we can make the same comments as in the case of the Vasicek model, except for the skewness, that decreases in absolute value as the time to maturity increases and it increases in absolute value over time. Moreover, the skewness is negative for the first year, because the initial zero-coupon bond prices are deterministic, but it becomes positive over time. The mean is smaller than in the Vasicek model and the standard deviation is smaller as well, except for low times of maturity in the long period. In conclusion, the minimum is lower than in the Vasicek model, except for low times of maturity in the medium-long period, and it implies that the downside risk is higher.



**Figure 5.10:** Samples of 100,000 possible trajectories of the Cox-Ingersoll-Ross short rate on weekly basis over a period of three years and, superimposed in red, the long-run mean interest rate level

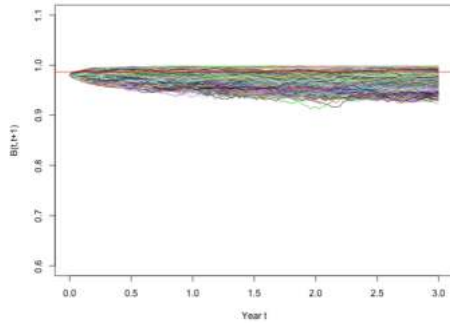


**Figure 5.11:** Simulated distribution of the Cox-Ingersoll-Ross short rate after one, two and three years

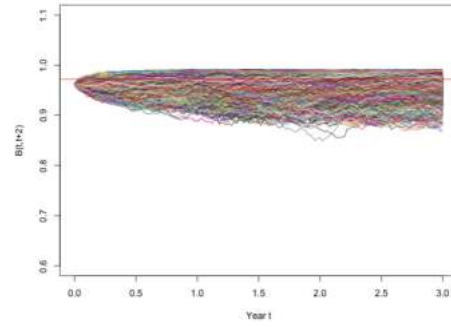
**Table 5.16:** Descriptive statistics of the simulated Cox-Ingersoll-Ross short rate after one, two and three years

$r(t)$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	1.76%	0.80%	0.7412	0.01%	1.17%	1.66%	2.23%	7.23%
$t = 2$	1.59%	0.92%	1.0137	0%	0.91%	1.44%	2.10%	9.74%
$t = 3$	1.49%	0.95%	1.1812	0%	0.78%	1.30%	1.99%	8.73%

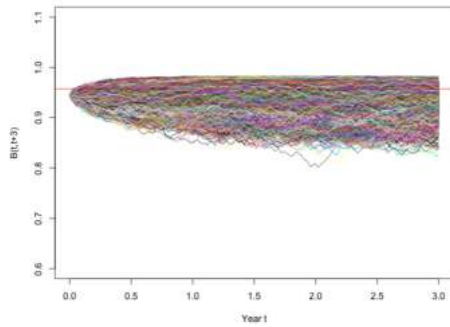




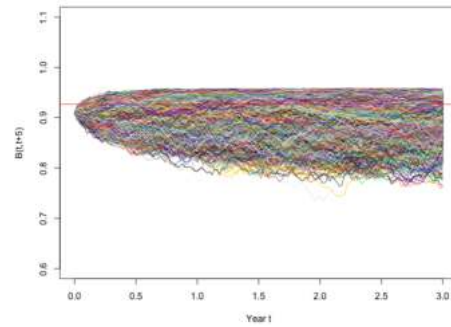
(a) 1 year bond



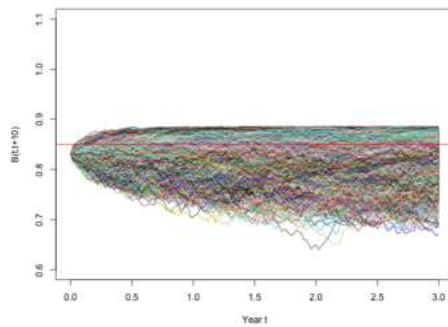
(b) 2 years bond



(c) 3 years bond

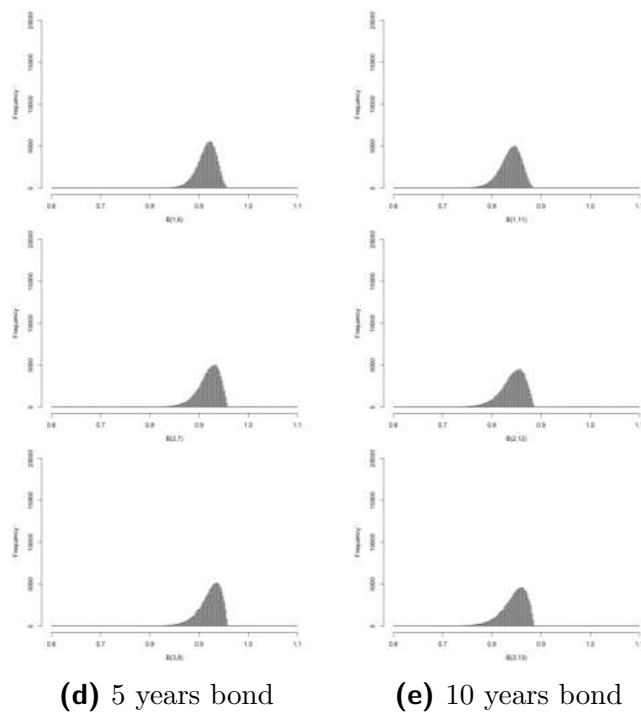
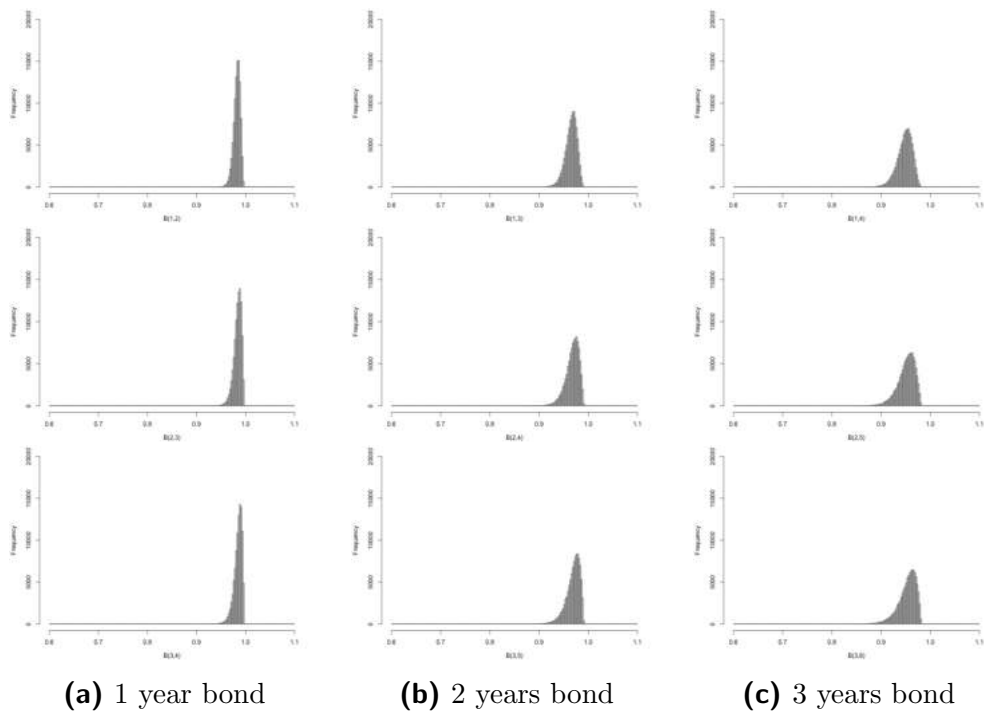


(d) 5 years bond



(e) 10 years bond

**Figure 5.12:** Samples of 100,000 possible trajectories of the zero-coupon bond prices on weekly basis given by the Cox-Ingersoll-Ross model over a period of three years and, superimposed in red, the long-run mean bond price levels



**Figure 5.13:** Simulated distributions of the zero-coupon bond prices given by the Cox-Ingersoll-Ross model after one, two and three years

**Table 5.17:** Descriptive statistics of the simulated zero-coupon bond prices given by the Cox-Ingersoll-Ross model after one, two and three years

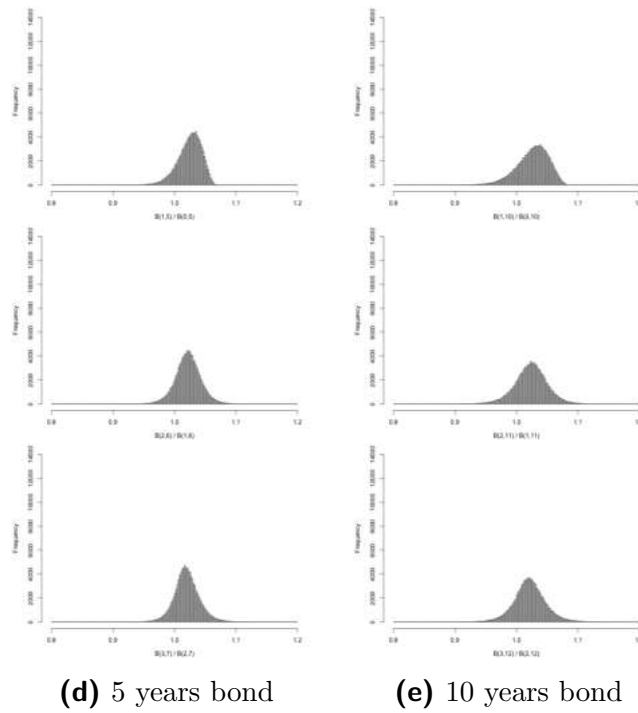
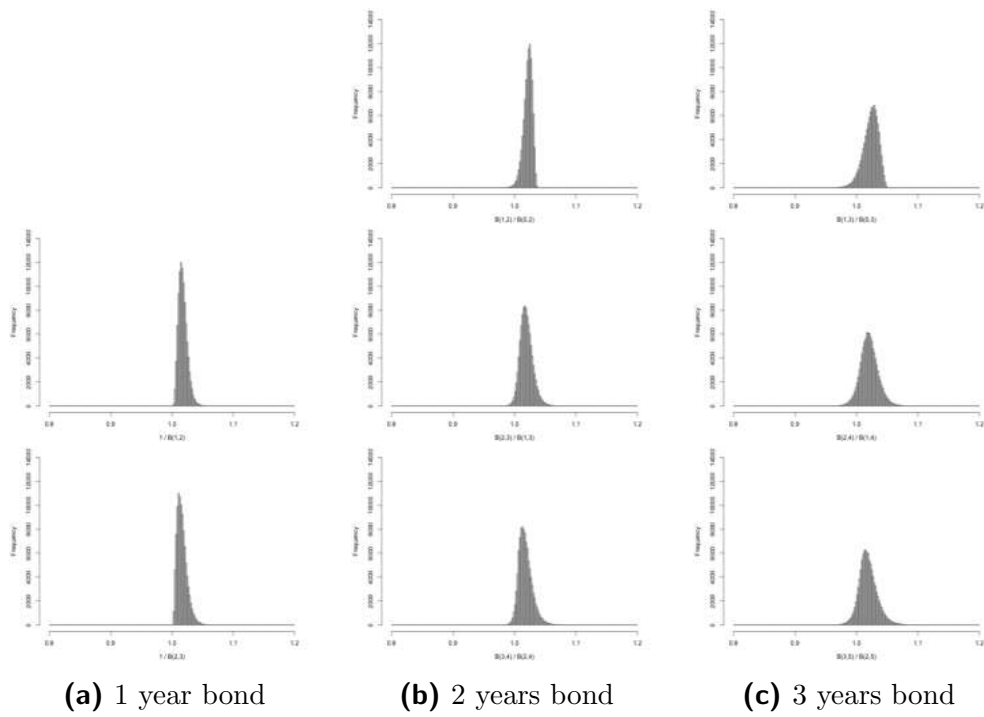
$B(1, 1+i)$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$i = 1$	0.9826	0.0068	-0.7181	0.9370	0.9785	0.9834	0.9875	0.9975
$i = 2$	0.9653	0.0116	-0.7011	0.8886	0.9584	0.9667	0.9738	0.9911
$i = 3$	0.9484	0.0150	-0.6886	0.8504	0.9394	0.9502	0.9593	0.9818
$i = 5$	0.9154	0.0189	-0.6727	0.7936	0.9039	0.9176	0.9292	0.9578
$i = 10$	0.8378	0.0211	-0.6580	0.7040	0.8250	0.8402	0.8532	0.8853

$B(2, 2+i)$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$i = 1$	0.9840	0.0078	-0.9842	0.9168	0.9796	0.9852	0.9898	0.9976
$i = 2$	0.9678	0.0134	-0.9627	0.8555	0.9602	0.9699	0.9777	0.9912
$i = 3$	0.9516	0.0173	-0.9470	0.8090	0.9418	0.9543	0.9644	0.9820
$i = 5$	0.9194	0.0218	-0.9270	0.7434	0.9070	0.9227	0.9356	0.9580
$i = 10$	0.8423	0.0243	-0.9088	0.6501	0.8284	0.8460	0.8604	0.8855

$B(3, 3+i)$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$i = 1$	0.9848	0.0081	-1.1491	0.9249	0.9805	0.9864	0.9908	0.9976
$i = 2$	0.9693	0.0138	-1.1256	0.8687	0.9618	0.9719	0.9796	0.9912
$i = 3$	0.9535	0.0178	-1.1086	0.8254	0.9438	0.9569	0.9668	0.9820
$i = 5$	0.9219	0.0225	-1.0869	0.7632	0.9095	0.9261	0.9386	0.9580
$i = 10$	0.8451	0.0250	-1.0671	0.6712	0.8313	0.8497	0.8638	0.8855



**Figure 5.14:** Simulated distributions of the annual zero-coupon bond capitalization factors given by the Cox-Ingersoll-Ross model for the first, second and third year

**Table 5.18:** Descriptive statistics of the simulated annual zero-coupon bond capitalization factors given by the Cox-Ingersoll-Ross model for the first, second and third year

$\frac{B(1,0+i)}{B(0,0+i)}$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$i = 1$	1.0200	-	-	-	-	-	-	-
$i = 2$	1.0218	0.0071	-0.7181	0.9744	1.0176	1.0227	1.0270	1.0374
$i = 3$	1.0232	0.0123	-0.7011	0.9418	1.0157	1.0246	1.0321	1.0505
$i = 5$	1.0249	0.0191	-0.6794	0.9013	1.0133	1.0271	1.0387	1.0675
$i = 10$	1.0265	0.0254	-0.6594	0.8651	1.0110	1.0293	1.0449	1.0836

$\frac{B(2,1+i)}{B(1,1+i)}$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$i = 1$	1.0178	0.0071	0.7647	1.0025	1.0127	1.0169	1.0220	1.0673
$i = 2$	1.0194	0.0102	0.5085	0.9711	1.0124	1.0185	1.0255	1.0933
$i = 3$	1.0205	0.0143	0.2047	0.9288	1.0113	1.0200	1.0293	1.1130
$i = 5$	1.0220	0.0201	-0.0277	0.8771	1.0095	1.0220	1.0346	1.1387
$i = 10$	1.0234	0.0258	-0.1423	0.8313	1.0076	1.0240	1.0397	1.1631

$\frac{B(3,2+i)}{B(2,2+i)}$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$i = 1$	1.0164	0.0081	1.0438	1.0024	1.0103	1.0150	1.0209	1.0907
$i = 2$	1.0178	0.0109	0.8314	0.9714	1.0101	1.0163	1.0239	1.0968
$i = 3$	1.0188	0.0146	0.5254	0.9405	1.0093	1.0175	1.0272	1.1215
$i = 5$	1.0202	0.0201	0.2474	0.9020	1.0078	1.0193	1.0320	1.1541
$i = 10$	1.0214	0.0254	0.0936	0.8669	1.0060	1.0210	1.0366	1.1851

## 5.2 Aggregate claim amount

In this section we describe the distribution of the aggregate claim amount over time, using the collective risk model presented in section 4.1 and making use of the Monte Carlo simulation. We assume that the number of claims has a Negative Binomial distribution and the single claim amount has a Lognormal distribution. Once again, for reasons of simplicity, the parameters are estimated by the Italian market data, provided by ANIA and IVASS.<sup>4</sup> Table 5.19 shows the loss ratio on accrual basis of the MTPL line of business.

**Table 5.19:** Loss ratio on accrual basis of the MTPL line of business

2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
87.7%	83.5%	76.8%	68.4%	68.5%	71.8%	76.3%	80.5%	81.3%	80.2%

Through empirical evidence, the initial mean of the single claim amount distribution is found to be:

$$m_0 = 4,000$$

We want the initial expected number of claims to fulfill the assumption about the initial gross premium amount, namely:

$$\pi_0 = n_0 \cdot m_0 \cdot \frac{1 + \varphi}{1 - c}$$

then:

$$n_0 = \frac{\pi_0}{m_0} \cdot \frac{1 - c}{1 + \varphi} = 19,514.37$$

Through empirical evidence, the initial coefficient of variation of the single claim amount distribution is found to be:

$$c_{\bar{Z}} = 7$$

The standard deviation of the structure variable is estimated by the product of the standard deviation of the loss ratio on accrual basis and the ratio of

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<sup>4</sup>See ANIA [1], and IVASS [20].

gross and risk premium amount:

$$\sigma_{\tilde{q}} = \sigma \left( \frac{\tilde{X}_t}{\pi_t} \right) \cdot \frac{\pi_t}{P_t} = 0.0820$$

This result holds because the Italian market data can be seen as the portfolio of a huge insurance company, and thus the standard deviation of the pure loss ratio (i.e. the coefficient of variation of the aggregate claim amount) approaches the standard deviation of the structure variable. Consequently, the skewness of the structure variable is found to be:

$$\gamma_{\tilde{q}} = 2 \cdot \sigma_{\tilde{q}} = 0.1640$$

Therefore, the initial parameters of the Negative Binomial distribution are found to be:

$$h = \frac{1}{\sigma_{\tilde{q}}^2} = 148.72$$

and:

$$p_0 = \frac{h}{h + n_0} = 0.76\%$$

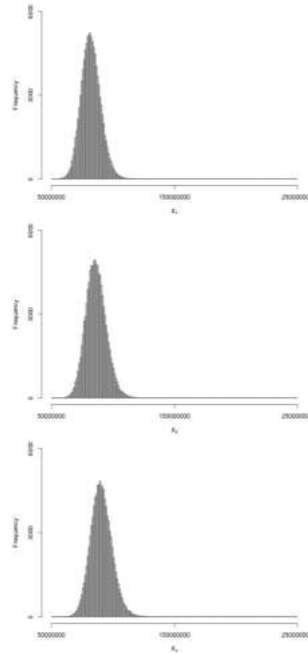
The parameters of the initial Lognormal distribution are estimated by the method of moments:

$$\sigma = \sqrt{\ln(1 + c_Z^2)} = 1.98$$

and:

$$\mu_0 = \ln(m_0) - \frac{\sigma^2}{2} = 6.34$$

Using equation (4.1), we are able to simulate the aggregate claim amount over time. Figure 5.15 shows the simulated distribution of the aggregate claim amount after one, two and three years, and Table 5.20 shows some elements of descriptive statistics. We point out that, because of the dynamic portfolio assumption, the mean and standard deviation increase over time, and the skewness decreases over time, approaching the skewness of the structure variable.



**Figure 5.15:** Simulated distribution of the aggregate claim amount after one, two and three years (100,000 simulations)

**Table 5.20:** Descriptive statistics of the simulated aggregate claim amount after one, two and three years (amounts in millions)

$X_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	82.07	7.86	0.3787	54.57	76.67	81.72	87.09	186.18
$t = 2$	86.12	8.26	0.4146	52.27	80.44	85.75	91.36	248.14
$t = 3$	90.50	8.61	0.3101	58.40	84.61	90.12	96.01	155.37



## 5.3 Market risk

In this section we isolate the effect of the market risk and so we neglect the non-life premium risk. In this regard, we drop the underwriting result by the risk reserve, but we still consider the interest on its investment. Moreover, we assume that the aggregate claim amount is deterministic and equal to its mean, so that the underwriting result is equal to the safety loadings. As a result, starting from equation (1.4), the risk reserve is found to be:

$$\tilde{U}_t = (1 + \tilde{j}_t) \cdot \tilde{U}_{t-1} + \tilde{j}_t \cdot \delta \cdot \pi_{t-1} + \tilde{j}_t \cdot \sum_{k=1}^{t-1} \varphi \cdot P_k$$

and, starting from equation (1.15), the risk reserve ratio is found to be:

$$\tilde{u}_t = \frac{(1 + \tilde{j}_t)}{(1 + i) \cdot (1 + g)} \cdot \tilde{u}_{t-1} + \frac{\tilde{j}_t \cdot \delta}{(1 + i) \cdot (1 + g)} + \sum_{k=1}^{t-1} \frac{1 - c}{1 + \varphi} \cdot \frac{\tilde{j}_t \cdot \varphi}{(1 + i)^k \cdot (1 + g)^k}$$

Furthermore, starting from equation (1.7), the annual net cash flows are found to be:

$$F_t = \pi_t \cdot \left[ (1 - c) + \delta \cdot \left( 1 - \frac{1}{(1 + i) \cdot (1 + g)} \right) \right] - P_t$$

The annual net cash flows are now deterministic, hence Table 5.21 shows the resulting value after one, two and three years.

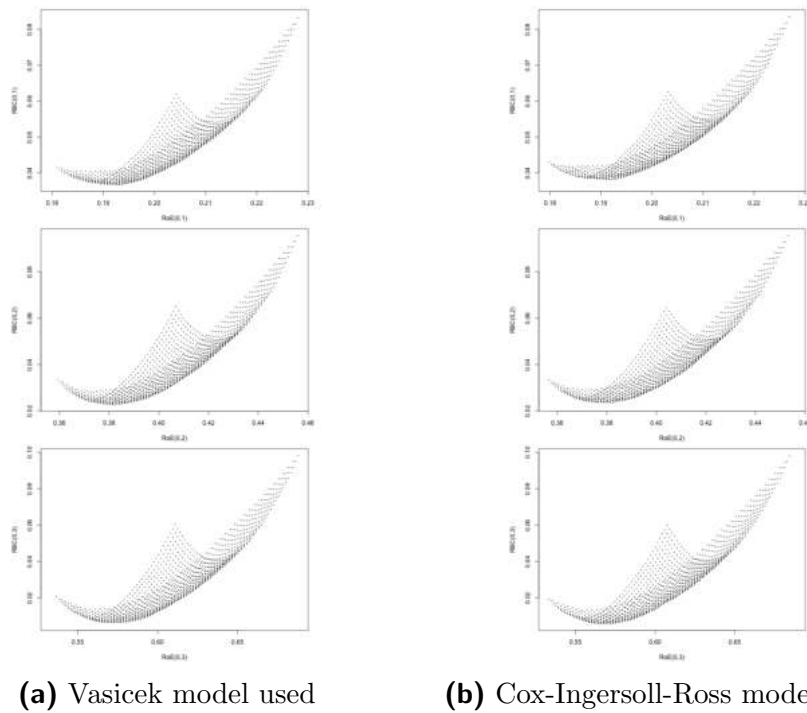
**Table 5.21:** Value of the annual net cash flows after one, two and three years

$F_1$	$F_2$	$F_3$
8,636,725	9,073,743	9,532,874

### 5.3.1 Portfolio optimization

We are now able to obtain the distribution of the annual rate of return over time, using equation (1.12), and considering the zero-coupon bond prices given by the Vasicek model first and then the Cox-Ingersoll-Ross model. For

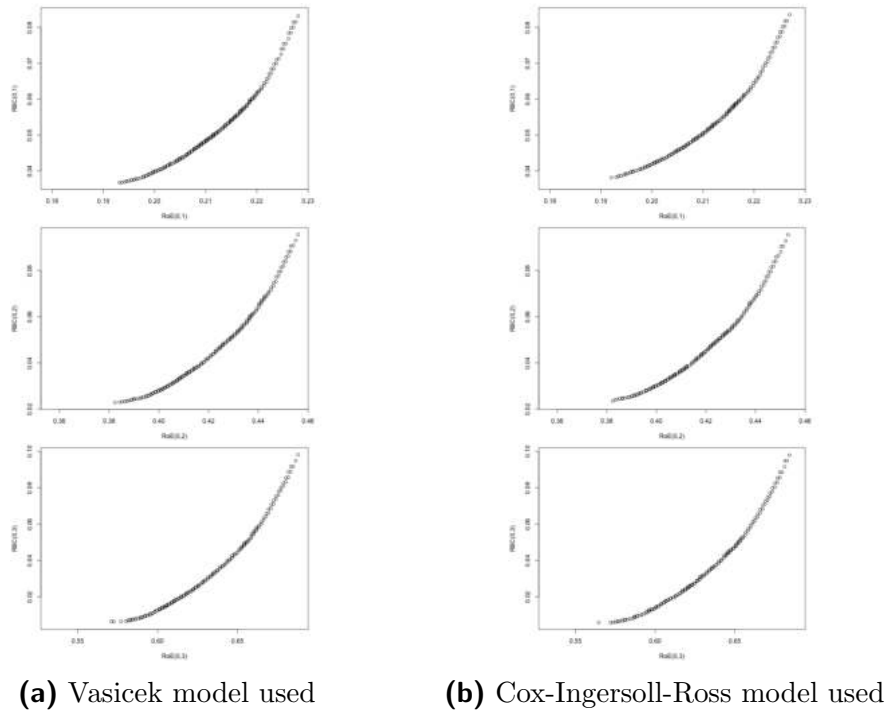
this reason, we need to select the percentage invested in each category of the stock portfolio. This choice is made by a procedure of optimization in terms of risk-return profile. We iteratively change each stock investment of 2% so that, for each combination, we determine the distribution of the annual rate of return, and thus the distribution of the risk reserve and risk reserve ratio over time. Figure 5.16 shows, for each portfolio, the pairs of expected spot Return on Equity, shown in equation (1.22), and minimum Risk-Based Capital as a percentage of the initial gross premium amount, shown in equation (1.21), over a period of one, two and three years. For capital requirement purposes, we deal with the theoretical annual rate of return that is obtained using equations (3.21) and (3.33) or (3.41), depending on the short rate model considered (Vasicek model or Cox-Ingersoll-Ross model).



**Figure 5.16:** Combinations of expected spot Return on Equity and ratio of minimum Risk-Based Capital and initial gross premium amount over a period of one, two and three years

Figure 5.17 shows the efficient frontier, i.e. the optimal portfolios that offer the highest expected return for a defined level of risk or the lowest risk

for a given level of expected return, over a period of one, two and three years. We point out that each portfolio is efficient from a risk-return perspective and the portfolio choice depends on the risk-return preferences. In this thesis we take into account an extremely prudent insurance company, so we consider the portfolio that minimizes the risk, even though it is the least profitable.



**Figure 5.17:** Efficient frontier over a period of one, two and three years

Table 5.22 shows the composition of the stock portfolio with the smallest risk, the minimum Risk-Based Capital as a percentage of the initial gross premium amount and the expected spot Return on Equity over a period of one, two and three years. The least risky optimal portfolio is mostly composed by the low-risk stock and, for the rest, by the medium and high-risk stocks. In Table 5.22b, we have less investments in the low and/or high-risk stocks, and more investments in the medium-risk stock than in Table 5.22a. In Table 5.22b, moreover, the minimum Risk-Based Capital is lower (higher) and the Return on Equity is lower (higher) than in Table 5.22a, since the different stock composition allows to minimize the risk, obtaining a smaller

(bigger) return. It is not the case over a one-year period, when the skewness of the zero-coupon bond capitalization factors is negative and the downside risk is higher, so that the minimization of the risk corresponds to a bigger minimum Risk-Based Capital and a smaller Return on Equity. Hence, except over a one-year period, the Vasicek model does not imply a better risk-return profile than the Cox-Ingersoll-Ross model, and vice versa.

**Table 5.22:** Composition of the stock portfolio, ratio of minimum Risk-Based Capital and initial gross premium amount, and expected spot Return on Equity of the optimal portfolio with the smallest risk over a period of one, two and three years

**(a)** Vasicek model used

	$\beta_1$	$\beta_2$	$\beta_3$	$u_{RBC}^{market}(0, t)$	$\overline{RoE}_{market}(0, t)$
$t = 1$	62%	24%	14%	3.67%	19.31%
$t = 2$	64%	24%	12%	2.27%	38.23%
$t = 3$	66%	22%	12%	0.64%	57.11%

**(b)** Cox-Ingersoll-Ross model used

	$\beta_1$	$\beta_2$	$\beta_3$	$u_{RBC}^{market}(0, t)$	$\overline{RoE}_{market}(0, t)$
$t = 1$	60%	28%	12%	3.81%	19.20%
$t = 2$	60%	26%	14%	2.34%	38.24%
$t = 3$	68%	22%	10%	0.59%	56.45%

The preceding result holds because the three stocks are low correlated, so that we have a risk diversification effect. Therefore, if the stocks were perfectly correlated, in order to minimize the risk, we would only invest in the low-risk stock. Moreover, we stress that the portfolio composition depends on the time horizon. For example, the composition that reduces the minimum Risk-Based Capital over a one-year period, does not reduce it over a period of two and three years. Hence, the portfolio choice depends on whether the view is short or medium-term. A short-term view is usually preferred, because it is easier to see the results of the activity. However, in this situation the risk is not properly managed, since problems could emerge in the future. Let us take the example of the stock options. Top managers are tempted to maximize the profit of the company without taking account

of future risks. Hence, companies could get in troubles and eventually fail, as it happened during the financial crisis of 2007/2008. Solvency II has a short-term view, because capital requirements shall be calculated over a one-year period. Nevertheless, because of the Own Risk and Solvency Assessment and Forward Looking Assessment of Own Risks (see section 2.7.1), insurance companies are encouraged to have a medium-term view as well.

In this thesis we take into account an exemplary insurance company, so we consider the portfolio that minimizes the risk over a period of three years. Table 5.23 shows the minimum Risk-Based Capital as a percentage of the initial gross premium amount and the expected spot Return on Equity over a period of one, two and three years, with the stock composition of the chosen optimal portfolio. Obviously, over a period of one and two years, the chosen optimal portfolio has a bigger minimum Risk-Based Capital than before. However, the situation is not so different from above. In Table 5.23b, the risk-return profile of the chosen optimal portfolio is worse over a period of one and two years than in Table 5.23a. This is because the distributions of the zero-coupon bond capitalization factors usually have a higher downside risk if the Cox-Ingersoll-Ross model is used. However, over a period of three years, the composition of the stock portfolio implies a better diversification effect in the case of the Cox-Ingersoll-Ross model rather than in the Vasicek model.

Furthermore, Figure 5.18 shows the resulting distribution of the annual rate of return after one, two and three years, and Table 5.24 shows some elements of descriptive statistics. We point out that the standard deviation raises over time and thus the mean decreases to offset the increased risk. In Table 5.24b, the mean and standard deviation are always smaller than in Table 5.24a. The skewness increases over time, as opposed to Table 5.24a.

In conclusion, Figure 5.19 shows the simulated risk reserve ratio over a period of three years. Figure 5.20 shows the resulting distribution of the risk reserve ratio after one, two and three years, and Table 5.25 shows some elements of descriptive statistics. We point out that the mean and standard deviation increase over time. The differences between Tables 5.25a and 5.25b are explained by the results in Table 5.24. In particular, the mean and standard deviation are higher in the first table than in the second.

**Table 5.23:** Stock composition of the chosen optimal portfolio, ratio of minimum Risk-Based Capital and initial gross premium amount, and expected spot Return on Equity over a period of one, two and three years

**(a)** Vasicek model used

$\beta_1$	$\beta_2$	$\beta_3$
66%	22%	12%

	$u_{RBC}^{market}(0, t)$	$\overline{RoE}_{market}(0, t)$
$t = 1$	3.69%	19.17%
$t = 2$	2.28%	38.14%
$t = 3$	0.64%	57.11%

**(b)** Cox-Ingersoll-Ross model used

$\beta_1$	$\beta_2$	$\beta_3$
68%	22%	10%

	$u_{RBC}^{market}(0, t)$	$\overline{RoE}_{market}(0, t)$
$t = 1$	3.84%	18.97%
$t = 2$	2.37%	37.66%
$t = 3$	0.59%	56.45%

### 5.3.2 Capital requirements according to the model

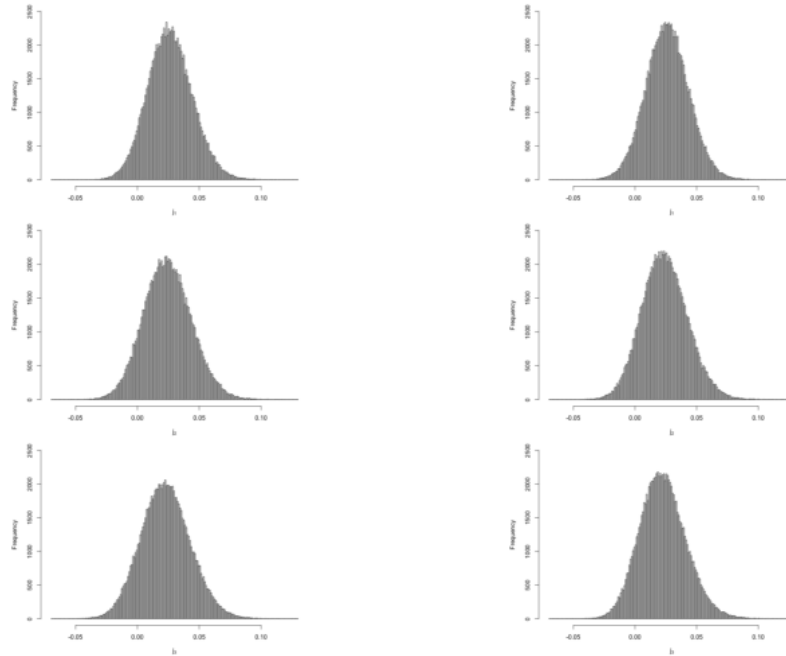
We are now able to calculate the capital requirements over a period of one, two or three years. In doing so, we compute the result for each investment category and we determine the overall diversification benefit.

Table 5.26 shows the initial value of each stock investment, given by the stock composition of the chosen optimal portfolio.

Now, we need to be coherent with equation (1.20). Hence, the minimum Risk-Based Capital of a stock investment is found to be:

$$RBC_{S_h}(0, t) = \alpha \cdot \beta_h \cdot \left[ U_0 + \frac{L_t + \sum_{k=1}^t \varphi \cdot P_k}{\prod_{k=1}^t 1 + E(\tilde{J}_k)} \right] - \frac{A_\varepsilon^{S_h}(t)}{\prod_{k=1}^t 1 + E(\tilde{J}_k)}$$

with  $h = 1, 2, 3$



(a) Vasicek model used

(b) Cox-Ingersoll-Ross model used

**Figure 5.18:** Simulated distribution of the annual rate of return (annually compounded) after one, two and three years (100,000 simulations)

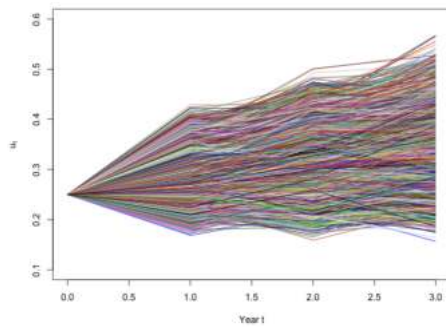
**Table 5.24:** Descriptive statistics of the simulated annual rate of return (annually compounded) after one, two and three years

(a) Vasicek model used

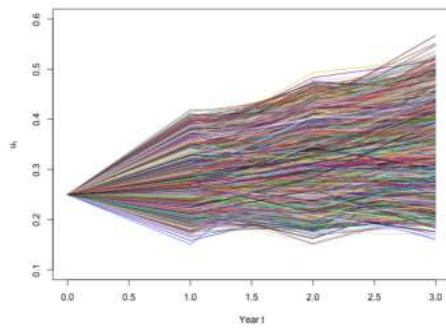
$j_t$	Mean	St.Dev.	Skew.	Min.	1 <sup>st</sup> Qu.	Median	3 <sup>rd</sup> Qu.	Max.
$t = 1$	2.65%	1.80%	0.1766	-4.02%	1.41%	2.60%	3.82%	11.10%
$t = 2$	2.44%	1.93%	0.1557	-5.40%	1.11%	2.40%	3.71%	11.63%
$t = 3$	2.29%	1.99%	0.1538	-6.21%	0.92%	2.24%	3.59%	12.72%

(b) Cox-Ingersoll-Ross model used

$j_t$	Mean	St.Dev.	Skew.	Min.	1 <sup>st</sup> Qu.	Median	3 <sup>rd</sup> Qu.	Max.
$t = 1$	2.62%	1.74%	0.0720	-5.05%	1.44%	2.60%	3.77%	10.55%
$t = 2$	2.41%	1.85%	0.1904	-4.86%	1.14%	2.35%	3.62%	11.93%
$t = 3$	2.26%	1.87%	0.2687	-5.55%	0.97%	2.19%	3.46%	11.84%



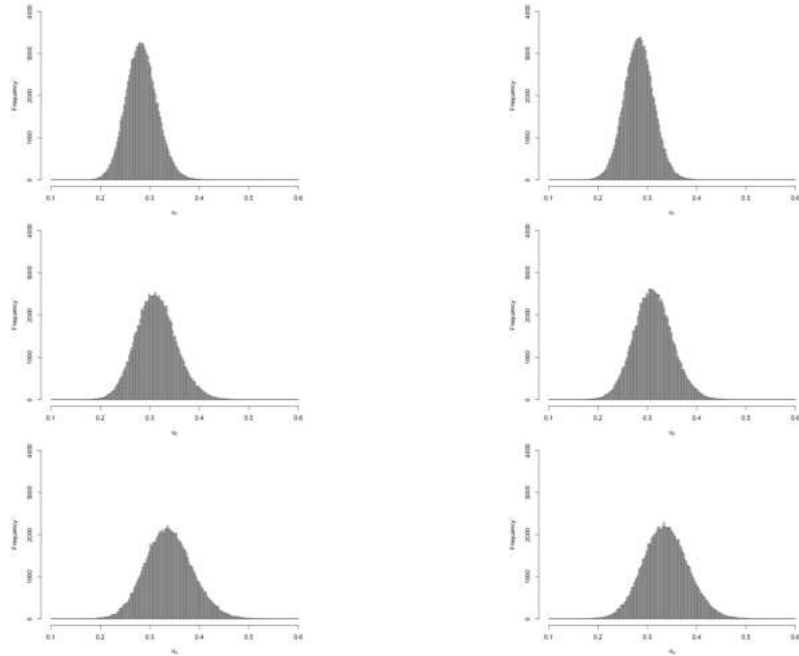
**(a)** Vasicek model used



**(b)** Cox-Ingersoll-Ross model used

**Figure 5.19:** Samples of 100,000 possible trajectories of the risk reserve ratio over a period of three years





(a) Vasicek model used

(b) Cox-Ingersoll-Ross model used

**Figure 5.20:** Simulated distribution of the risk reserve ratio after one, two and three years

**Table 5.25:** Descriptive statistics of the simulated risk reserve ratio after one, two and three years

(a) Vasicek model used

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	28.36%	3.10%	0.1766	16.86%	26.23%	28.27%	30.38%	42.93%
$t = 2$	31.29%	4.03%	0.1935	15.90%	28.52%	31.16%	33.90%	50.10%
$t = 3$	33.87%	4.69%	0.1764	15.65%	30.63%	33.73%	36.95%	56.72%

(b) Cox-Ingersoll-Ross model used

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	28.31%	3.00%	0.0720	15.09%	26.28%	28.28%	30.29%	41.98%
$t = 2$	31.18%	3.88%	0.1213	15.03%	28.56%	31.10%	33.72%	49.31%
$t = 3$	33.73%	4.53%	0.1373	16.07%	30.62%	33.62%	36.72%	56.71%

**Table 5.26:** Initial values of the stock investments**(a)** Vasicek model used

$A_0^{S_1}$	$A_0^{S_2}$	$A_0^{S_3}$
17,928,900	5,976,300	3,259,800

**(b)** Cox-Ingersoll-Ross model used

$A_0^{S_1}$	$A_0^{S_2}$	$A_0^{S_3}$
18,472,200	5,976,300	2,716,500

therefore, the result above as a percentage of the initial gross premium amount is found to be:

$$u_{RBC}^{S_h}(0, t) = \frac{RBC_{S_h}(0, t)}{\pi_0} \quad (5.5)$$

and the minimum Risk-Based Capital of a bond investment is found to be:

$$RBC_{B_i}(0, t) = (1 - \alpha) \cdot \gamma_i \cdot \left[ U_0 + \frac{L_t + \sum_{k=1}^t \varphi \cdot P_k}{\prod_{k=1}^t 1 + E(\tilde{J}_k)} \right] - \frac{A_\varepsilon^{B_i}(t)}{\prod_{k=1}^t 1 + E(\tilde{J}_k)}$$

with  $i = 1, 2, 3, 5, 10$

therefore, the result above as a percentage of the initial gross premium amount is found to be:

$$u_{RBC}^{B_i}(0, t) = \frac{RBC_{B_i}(0, t)}{\pi_0} \quad (5.6)$$

where  $A_\varepsilon^{S_h}(t)$  and  $A_\varepsilon^{B_i}(t)$  are the  $\varepsilon$ -th order quantiles of the values of the stock and bond investments. The claims reserve, as well as the initial risk reserve, are not resources of the insurance company, but they belong to the customers and shareholders, hence we have to drop them. Furthermore, we take off the safety loadings, because they belong to the underwriting side, and finally we discount all the amounts to time zero.

We are able to obtain the distributions of the values of the stock and bond

investments over time, using equations (1.9) and (1.11), and considering the annual stock capitalization factors and the annual zero-coupon bond capitalization factors given by the Vasicek model or Cox-Ingersoll-Ross model. Tables 5.27 and 5.28 show the minimum Risk-Based Capital of each stock and bond investment as a percentage of the initial gross premium amount over a period of one, two and three years. We point out that not only do the capital requirements depend on the riskiness of the investment, but also on the amount invested, hence the minimum Risk-Based Capital of the stock investments is bigger in the case of the low-risk stocks. Moreover, the minimum Risk-Based Capital of the bond investments increases as the time to maturity raises. As we explained before, for the first year, the 1 year zero-coupon bond factor is not stochastic, hence it is reasonable to use a risk measure, such as the minimum Risk-Based Capital, that discounts for the expected rate of return. Despite this, there is an expected profit, due to the investment of the claims reserve and safety loadings. As a result, the 1 year zero-coupon bond always produces a negative minimum Risk-Based Capital, i.e. a reduction of capital. The other zero-coupon bonds that produce negative capital requirements are those where the risk is small compared to the expected profit of the investment. In conclusion, the differences between Tables 5.27a and 5.27b are only due to the differences in the initial values of the stock investments, while the differences between Tables 5.28a and 5.28b are explained by the results in Tables 5.14 and 5.18.

The Degree of Diversification of the risk reserve, calculated on the basis of the minimum Risk-Based Capital, is given by:

$$DoD_{RBC}^{market}(0, t) = \frac{\sum_{h=1}^3 u_{RBC}^{S_h}(0, t) + \sum_{\{i=1,2,3,5,10\}} u_{RBC}^{B_i}(0, t) - u_{RBC}^{market}(0, t)}{\sum_{h=1}^3 u_{RBC}^{S_h}(0, t) + \sum_{\{i=1,2,3,5,10\}} u_{RBC}^{B_i}(0, t)} \quad (5.7)$$

Table 5.29 shows the Degree of Diversification of the risk reserve over a period of one, two and three years. Obviously the Degree of Diversification is always quite high, since the correlation coefficients in Table 5.7 are low or negative. In Table 5.29b, the Degree of Diversification is higher than in Table 5.29a, except over a period of two years, because the composition of the stock

**Table 5.27:** Ratios of minimum Risk-Based Capital of the stock investments and initial gross premium amount over a period of one, two and three years

**(a)** Vasicek model used

$u_{RBC}^{S_h}(0, t)$	$h = 1$	$h = 2$	$h = 3$	Total
$t = 1$	3.57%	1.65%	1.14%	6.36%
$t = 2$	3.36%	1.60%	1.09%	6.06%
$t = 3$	3.18%	1.53%	1.05%	5.76%

**(b)** Cox-Ingersoll-Ross model used

$u_{RBC}^{S_h}(0, t)$	$h = 1$	$h = 2$	$h = 3$	Total
$t = 1$	3.68%	1.65%	0.95%	6.28%
$t = 2$	3.47%	1.60%	0.91%	5.98%
$t = 3$	3.29%	1.53%	0.88%	5.70%

**Table 5.28:** Ratios of minimum Risk-Based Capital of the bond investments and initial gross premium amount over a period of one, two and three years

**(a)** Vasicek model used

$u_{RBC}^{B_i}(0, t)$	$i = 1$	$i = 2$	$i = 3$	$i = 5$	$i = 10$	Total
$t = 1$	-1.04%	0.04%	0.33%	0.48%	0.73%	0.54%
$t = 2$	-0.36%	-0.03%	0.18%	0.34%	0.57%	0.70%
$t = 3$	-0.72%	-0.31%	-0.02%	0.19%	0.42%	-0.45%

**(b)** Cox-Ingersoll-Ross model used

$u_{RBC}^{B_i}(0, t)$	$i = 1$	$i = 2$	$i = 3$	$i = 5$	$i = 10$	Total
$t = 1$	-1.04%	0.15%	0.44%	0.59%	0.86%	1.01%
$t = 2$	-0.35%	0%	0.24%	0.41%	0.67%	0.96%
$t = 3$	-0.72%	-0.32%	-0.01%	0.22%	0.47%	-0.37%

portfolio implies a better diversification in the case of the Cox-Ingersoll-Ross model rather than in the Vasicek model.

**Table 5.29:** Degree of Diversification of the risk reserve over a period of one, two and three years

(a) Vasicek model used

	$DoD_{RBC}^{market}(0, t)$
$t = 1$	46.52%
$t = 2$	66.28%
$t = 3$	87.89%

(b) Cox-Ingersoll-Ross model used

	$DoD_{RBC}^{market}(0, t)$
$t = 1$	47.40%
$t = 2$	65.79%
$t = 3$	88.99%

### 5.3.3 Capital requirements according to the standard formula

We can compare the results according to our model with the results according to the Solvency II standard formula. In doing so, we compute the Solvency Capital Requirement for the equity risk sub-module and the Solvency Capital Requirement for the interest rate risk sub-module, i.e. the sub-module that affects the zero-coupon bonds. We point out that the standard formula allows to calculate the capital requirements over a one-year period only.

We remember that the stock investments can be considered as type 1 equities not having a strategic nature and a long-term holding strategy, and that the symmetric adjustment is equal to zero. Hence, the Solvency Capital Requirement for equity risk, calculated on a single stock investment, is found to be:

$$SCR_{S_h} = 39\% \cdot A_0^{S_h} \quad \text{with } h = 1, 2, 3 \quad (5.8)$$

therefore, the result above as a percentage of the initial gross premium amount is found to be:

$$u_{SCR}^{S_h} = \frac{SCR_{S_h}}{\pi_0}$$

The Solvency Capital Requirement for interest rate risk, calculated on a single bond investment, is found to be:

$$SCR_{B_i} = A_0^{B_i} - A_0^{B_i} \cdot \frac{[1 + R_M(0, i)]^i}{[1 + R_M^+(0, i)]^i} \quad \text{with } i = 1, 2, 3, 5, 10 \quad (5.9)$$

therefore, the result above as a percentage of the initial gross premium amount is found to be:

$$u_{SCR}^{B_i} = \frac{SCR_{B_i}}{\pi_0}$$

where  $R_M(0, i)$  and  $R_M^+(0, i)$  are the spot interest rate with maturity  $i$  according to the market and according to the positive interest rate shock (see Table 5.30). Indeed, when the market interest rates increase, the bond prices decrease and thus we have a loss.

**Table 5.30:** Market interest rates and increased market interest rates (annually compounded)

Maturity (years)	Market rate	Increased market rate
0.5	1.89%	3.21%
1	1.80%	3.06%
2	1.69%	2.87%
3	1.63%	2.67%
5	1.61%	2.50%
7	1.69%	2.52%
10	1.76%	2.50%
20	2.03%	2.56%
30	2.23%	2.79%

Tables 5.31 and 5.32 show the Solvency Capital Requirement for equity risk, calculated on a single stock investment, and the Solvency Capital Requirement for interest rate risk, calculated on a single bond investment, as percentages of the initial gross premium amount. We point out that the differences between Tables 5.31a and 5.31b are only due to the differences in the initial values of the stock investments. Furthermore, all the capital requirements are bigger in the case of the standard formula rather than in

our model. This is because the standard formula was calibrated on riskier distributions than the distributions of our model.

**Table 5.31:** Ratios of Solvency Capital Requirement of the stock investments and initial gross premium amount

(a) Vasicek model used

	$h = 1$	$h = 2$	$h = 3$	Total
$u_{SCR}^{S_h}$	6.99%	2.33%	1.27%	10.59%

(b) Cox-Ingersoll-Ross model used

	$h = 1$	$h = 2$	$h = 3$	Total
$u_{SCR}^{S_h}$	7.20%	2.33%	1.06%	10.59%

**Table 5.32:** Ratios of Solvency Capital Requirement of the bond investments and initial gross premium amount

	$i = 1$	$i = 2$	$i = 3$	$i = 5$	$i = 10$	Total
$u_{SCR}^{B_i}$	0.75%	0.88%	0.70%	0.65%	1.07%	4.06%

Furthermore, we remember that the scenario of the interest rate risk sub-module only affects the bond investments and the scenario of the equity risk sub-module only affects the stock investments. Hence, the Solvency Capital Requirement for equity risk is found to be:

$$SCR_{equity} = \sum_{h=1}^3 SCR_{S_h} = 39\% \cdot A_0^S \quad (5.10)$$

and the Solvency Capital Requirement for interest rate risk is found to be:

$$SCR_{interest\ rate} = \sum_{\{i=1,2,3,5,10\}} SCR_{B_i} \quad (5.11)$$

The Solvency Capital Requirement for equity risk is the same if we consider the initial values of the stock investments found by using the Vasicek model or Cox-Ingersoll-Ross model. As a result, also the Solvency Capital Requirement for market risk is the same in both the cases.

We point out that the equity risk and interest rate risk are the only sources of market risk we have. Hence, the Solvency Capital Requirement for market risk is found to be:

$$SCR_{market} = \sqrt{SCR_{equity}^2 + SCR_{interest\ rate}^2} \quad (5.12)$$

therefore, the result above as a percentage of the initial gross premium amount is found to be:

$$u_{SCR}^{market} = \frac{SCR_{market}}{\pi_0} = 11.34\%$$

Hence, the Degree of Diversification of the risk reserve, calculated on the basis of the Solvency Capital Requirement, is found to be:

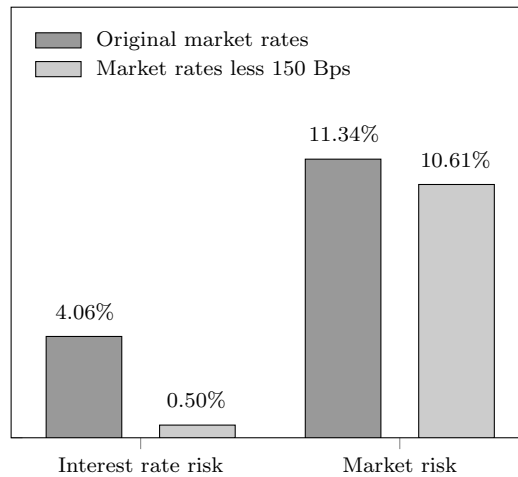
$$DoD_{SCR}^{market} = \frac{\sum_{h=1}^3 u_{SCR}^{S_h} + \sum_{\{i=1,2,3,5,10\}} u_{SCR}^{B_i} - u_{SCR}^{market}}{\sum_{h=1}^3 u_{SCR}^{S_h} + \sum_{\{i=1,2,3,5,10\}} u_{SCR}^{B_i}} = 22.57\%$$

We point out that the overall capital requirement is bigger in the case of the standard formula rather than in our model. Furthermore, the Degree of Diversification is significant, because of zero correlation between the equity and interest rate risk sub-modules, but it is quite smaller than in the case of our model.

The interest rate curve, according to our model, was estimated by Treasury rates in the United States, assuming that the credit risk and currency risk were absent. The government bonds, e.g. Treasury rates in the United States, are close to be risk-free investments. Nevertheless, government bonds are not completely risk-free, hence we should consider the credit risk in our model. Furthermore, Treasury rates in the United States are quoted in US dollars. Nevertheless, the main currency of European countries, e.g. in Italy, is the euro, hence we should also consider the currency risk in our model. As a result, the capital requirements for market risk, according to our model, should be bigger. Moreover, the standard formula takes into account the EIOPA risk-free interest rate curve. Nowadays, it is lower than what we are considering in this chapter. As a result, the



Solvency Capital Requirement for interest rate risk and the Solvency Capital Requirement for market risk should be smaller, because the interest rate shock would be smaller as well. Figure 5.21 shows a sensitivity analysis on the Solvency Capital Requirement for interest rate risk and on the Solvency Capital Requirement for market risk. We compare the results above with the results obtained after having applied a parallel negative shift of 150 Bps in the interest rate curve. We point out that, after the shift, the Solvency Capital Requirement for interest rate risk is found to be significantly smaller. As a result, there is a non-negligible reduction in the Solvency Capital Requirement for market risk.



**Figure 5.21:** Sensitivity analysis on the ratios of Solvency Capital Requirements and initial gross premium amount

### 5.3.4 Portfolio composition sensitivity

In section 5.3.2 we observed that the capital requirements decrease, because of the expected profit represented by the investment of the claims reserve and safety loadings. The low-risk zero-coupon bonds are thus likely to have a negative minimum Risk-Based Capital, e.g. for 1 year zero-coupon bonds. As a consequence, we have a drop in the overall capital requirements calculated according to our model. Moreover, we assumed that the bond portfolio is mostly composed by the low-risk investments, hence the capital requirements

drop further. On the contrary, the capital requirements calculated according to the standard formula are not reduced by the expected profit and they can never be negative.

We now make a sensitivity analysis to reduce the impact of those zero-coupon bonds with negative capital requirements, keeping fixed the stock composition found through the optimization procedure. Hence, Table 5.33 shows an alternative composition of the bond portfolio.

**Table 5.33:** Alternative composition of the bond portfolio

$\gamma_1$	$\gamma_2$	$\gamma_3$	$\gamma_5$	$\gamma_{10}$
10%	10%	15%	25%	40%

Table 5.34 shows the initial value of each alternative bond investment.

**Table 5.34:** Initial values of the alternative bond investments

$A_0^{B_1}$	$A_0^{B_2}$	$A_0^{B_3}$	$A_0^{B_5}$	$A_0^{B_{10}}$
15,393,500	15,393,500	23,090,250	38,483,750	61,574,000

Table 5.35 shows the minimum Risk-Based Capital of each alternative bond investment as a percentage of the initial gross premium amount over a one-year period. We point out that, the negative minimum Risk-Based Capital of the 1 year zero-coupon bond is now very low.

**Table 5.35:** Ratios of minimum Risk-Based Capital of the alternative bond investments and initial gross premium amount over a one-year period

**(a)** Vasicek model used

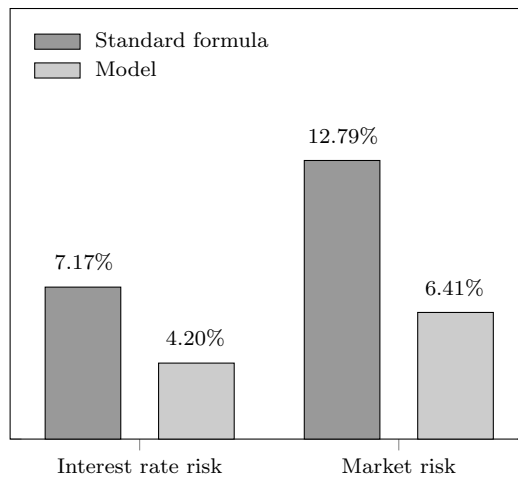
$w_{RBC}^{B_i}(0, t)$	$i = 1$	$i = 2$	$i = 3$	$i = 5$	$i = 10$	Total
$t = 1$	-0.26%	0.02%	0.33%	1.20%	2.91%	4.20%

**(b)** Cox-Ingersoll-Ross model used

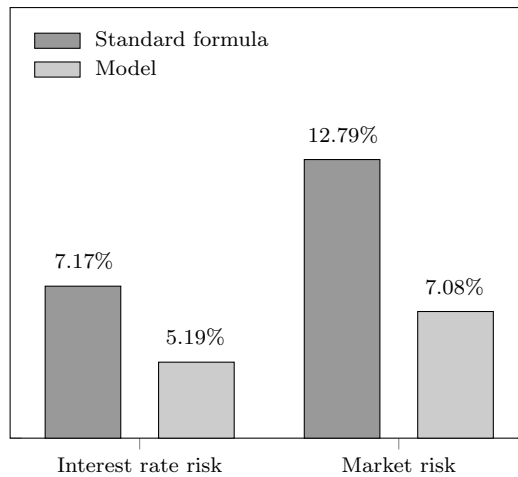
$w_{RBC}^{B_i}(0, t)$	$i = 1$	$i = 2$	$i = 3$	$i = 5$	$i = 10$	Total
$t = 1$	-0.26%	0.06%	0.44%	1.48%	3.46%	5.19%

Figure 5.22 shows the sensitivity analysis on the capital requirements for interest rate risk and on the capital requirements for market risk. We

compare the results according to the standard formula and according to our model. We point out that, the Solvency Capital Requirement for interest rate risk, according to the standard formula, increases significantly with respect to the original bond composition, as well as the minimum Risk-Based Capital according to our model. As a result, the capital requirements for market risk increase with respect to the original bond composition. Moreover, the gap between the standard formula and our model is reduced.



(a) Vasicek model used



(b) Cox-Ingersoll-Ross model used

**Figure 5.22:** Sensitivity analysis on the ratios of capital requirements and initial gross premium amount

## 5.4 Non-life premium risk

In this section we isolate the effect of the non-life premium risk and so we neglect the market risk. In this regard, we drop the investment result by the risk reserve, assuming that the annual rate of return is equal to zero. As a result, the risk reserve is found to be:

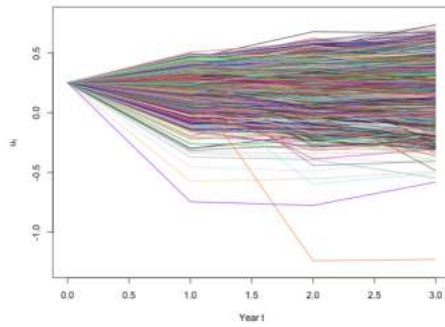
$$\tilde{U}_t = \tilde{U}_{t-1} + \left[ (1 + \varphi) \cdot P_t - \tilde{X}_t \right]$$

and the risk reserve ratio is found to be:

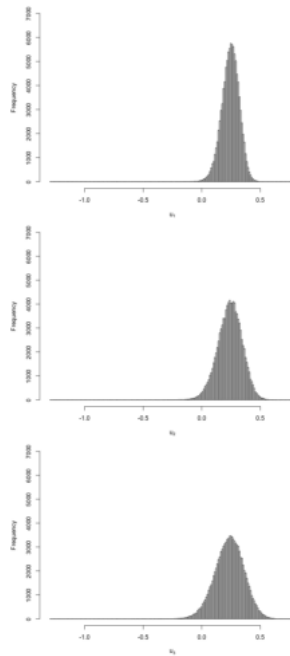
$$\tilde{u}_t = \frac{1}{(1 + i) \cdot (1 + g)} \cdot \tilde{u}_{t-1} + \frac{1 - c}{1 + \varphi} \cdot \left[ (1 + \varphi) - \frac{\tilde{X}_t}{P_t} \right]$$

We now consider the aggregate claim amount given by the collective risk model. Figure 5.23 shows the simulated risk reserve ratio over a period of three years. Figure 5.24 shows the resulting distribution of the risk reserve ratio after one, two and three years, and Table 5.36 shows some elements of descriptive statistics. We point out that the standard deviation increases over time and it is bigger than in presence of the market risk only. On the contrary, the mean decreases over time and it is smaller than in presence of the market risk only. This is because the underwriting result is not big enough to compensate the increase in the gross premium amount, that depends on the claims inflation and, especially, on the real growth. This situation represents the risk of rapid growth and it can be avoided by the ability of the shareholders of adding fresh capital. Moreover, the skewness is negative.

Figure 5.25 shows the resulting distribution of the annual net cash flows after one, two and three years, according to equation (1.7), and Table 5.37 shows some elements of descriptive statistics. We point out that the mean and standard deviation increase over time, and the skewness decreases in absolute value. Moreover, we stress that the standard deviation and absolute value of the skewness are the same as in the case of the distribution of the aggregate claim amount.



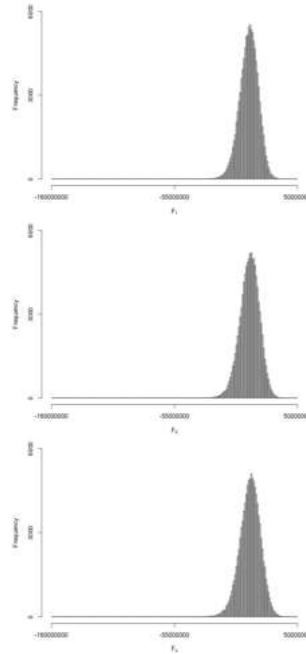
**Figure 5.23:** Samples of 100,000 possible trajectories of the risk reserve ratio over a period of three years



**Figure 5.24:** Simulated distribution of the risk reserve ratio after one, two and three years

**Table 5.36:** Descriptive statistics of the simulated risk reserve ratio after one, two and three years

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sup>st</sup> Qu.	Median	3 <sup>rd</sup> Qu.	Max.
$t = 1$	24.44%	7.48%	-0.3787	-74.66%	19.66%	24.77%	29.58%	50.62%
$t = 2$	23.99%	10.36%	-0.2924	-124.09%	17.30%	24.34%	31.09%	67.69%
$t = 3$	23.56%	12.34%	-0.2124	-122.94%	15.48%	23.91%	32.00%	73.55%



**Figure 5.25:** Simulated distribution of the annual net cash flows after one, two and three years (100,000 simulations)

**Table 5.37:** Descriptive statistics of the simulated annual net cash flows after one, two and three years (amounts in millions)

$F_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	8.57	7.86	-0.3787	-95.54	3.56	8.92	13.97	36.08
$t = 2$	9.11	8.26	-0.4146	-152.91	3.87	9.48	14.79	42.96
$t = 3$	9.55	8.61	-0.3101	-55.32	4.04	9.93	15.44	41.65

### 5.4.1 Capital requirements according to the model

We are now able to calculate the capital requirements over a period of one, two or three years.

Table 5.38 shows the minimum Risk-Based Capital as a percentage of the initial gross premium amount over a period of one, two and three years. We point out that the minimum Risk-Based Capital increases over time and it is much bigger than in the previous section. The aggregate claim amount is a loss, and it ranges between zero and infinity, while the annual rate of return is a profit, and it ranges between plus and minus infinity. In the best case scenario, we produce a zero loss and a high profit and, in the worst case scenario, we produce a high loss and a high negative profit. In the previous section, we worked to find an investment strategy that might reduce the possibility to have a big negative profit, while in this section we do not care about any risk reduction strategy. Furthermore, the differences between Tables 5.38a and 5.38b are only due to the different theoretical rates used for the minimum Risk-Based Capital calculation.

**Table 5.38:** Ratio of minimum Risk-Based Capital and initial gross premium amount over a period of one, two and three years

**(a)** Vasicek model used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}^{non-life}(0, t)$	21.87%	30.31%	36.44%

**(b)** Cox-Ingersoll-Ross model used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}^{non-life}(0, t)$	21.87%	30.32%	36.45%

### 5.4.2 Capital requirements according to the standard formula

We can compare the results according to our model with the results according to the Solvency II standard formula.

We remember that the reserve risk is neglected, as well as reinsurance. Furthermore, the insurance company only works in the MTPL line of business, hence the standard deviation for non-life premium and reserve risk is found to be:

$$\sigma_{nl} = 10\%$$

The volume measure for non-life premium and reserve risk is found to be:

$$V_{nl} = \pi_1 = 105,060,000$$

Using equation (2.1), the Solvency Capital Requirement for non-life premium and reserve risk as a percentage of the initial gross premium amount is found to be:

$$u_{SCR}^{nl\ prem\ res} = \frac{SCR_{nl\ prem\ res}}{\pi_0} = 31.52\%$$

Once again, the capital requirement is bigger in the case of the standard formula rather than in our model. This is because the safety loadings and size factor are neglected, and because the assumption of Lognormal distribution for non-life premium and reserve risk is erroneous if we consider our distribution of the aggregate claim amount.

Furthermore, we point out that the non-life premium risk is the only source of non-life underwriting risk we have. Hence, the Solvency Capital Requirement for non-life underwriting risk is found to be the same, namely:

$$u_{SCR}^{non-life} = \frac{SCR_{non-life}}{\pi_0} = 31.52\%$$

## 5.5 Market and non-life premium risk

In section 5.3 we assumed that the underwriting result was equal to the safety loadings and that it produced interests year by year. Actually, the interest is obtained from the investment of the real value of the underwriting result. We now finally take into account both the market risk and non-life premium risk, either according to the integrated model or stand-alone model, so that we can deepen the effect of the small approximation that we previously made. We



point out that in this case the portfolio optimization would give us another annual rate of return. Nevertheless, we are now interested in the analysis of the combination of the results above, hence we keep the rate that we have previously obtained.

### **5.5.1 Integrated model**

We now deal with the integrated model, considering equations (1.4) and (1.15).

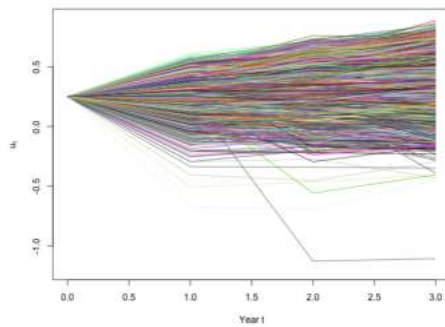
Figure 5.26 shows the simulated risk reserve ratio over a period of three years. Figure 5.27 shows the resulting distribution of the risk reserve ratio after one, two and three years, and Table 5.39 shows some elements of descriptive statistics. We point out that the market and underwriting effects are now combined together. Hence, the mean increases over time, because the financial result is added to the underwriting result, so that they compensate the increase in the gross premium amount, avoiding the risk of rapid growth. The standard deviation increases over time and it is bigger than in presence of the market risk only or non-life premium risk only. The skewness is negative, as in presence of the non-life premium risk only, but it is lower in absolute value. Once again, the differences between Tables 5.39a and 5.39b are explained by the results in Table 5.24, and they are the same as in presence of the market risk only.

The annual net cash flows are the same as in presence of the non-life premium risk only.

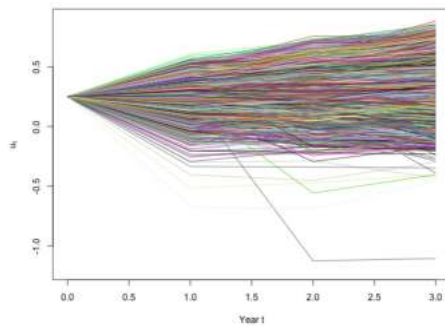
### **5.5.2 Capital requirements according to the integrated model**

We are now able to calculate the capital requirements over a period of one, two or three years.

Table 5.40 shows the minimum Risk-Based Capital as a percentage of the initial gross premium amount over a period of one, two and three years. We point out that the minimum Risk-Based Capital increases over time and it is smaller than the sum of the minimum Risk-Based Capital in presence

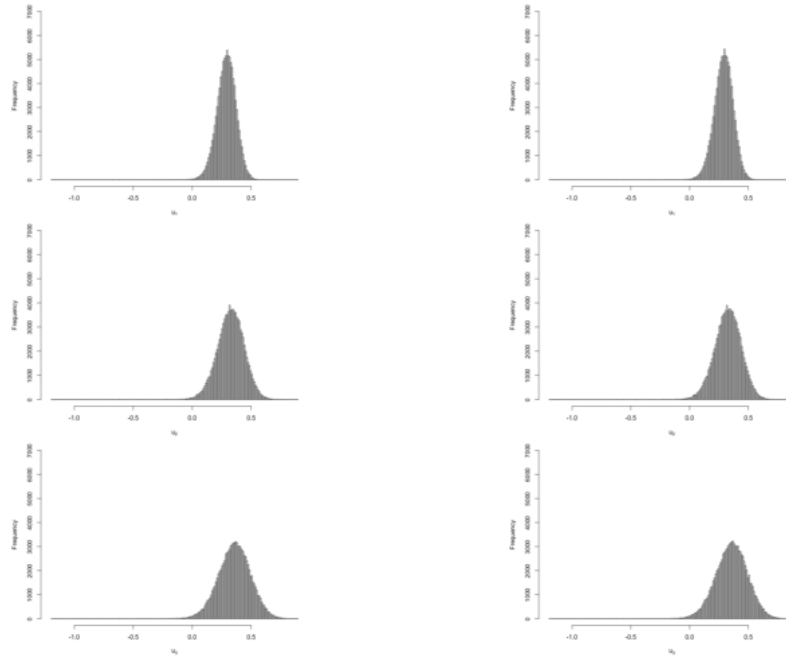


**(a)** Vasicek model used



**(b)** Cox-Ingersoll-Ross model used

**Figure 5.26:** Samples of 100,000 possible trajectories of the risk reserve ratio over a period of three years



(a) Vasicek model used

(b) Cox-Ingersoll-Ross model used

**Figure 5.27:** Simulated distribution of the risk reserve ratio after one, two and three years

**Table 5.39:** Descriptive statistics of the simulated risk reserve ratio after one, two and three years

(a) Vasicek model used

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	29.00%	8.09%	-0.2829	-67.52%	23.79%	29.27%	34.50%	60.69%
$t = 2$	32.63%	11.22%	-0.2113	-112.74%	25.34%	32.88%	40.27%	76.11%
$t = 3$	35.87%	13.46%	-0.1494	-110.65%	27.01%	36.13%	45.01%	88.86%

(b) Cox-Ingersoll-Ross model used

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	28.95%	8.05%	-0.2938	-67.36%	23.77%	29.23%	34.43%	60.65%
$t = 2$	32.52%	11.16%	-0.2192	-112.50%	25.29%	32.79%	40.14%	75.78%
$t = 3$	35.73%	13.40%	-0.1562	-110.43%	26.91%	36.00%	44.83%	88.40%

of the market risk only and non-life premium risk only. This is because we have a diversification effect, since the annual rate of return is simulated independently of the aggregate claim amount. Furthermore, in Table 5.40b, the minimum Risk-Based Capital is smaller over a one-year period and bigger over a period of two and three years than in Table 5.40a, even though it was different in presence of the market risk only. Over a one-year period, the distributions of the zero-coupon bond capitalization factors usually have a higher downside risk if the Cox-Ingersoll-Ross model is used, but now we have a better diversification effect, because of the presence of the aggregate claim amount.

**Table 5.40:** Ratio of minimum Risk-Based Capital and initial gross premium amount over a period of one, two and three years

**(a)** Vasicek model used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}(0, t)$	18.51%	22.90%	25.66%

**(b)** Cox-Ingersoll-Ross model used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}(0, t)$	18.50%	22.96%	25.79%

Now, we can calculate the Degree of Diversification of the risk reserve on the level of the minimum Risk-Based Capital for market risk or on the level of the minimum Risk-Based Capital of the stock or bond investments, so that we can observe the partial diversification effect, due to the last aggregation phase, or the total diversification effect. In the first case, it is given by:

$$DoD_{RBC}(0, t) = \frac{u_{RBC}^{market}(0, t) + u_{RBC}^{non-life}(0, t) - u_{RBC}(0, t)}{u_{RBC}^{market}(0, t) + u_{RBC}^{non-life}(0, t)} \quad (5.13)$$

and, in the second case, it is given by:

$$DoD_{RBC}^{total}(0, t) = \frac{u-sum_{RBC}^{market}(0, t) + u_{RBC}^{non-life}(0, t) - u_{RBC}(0, t)}{u-sum_{RBC}^{market}(0, t) + u_{RBC}^{non-life}(0, t)}$$

where:

$$u\text{-}sum_{RBC}^{market}(0, t) = \sum_{h=1}^3 u_{RBC}^{S_h}(0, t) + \sum_{\{i=1,2,3,5,10\}} u_{RBC}^{B_i}(0, t)$$

Table 5.41 shows the partial and total Degrees of Diversification of the risk reserve over a period of one, two and three years. The partial and total Degrees of Diversification are not extremely high, although the annual rate of return is simulated independently of the aggregate claim amount. This is because the minimum Risk-Based Capital of the market risk is very small compared with the minimum Risk-Based Capital of the non-life premium risk and thus the diversification effect is not fully exploited. Furthermore, in Table 5.41b, the partial and total Degrees of Diversification are higher over a period of one and two years and lower over a period of three years than in Table 5.41a, even though it was different in presence of the market risk only. This is because the presence of the aggregate claim amount implies a different diversification effect in the case of the Vasicek model or Cox-Ingersoll-Ross model.

**Table 5.41:** Degree of Diversification of the risk reserve over a period of one, two and three years

**(a)** Vasicek model used

	$DoD_{RBC}(0, t)$	$DoD_{RBC}^{total}(0, t)$
$t = 1$	27.60%	35.68%
$t = 2$	29.72%	38.21%
$t = 3$	30.80%	38.54%

**(b)** Cox-Ingersoll-Ross model used

	$DoD_{RBC}(0, t)$	$DoD_{RBC}^{total}(0, t)$
$t = 1$	28.04%	36.57%
$t = 2$	29.76%	38.36%
$t = 3$	30.37%	38.27%

### 5.5.3 Stand-alone model

We now deal with the stand-alone model, considering the following risk reserve:

$$\tilde{U}_t = \tilde{U}_t^M + \tilde{U}_t^{NL} \quad (5.14)$$

where:

$$\tilde{U}_t^M = (1 + \tilde{j}_t) \cdot \tilde{U}_{t-1}^M + \tilde{j}_t \cdot \delta \cdot \pi_{t-1} + \tilde{j}_t \cdot \sum_{k=1}^{t-1} \varphi \cdot P_k$$

and:

$$\tilde{U}_t^{NL} = \tilde{U}_{t-1}^{NL} + \left[ (1 + \varphi) \cdot P_t - \tilde{X}_t \right]$$

Furthermore, the initial values of the risk reserves are given by:

$$U_0^M = U_0 = 25,000,000$$

and:

$$U_0^{NL} = 0$$

The risk reserve ratio is found to be:

$$\tilde{u}_t = \tilde{u}_t^M + \tilde{u}_t^{NL} \quad (5.15)$$

where:

$$\tilde{u}_t^M = \frac{(1 + \tilde{j}_t)}{(1 + i) \cdot (1 + g)} \cdot \tilde{u}_{t-1}^M + \frac{\tilde{j}_t \cdot \delta}{(1 + i) \cdot (1 + g)} + \sum_{k=1}^{t-1} \frac{1 - c}{1 + \varphi} \cdot \frac{\tilde{j}_t \cdot \varphi}{(1 + i)^k \cdot (1 + g)^k}$$

and:

$$\tilde{u}_t^{NL} = \frac{1}{(1 + i) \cdot (1 + g)} \cdot \tilde{u}_{t-1}^{NL} + \frac{1 - c}{1 + \varphi} \cdot \left[ (1 + \varphi) - \frac{\tilde{X}_t}{P_t} \right]$$

Figure 5.28 shows the simulated risk reserve ratio over a period of three years. Figure 5.29 shows the resulting distribution of the risk reserve ratio after one, two and three years, and Table 5.42 shows some elements of descriptive statistics. We point out that we can make the same comments as in the case of the integrated model. As we could expect, after one year, the

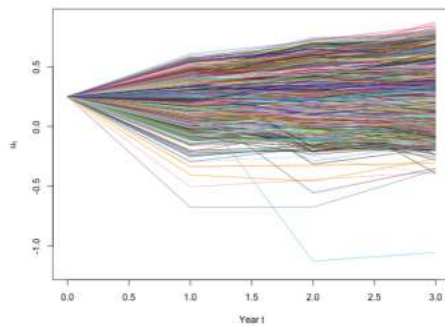
descriptive statistics of the stand-alone model are the same as in the case of integrated model, because the risk reserve equations are found to be the same as well. On the other hand, after two and three years, we have some small differences, especially in the standard deviation, that is smaller than in the case of the integrated model, and skewness, that is bigger in absolute value than in the case of the integrated model. Obviously the differences become bigger over time, because we repeat the approximation several times. In conclusion, the mean is roughly the same as in the case of the integrated model, since we assume that the underwriting result is equal to its mean.

Figure 5.30 shows the QQ-plot of the integrated and stand-alone distributions of the risk reserve ratios after one, two and three years. We point out that, as we could expect, the integrated and stand-alone quantiles are found to be exactly the same after one year, because the integrated and stand-alone distributions of the risk reserves are found to be the same as well. On the other hand, after two and three years, the integrated and stand-alone quantiles are roughly equal, even though there are some small differences, especially on the distribution tails. As a result, the capital requirements of the stand-alone model will be slightly different from the integrated model.

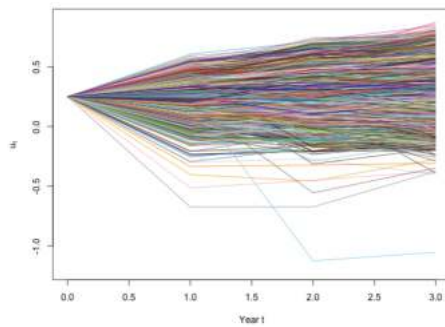
#### **5.5.4 Capital requirements according to the stand-alone model**

We are now able to calculate the capital requirements over a period of one, two or three years.

Table 5.43 shows the minimum Risk-Based Capital as a percentage of the initial gross premium amount over a period of one, two and three years. We point out that we can make the same comments as in the case of the integrated model. Once again, as we could expect, the minimum Risk-Based Capital over a one-year period is the same as in the case of the integrated model. On the other hand, over a period of two and three years, the minimum Risk-Based Capital is smaller than in the case of the integrated model, because we partially neglect the risk represented by the investment of the underwriting result, since we assume that it is equal to the safety loadings.



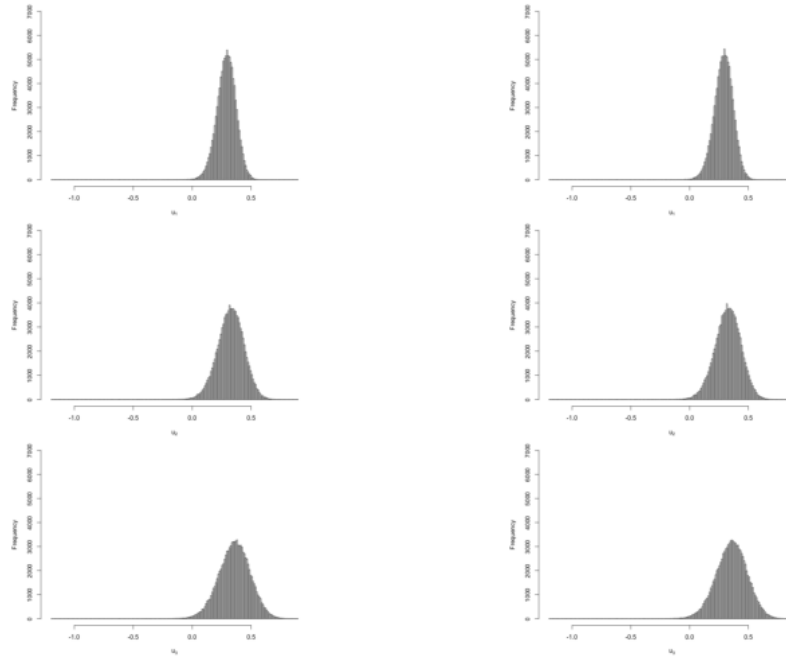
(a) Vasicek model used



(b) Cox-Ingersoll-Ross model used

**Figure 5.28:** Samples of 100,000 possible trajectories of the risk reserve ratio over a period of three years





(a) Vasicek model used

(b) Cox-Ingersoll-Ross model used

**Figure 5.29:** Simulated distribution of the risk reserve ratio after one, two and three years

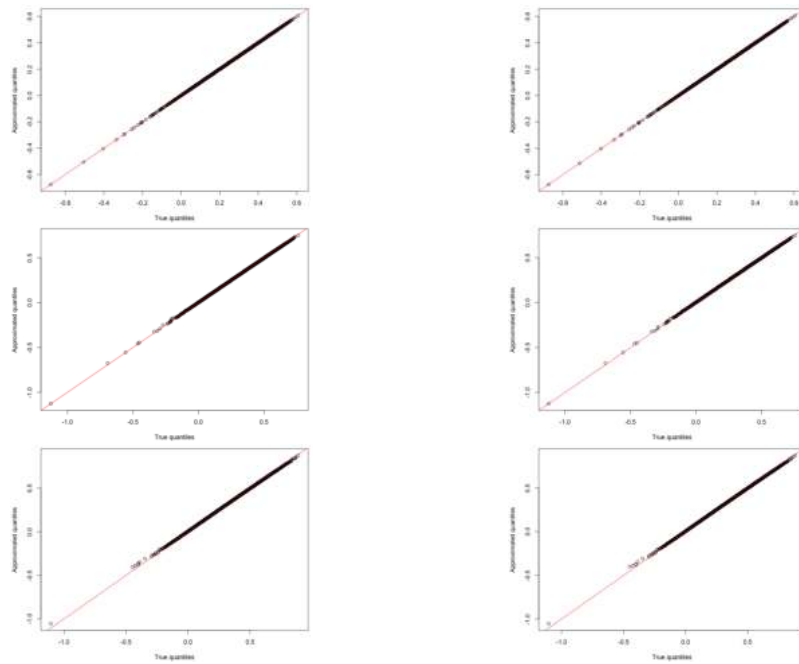
**Table 5.42:** Descriptive statistics of the simulated risk reserve ratio after one, two and three years

(a) Vasicek model used

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	29.00%	8.09%	-0.2829	-67.52%	23.79%	29.27%	34.50%	60.69%
$t = 2$	32.63%	11.10%	-0.2218	-112.66%	25.43%	32.89%	40.21%	74.92%
$t = 3$	35.87%	13.20%	-0.1647	-105.47%	27.20%	36.16%	44.85%	87.61%

(b) Cox-Ingersoll-Ross model used

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	28.95%	8.05%	-0.2938	-67.36%	23.77%	29.23%	34.43%	60.65%
$t = 2$	32.52%	11.05%	-0.2290	-112.43%	25.36%	32.81%	40.08%	74.65%
$t = 3$	35.73%	13.15%	-0.1703	-105.32%	27.10%	36.02%	44.68%	87.18%



(a) Vasicek model used

(b) Cox-Ingersoll-Ross model used

**Figure 5.30:** QQ-plot of the integrated and stand-alone distributions of the risk reserve ratios after one, two and three years

**Table 5.43:** Ratio of minimum Risk-Based Capital and initial gross premium amount over a period of one, two and three years

(a) Vasicek model used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}(0, t)$	18.51%	22.64%	25.22%

(b) Cox-Ingersoll-Ross model used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}(0, t)$	18.50%	22.74%	25.27%

Table 5.44 shows the partial and total Degrees of Diversification of the risk reserve over a period of one, two and three years. Once again, we can make the same comments as in the case of the integrated model. However, in Table 5.44b, the partial Degree of Diversification is higher over a one-year period and lower over a period of two and three years than in Table 5.44a,

while the total Degree of Diversification is higher over a period of one and two years and lower over a period of three years than in Table 5.44a. Furthermore, as we could expect, the partial and total Degrees of Diversification over a one-year period are the same as in the case of the integrated model. On the other hand, over a period of two and three years, the partial and total Degrees of Diversification are bigger than in the case of the integrated model, because the minimum Risk-Based Capital is lower than in the case of the integrated model.

**Table 5.44:** Degree of Diversification of the risk reserve over a period of one, two and three years

**(a)** Vasicek model used

	$DoD_{RBC}(0, t)$	$DoD_{RBC}^{total}(0, t)$
$t = 1$	27.60%	35.68%
$t = 2$	30.51%	38.90%
$t = 3$	31.98%	39.58%

**(b)** Cox-Ingersoll-Ross model used

	$DoD_{RBC}(0, t)$	$DoD_{RBC}^{total}(0, t)$
$t = 1$	28.04%	36.57%
$t = 2$	30.43%	38.95%
$t = 3$	31.77%	39.52%

### 5.5.5 Capital requirements according to the standard formula

We can compare the results according to our model with the results according to the Solvency II standard formula.

The Solvency Capital Requirement for market and non-life underwriting risk is found to be:

$$SCR = \sqrt{SCR_{market}^2 + 2 \cdot 0.25 \cdot SCR_{market} \cdot SCR_{non-life} + SCR_{non-life}^2} \quad (5.16)$$

therefore, the result above as a percentage of the initial gross premium amount is found to be:

$$u_{SCR} = \frac{SCR}{\pi_0} = 36.07\%$$

Hence, the Degree of Diversification of the risk reserve, calculated on the basis of the Solvency Capital Requirement and on the level of risk modules, is found to be:

$$D_oD_{SCR} = \frac{u_{SCR}^{market} + u_{SCR}^{non-life} - u_{SCR}}{u_{SCR}^{market} + u_{SCR}^{non-life}} = 15.85\%$$

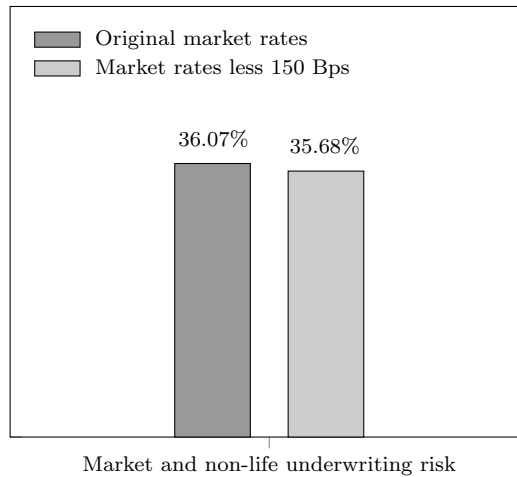
and, the Degree of Diversification of the risk reserve, calculated on the basis of the Solvency Capital Requirement and on the level of risk sub-modules, is found to be:

$$D_oD_{SCR}^{total} = \frac{\sum_{h=1}^3 u_{SCR}^{S_h} + \sum_{\{i=1,2,3,5,10\}} u_{SCR}^{B_i} + u_{SCR}^{non-life} - u_{SCR}}{\sum_{h=1}^3 u_{SCR}^{S_h} + \sum_{\{i=1,2,3,5,10\}} u_{SCR}^{B_i} + u_{SCR}^{non-life}} = 21.88\%$$

We point out that the overall capital requirement is bigger in the case of the standard formula rather than in our model. Furthermore, the partial Degree of Diversification is smaller than in the case of our model, but it is significant, although the correlation is not equal to zero. That is also because the Solvency Capital Requirement for market risk is small, but it is not so small compared with the Solvency Capital Requirement for non-life underwriting risk. In this regard, we can understand that the diversification is less effective when the risk mostly depends on a single source. As a result, if we wanted to further improve the Degree of Diversification, we should have had risk modules with balanced Solvency Capital Requirements. In conclusion, also the total Degree of Diversification is smaller than in the case of our model.

Figure 5.31 shows a sensitivity analysis on the Solvency Capital Requirement for market and non-life underwriting risk. We compare the result above with the result obtained after having applied a parallel negative

shift of 150 Bps in the interest rate curve. We point out that, after the shift, the Solvency Capital Requirement for market and non-life underwriting risk is found to be smaller, because the Solvency Capital Requirement for interest rate risk is found to be significantly smaller.



**Figure 5.31:** Sensitivity analysis on the ratio of Solvency Capital Requirement and initial gross premium amount

## 5.6 Interest rate exposure

In this chapter we assumed that the insurance liabilities are not affected by changes in the term structure of interest rates. We now consider that also the claims reserve is sensitive to interest rates, so that we can study the interest rate exposure of the portfolio.

For reasons of simplicity, the development triangles are estimated by the Italian market data, provided by ANIA and IVASS.<sup>5</sup> Table 5.45 shows the settlement speed for amount of the MTPL line of business.

Our development triangle of cumulative paid amounts shall give a Solvency II claims reserve equal to 156,100,000. In this regard, for each accident year, we multiply the settlement speed for amount and the expected ultimate cost at the starting time of our analysis. Furthermore, we use

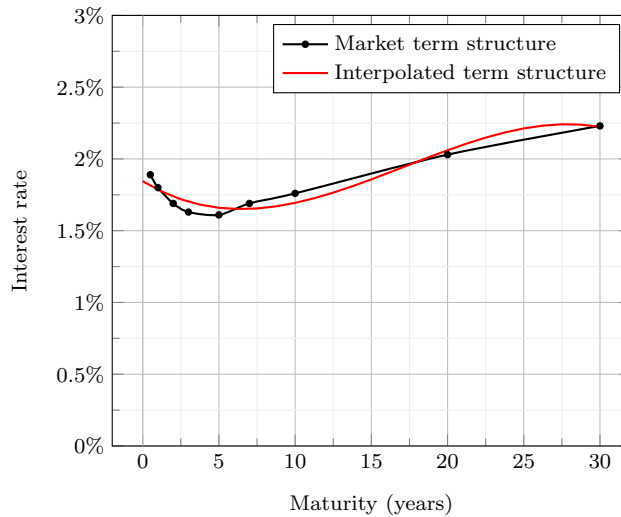
<sup>5</sup>See ANIA [1], and IVASS [20].

**Table 5.45:** Settlement speed for amount of the MTPL line of business

Accident year	Development year											
	0	1	2	3	4	5	6	7	8	9	10	11
2007	33.08%	67.34%	79.85%	85.95%	89.42%	91.89%	94.03%	95.57%	96.72%	97.61%	98.19%	98.65%
2008	36.47%	68.34%	80.26%	86.27%	89.84%	92.47%	94.34%	95.84%	96.95%	97.68%	98.30%	
2009	36.70%	68.73%	80.64%	86.55%	90.07%	92.59%	94.41%	95.88%	96.80%	97.47%		
2010	36.53%	68.62%	80.78%	86.47%	89.79%	92.27%	94.29%	95.66%	96.57%			
2011	36.82%	68.77%	80.72%	86.22%	89.65%	92.16%	94.03%	95.37%				
2012	35.35%	67.08%	79.53%	85.40%	88.85%	91.46%	93.17%					
2013	35.14%	66.19%	78.94%	84.79%	88.01%	90.29%						
2014	35.84%	67.21%	79.24%	84.44%	87.57%							
2015	36.12%	67.39%	79.55%	84.81%								
2016	37.18%	67.87%	79.93%									
2017	38.28%	69.85%										
2018	43.76%											

the Paid Chain-Ladder method to estimate the future cumulative and incremental paid amounts. Then, we discount the incremental paid amounts, so that we are able to calculate the best estimate of the claims reserve, and we sum up the risk margin. In this context, we assume that the expected ultimate costs are equal to the expected ultimate cost of the first accident year, otherwise they are bigger or smaller by 5%. Moreover, according to the market data provided by ANIA and IVASS, we assume that the risk margin is equal to 4.5% of the best estimate.<sup>6</sup> As a result, by changing the expected ultimate cost of the first accident year, and so the cumulative paid amounts, the claims reserve calculated so far can be found to be equal to 156,100,000.

For this analysis, we need the market interest rates with maturity from 1 to 12, but in Table 5.30 we do not have all of them. In order to obtain the missing rates, we apply a polynomial interpolation of third order. Figure 5.32 shows the market or interpolated interest rate term structure and Table 5.46 shows the value of the interpolated interest rates for the maturities where the market interest rates were absent.



**Figure 5.32:** Market interest rates and interpolated interest rates (annually compounded)

Table 5.47 shows the development triangle of undiscounted cumulative

<sup>6</sup>See ANIA [1], and IVASS [20].

**Table 5.46:** Interpolated interest rates (annually compounded)

Maturity (years)	Interpolated rate
4	1.68%
6	1.65%
8	1.66%
9	1.67%
11	1.72%
12	1.75%

paid amounts obtained through the procedure described so far, and the development factors. We assume that the unpaid claims are settled within one year, so that we have a one-year triangle tail. Table 5.48 shows the resulting development triangle of undiscounted incremental paid amounts and Table 5.49 shows the resulting lower development triangle of discounted incremental paid amounts. As we discussed, the Solvency II claims reserve is found to be equal to 156,100,000 of which 149,377,990 represents the best estimate and 6,722,010 represents the risk margin. On the other side, the Italian GAAP claims reserve is found to be equal to 156,925,978.

Starting from the development triangles of the incremental paid amounts, we are able to calculate some financial indicators of the liabilities. Table 5.50 shows the internal rate of return, duration and convexity. Since the term structure of interest rates is not flat, we compute the duration and convexity, using the internal rate of return. The risk margin is exposed to interest rates, because we assume that it is a percentage of the best estimate. Nevertheless, we remember that the interest rate shock for Solvency Capital Requirement purposes does not affect the risk margin. In any case, the financial indicators of the liabilities are the same if we consider or not the risk margin, because the latter is produced by an equal proportion of each incremental paid amount. We point out that the time to wait before paying the present value of the claim amounts is on average approximately equal to three years.

We now focus on the asset side of our portfolio. Zero-coupon bonds are the only investments in our portfolio to be exposed to interest rates. We remember that Table 5.6 shows the initial value of each bond investment.



**Table 5.47:** Undiscounted cumulative paid amounts and development factors (amounts in millions)

Accident year	Development year												
	0	1	2	3	4	5	6	7	8	9	10	11	12
0	29.81	60.69	71.96	77.46	80.59	82.81	84.74	86.13	87.16	87.97	88.49	88.91	90.12
1	34.51	64.67	75.95	81.63	85.02	87.50	89.27	90.69	91.74	92.43	93.02	93.46	94.73
2	31.42	58.84	69.04	74.10	77.11	79.27	80.83	82.09	82.88	83.45	83.96	84.36	85.51
3	32.92	61.84	72.80	77.93	80.92	83.16	84.97	86.21	87.03	87.72	88.26	88.67	89.88
4	34.85	65.08	76.38	81.59	84.83	87.21	88.98	90.25	91.22	91.94	92.50	92.94	94.20
5	30.26	57.43	68.09	73.11	76.07	78.30	79.77	80.99	81.86	82.50	83.01	83.40	84.54
6	31.67	59.66	71.14	76.41	79.31	81.37	83.06	84.33	85.23	85.90	86.43	86.84	88.02
7	33.91	63.60	74.98	79.90	82.86	85.18	86.95	88.28	89.22	89.93	90.48	90.91	92.15
8	30.92	57.70	68.10	72.61	75.48	77.59	79.20	80.41	81.27	81.91	82.41	82.80	83.93
9	33.51	61.17	72.04	77.18	80.23	82.47	84.18	85.47	86.39	87.07	87.60	88.02	89.22
10	36.23	66.10	77.99	83.56	86.86	89.29	91.13	92.53	93.52	94.26	94.84	95.29	96.59
11	37.46	70.42	83.09	89.02	92.54	95.12	97.09	98.58	99.64	100.42	101.04	101.52	102.90
Development factors	1.88	1.18	1.18	1.07	1.04	1.03	1.02	1.02	1.01	1.01	1.01	1.00	1.01

**Table 5.48:** Undiscounted incremental paid amounts (amounts in millions)

Accident year	Development year												
	0	1	2	3	4	5	6	7	8	9	10	11	12
0	29.81	30.88	11.27	5.50	3.13	2.22	1.93	1.38	1.04	0.80	0.52	0.42	1.21
1	34.51	30.16	11.28	5.68	3.38	2.48	1.77	1.42	1.05	0.69	0.58	0.44	1.27
2	31.42	27.42	10.20	5.06	3.01	2.16	1.56	1.26	0.79	0.57	0.51	0.40	1.15
3	32.92	28.92	10.96	5.13	2.99	2.24	1.82	1.23	0.82	0.69	0.54	0.42	1.21
4	34.85	30.23	11.31	5.21	3.24	2.38	1.77	1.27	0.97	0.72	0.56	0.44	1.27
5	30.26	27.16	10.67	5.02	2.96	2.23	1.47	1.22	0.87	0.65	0.51	0.39	1.14
6	31.67	27.99	11.49	5.27	2.90	2.06	1.69	1.27	0.90	0.67	0.53	0.41	1.18
7	33.91	29.69	11.38	4.92	2.96	2.32	1.76	1.33	0.95	0.71	0.55	0.43	1.24
8	30.92	26.78	10.40	4.51	2.87	2.11	1.61	1.21	0.86	0.64	0.50	0.39	1.13
9	33.51	27.66	10.87	5.14	3.05	2.24	1.71	1.29	0.92	0.68	0.53	0.41	1.20
10	36.23	29.87	11.89	5.57	3.30	2.43	1.85	1.39	0.99	0.74	0.58	0.45	1.30
11	37.46	32.96	12.67	5.93	3.52	2.59	1.97	1.49	1.06	0.79	0.62	0.48	1.38

**Table 5.49: Discounted incremental paid amounts (amounts in millions)**

Accident year	Development year												
	0	1	2	3	4	5	6	7	8	9	10	11	12
0													1.19
1												0.43	1.23
2											0.50	0.38	1.10
3										0.68	0.52	0.40	1.13
4									0.95	0.70	0.54	0.41	1.17
5								1.20	0.84	0.62	0.47	0.36	1.03
6							1.66	1.23	0.86	0.63	0.49	0.37	1.05
7						2.28	1.71	1.27	0.89	0.65	0.50	0.38	1.09
8					2.82	2.04	1.53	1.13	0.80	0.58	0.45	0.34	0.97
9				5.05	2.95	2.14	1.60	1.19	0.83	0.61	0.47	0.36	1.01
10			11.68	5.38	3.14	2.27	1.71	1.26	0.88	0.65	0.50	0.38	1.08
11		32.38	12.25	5.65	3.29	2.39	1.79	1.32	0.93	0.68	0.52	0.40	1.12

**Table 5.50:** Financial indicators of the liabilities

Internal rate of return	Duration	Convexity
1.69%	2.89	16.75

Using interest rates, we are able to calculate the face values (see Table 5.51).

**Table 5.51:** Face values of the bond investments

$A_1^{B_1}$	$A_2^{B_2}$	$A_3^{B_3}$	$A_5^{B_5}$	$A_{10}^{B_{10}}$
62,682,332	39,795,492	24,237,868	16,673,226	18,327,717

Hence, we can calculate some financial indicators of the assets. Table 5.52 shows the internal rate of return, duration and convexity. We point out that the time to wait before receiving the present value of our investments is on average approximately equal to three years.

**Table 5.52:** Financial indicators of the assets

Internal rate of return	Duration	Convexity
1.71%	2.85	17.54

In conclusion, we can calculate some financial indicators of the portfolio of assets and liabilities. Table 5.53 shows the internal rate of return, duration and convexity, considering or not the risk margin. We point out that the present value of the portfolio, i.e. the difference between the present value of the assets and the present value of the liabilities, is found to be equal to 4,557,010 when we neglect the risk margin and  $-2,165,000$  when we consider the risk margin. As a result, the internal rate of return is bigger in the first case than in the second. Moreover, the duration is positive in both the cases, i.e. there is not interest rate immunization, so that the portfolio is exposed to a parallel positive shift in the term structure in the first case and it is exposed to a parallel negative shift in the term structure in the second case (we see that the present value of the portfolio is negative). The effect of an increase (decrease) in interest rates is smaller in absolute value when we neglect the risk margin, because the duration is smaller as well. The convexity is positive

in the first case and negative in the second case, so that the loss is contained and the gain is further increased.

**Table 5.53:** Financial indicators of the portfolio

	Internal rate of return	Duration	Convexity
Risk margin neglected	2.93%	1.16	34.81
Risk margin considered	1.01%	5.17	-50.55

Using relation (1.24), we are now able to calculate the percentage variation in the value of the portfolio and, consequently, the absolute variation. Let us assume a parallel positive shift of 50 Bps in the interest rate curve. When we neglect the risk margin, we obtain:

$$\frac{\Delta V}{V} \approx -0.52\% \quad \text{so that} \quad \Delta V \approx -23,740.14$$

and when we consider the risk margin, we obtain:

$$\frac{\Delta V}{V} \approx -2.62\% \quad \text{so that} \quad \Delta V \approx 56,741.95$$

The real percentage variations in the value of the portfolio are given by  $-0.66\%$  and  $-2.96\%$  respectively, which are not perfectly equivalent to the percentages above, because we summarized the term structure of interest rates with the internal rate of return.

Let us now assume a parallel negative shift of 50 Bps in the interest rate curve. When we neglect the risk margin, we obtain:

$$\frac{\Delta V}{V} \approx 0.61\% \quad \text{so that} \quad \Delta V \approx 27,705.86$$

and when we consider the risk margin, we obtain:

$$\frac{\Delta V}{V} \approx 2.49\% \quad \text{so that} \quad \Delta V \approx -54,006.20$$

The real percentage variations in the value of the portfolio are given by  $0.76\%$  and  $2.87\%$  respectively, which are once again slightly different from the percentages above.

## Chapter 6

# Capital requirements for a multi-line insurance company

In this chapter we calculate the capital requirements for market and non-life premium risk of a multi-line insurance company. We lay down the same assumptions and we assume the same relevant parameters as in the case of the single-line insurance company. On the contrary, we assume that the insurance company is multi-line, because not only does it work in the MTPL line of business, but also in the Motor Other Damages (MOD) and General Third-Party Liability (GTPL) lines of business. Hence, the gross premium amount shown in Table 5.1 is found to come from three different sources. Furthermore, Table 6.1 shows the other relevant parameters, that are estimated by the Italian market data for the period 2014-2018, provided by ANIA and IVASS.<sup>1</sup> We point out that the parameters of the MTPL line of business are the same as in the case of the single-line insurance company, except for the gross premium amount. The safety loading coefficient of the MOD line of business is bigger than the safety loading coefficients of the other lines of business. Furthermore, the ratio of claims reserve and gross premium amount of the MOD line of business is very low, since the settlement speed is very high, while the ratio of claims reserve and gross premium amount of the GTPL line of business is very high, since the settlement speed is very low.

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<sup>1</sup>See ANIA [1], and IVASS [20].

This is because the MOD line of business deals with damages to property and the GTPL line of business deals mainly with damages to people, which have a settlement process really long.

**Table 6.1:** Relevant parameters of the multi-line insurance company

LoB	$\pi_0$	$\varphi$	$c$	$\delta$
MTPL	50,000,000	0.90%	21.24%	156.10%
MOD	25,000,000	13.85%	30.30%	22.88%
GTPL	25,000,000	6.65%	32.30%	415.62%

## 6.1 Aggregate claim amount

In this section we describe the distribution of the aggregate claim amount over time. In doing so, we firstly generate a sample for each line of business, using the collective risk model and Monte Carlo simulation, and making the same assumptions on the distributions as in the case of the single-line insurance company.

Table 6.2 shows the parameters for each line of business, that are estimated by the Italian market data for the period 2009-2018, provided by ANIA and IVASS.<sup>2</sup> We point out that the coefficient of variation of the single claim amount and the standard deviation of the structure variable of the GTPL line of business are the highest, because this business involves damages to people, which are really volatile.

**Table 6.2:** Parameters for the collective risk model of the multi-line insurance company

LoB	$m_0$	$n_0$	$c_{\bar{z}}$	$\sigma_{\bar{q}}$
MTPL	4,000	9,757.19	7	0.0820
MOD	2,500	6,122.09	2	0.0501
GTPL	10,000	1,586.97	12	0.1480

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<sup>2</sup>See ANIA [1], and IVASS [20].

Figure 6.1 shows the simulated distribution of the aggregate claim amount of each line of business after one, two and three years, and Table 6.3 shows some elements of descriptive statistics. We point out that we can make the same comments as in the case of the single-line insurance company. The MTPL line of business has the highest mean, because it is the main line of business of the insurance company. On the other side, the GTPL line of business has the highest standard deviation and skewness, even though the number of claims is not the biggest, because the coefficient of variation of the single claim amount and the standard deviation of the structure variable are really high. Furthermore, the MOD line of business has the lowest standard deviation and skewness, because it is not the main line of business, and the coefficient of variation of the single claim amount and the standard deviation of the structure variable are small compared to the others.

We thus use a copula function, i.e Gaussian copula or Gumbel copula, to inject some dependence structure. Through the copula function, we simulate different vectors of observations, that are translated in different pseudo aggregate claim amounts. As a result, the total aggregate claim amount is found to be:

$$\tilde{X}_t = \tilde{X}_t^{MTPL,C} + \tilde{X}_t^{MOD,C} + \tilde{X}_t^{GTPL,C} \quad (6.1)$$

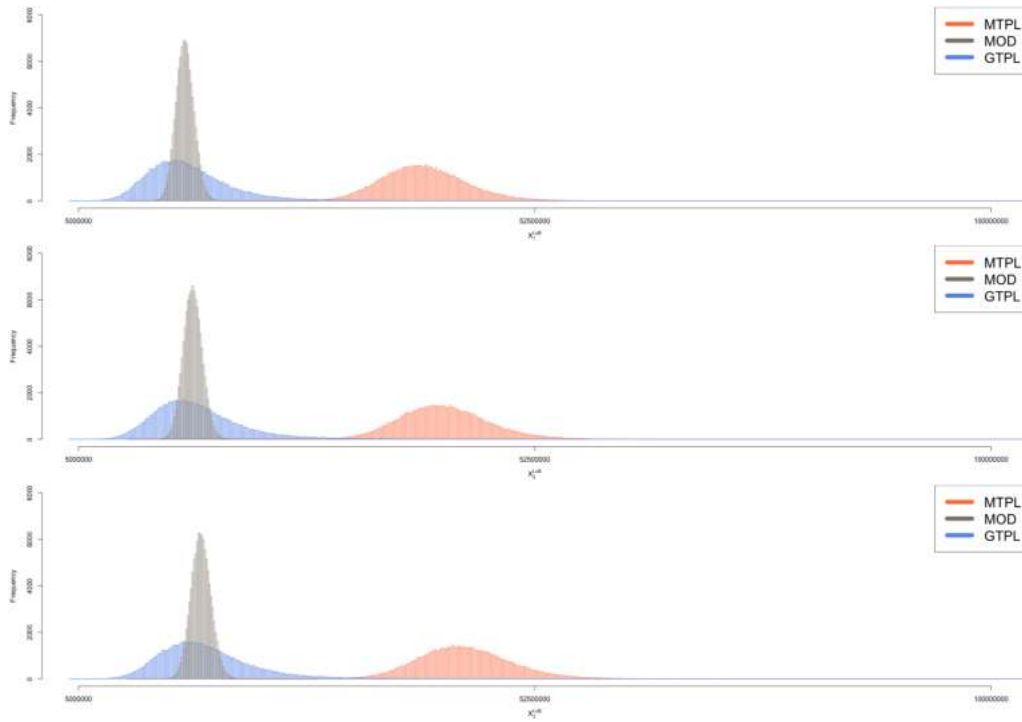
where  $\tilde{X}_t^{MTPL,C}$ ,  $\tilde{X}_t^{MOD,C}$  and  $\tilde{X}_t^{GTPL,C}$  are the pseudo aggregate claim amounts, that are translated from the simulated vectors of observations given by the copula function.

### 6.1.1 Gaussian copula

We now consider a Gaussian copula with parameters equal to the correlation coefficients of the Delegated Regulation. Table 6.4 shows the correlation matrix of the Gaussian copula.

Figure 6.2 shows, for each pair of lines of business, the simulated scatter plot given by the Gaussian copula after one, two and three years. We point out that, as we could expect, the scatter plots of the MTPL and MOD lines of business, and the scatter plots of the MTPL and GTPL lines of business,





**Figure 6.1:** Simulated distributions of the aggregate claim amounts after one, two and three years (100,000 simulations)

**Table 6.3:** Descriptive statistics of the simulated aggregate claim amounts after one, two and three years (amounts in millions)

$X_t^{MTPL}$	Mean	St.Dev.	Skew.	Min.	1 <sup>st</sup> Qu.	Median	3 <sup>rd</sup> Qu.	Max.
$t = 1$	41.00	4.43	0.6557	24.71	37.98	40.74	43.71	100.18
$t = 2$	43.08	4.60	0.6172	26.19	39.93	42.82	45.90	117.07
$t = 3$	45.25	4.85	0.7106	27.63	41.96	44.97	48.19	124.60

$X_t^{MOD}$	Mean	St.Dev.	Skew.	Min.	1 <sup>st</sup> Qu.	Median	3 <sup>rd</sup> Qu.	Max.
$t = 1$	16.08	0.92	0.1162	12.39	15.45	16.06	16.70	20.24
$t = 2$	16.89	0.97	0.1132	13.23	16.23	16.88	17.54	21.77
$t = 3$	17.75	1.02	0.1144	13.78	17.05	17.73	18.42	22.58

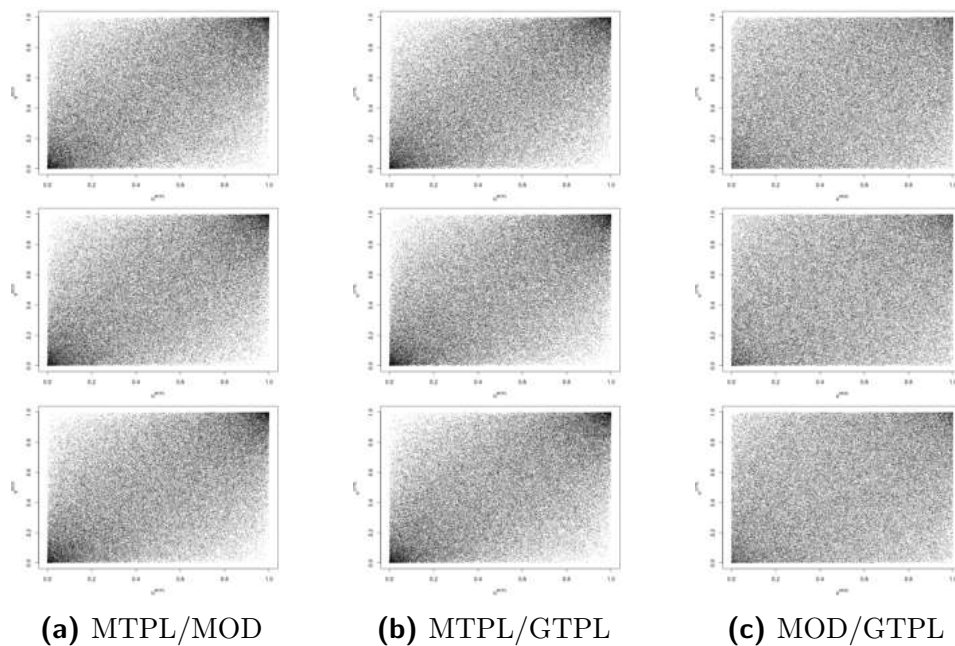
  

$X_t^{GTPL}$	Mean	St.Dev.	Skew.	Min.	1 <sup>st</sup> Qu.	Median	3 <sup>rd</sup> Qu.	Max.
$t = 1$	16.66	5.46	10.6987	4.72	13.50	15.85	18.75	470.28
$t = 2$	17.50	5.85	13.2866	5.60	14.22	16.67	19.67	519.80
$t = 3$	18.39	5.80	7.4075	5.78	14.94	17.55	20.70	392.09

**Table 6.4:** Correlation matrix of the Gaussian copula

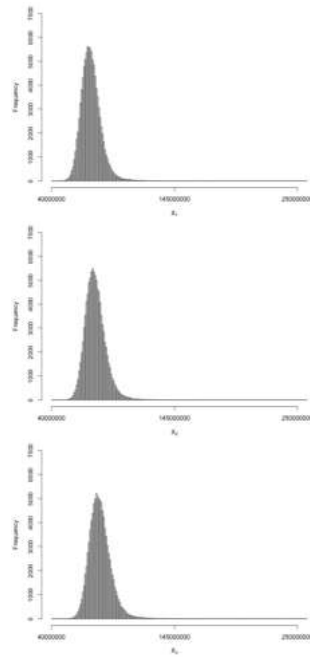
	MTPL	MOD	GTPL
MTPL	1	0.5	0.5
MOD	0.5	1	0.25
GTPL	0.5	0.25	1

show more positive dependence than the scatter plots of the MOD and GTPL lines of business.



**Figure 6.2:** Simulated scatter plots given by the Gaussian copula after one, two and three years (100,000 simulations)

Figure 6.3 shows the resulting distribution of the total aggregate claim amount after one, two and three years, and Table 6.5 shows some elements of descriptive statistics. We point out that the mean is lower than in the case of the single-line insurance company, but the standard deviation and skewness are higher.



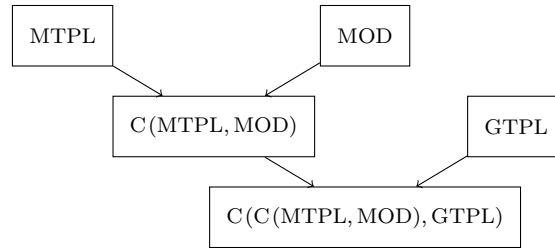
**Figure 6.3:** Simulated distribution of the total aggregate claim amount given by the Gaussian copula after one, two and three years (100,000 simulations)

**Table 6.5:** Descriptive statistics of the simulated total aggregate claim amount given by the Gaussian copula after one, two and three years (amounts in millions)

$X_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	73.76	8.79	2.9481	44.78	68.08	72.97	78.32	380.08
$t = 2$	77.43	9.01	2.0912	47.22	71.54	76.56	82.17	297.68
$t = 3$	81.40	9.40	1.7415	52.69	75.21	80.51	86.41	303.47

## 6.1.2 Gumbel copula

Archimedean copulas have the disadvantage to represent the dependence structure with few parameters only. As a result, we now consider Gumbel copulas with a hierarchical structure.<sup>3</sup> In doing so, we join the MTPL and MOD lines of business at the first step, and we add the GTPL line of business at the second step, as shown in Figure 6.4.



**Figure 6.4:** Hierarchical structure for the Gumbel copulas

Using equation (1.25), we start from a correlation coefficient equal to 0.5, i.e. the parameter of the Gaussian copula used to model the dependence between the MTPL and MOD lines of business, and we obtain a Kendall's rank correlation coefficient equal to 0.33. Moreover, using equation (1.26), we start from this Kendall's rank correlation coefficient and we obtain a parameter for the Gumbel copula equal to 1.5.

Actually we do not know the correlation between the joined MTPL and MOD lines of business and the GTPL line of business. The implicit correlation coefficient between the sum of the MTPL and MOD lines of business and the GTPL line of business is found to be:

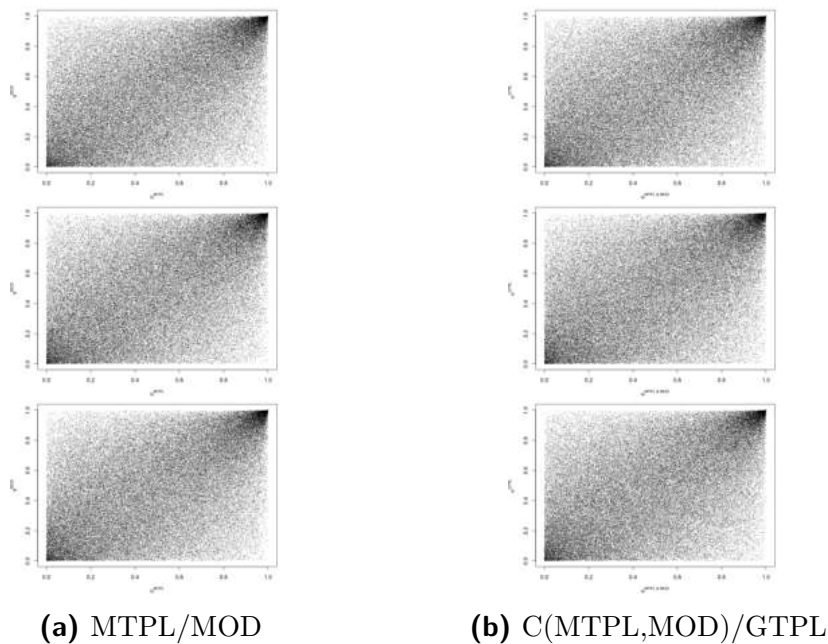
$$\begin{aligned}
 & \text{Corr}((\tilde{X}_t^{MTPL} + \tilde{X}_t^{MOD}), \tilde{X}_t^{GTPL}) = \\
 & \frac{\text{Corr}(\tilde{X}_t^{MTPL}, \tilde{X}_t^{GTPL}) \cdot \sigma_{\tilde{X}_t^{MTPL}} + \text{Corr}(\tilde{X}_t^{MOD}, \tilde{X}_t^{GTPL}) \cdot \sigma_{\tilde{X}_t^{MOD}}}{\sqrt{\text{Var}(\tilde{X}_t^{MTPL}) + 2 \cdot \text{Corr}(\tilde{X}_t^{MTPL}, \tilde{X}_t^{MOD}) \cdot \sigma_{\tilde{X}_t^{MTPL}} \cdot \sigma_{\tilde{X}_t^{MOD}} + \text{Var}(\tilde{X}_t^{MOD})}}
 \end{aligned}$$

Hence, using equation (4.6) and starting from the correlation coefficients of

<sup>3</sup>See Savelli and Clemente [31] for further details.

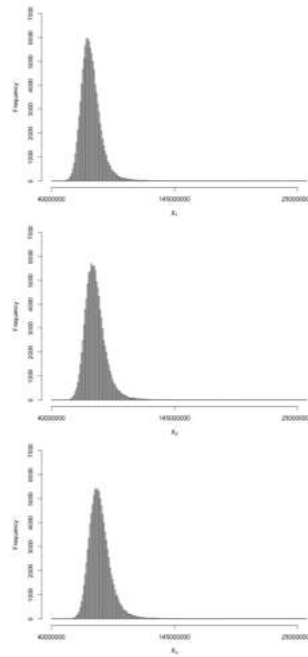
the Delegated Regulation, we obtain an implicit correlation coefficient equal to 0.4935, 0.4934 and 0.4934, and a parameter for the Gumbel copula equal to 1.4893, 1.4893 and 1.4892 after one, two and three years.

Figure 6.5 shows, for each step of the hierarchical structure, the simulated scatter plot given by the Gumbel copula after one, two and three years. We point out that, as we could expect, the scatter plots of the MTPL and MOD lines of business show a significant upper tail dependence as the scatter plots of the joined MTPL and MOD lines of business and GTPL line of business.



**Figure 6.5:** Simulated scatter plots given by the Gumbel copulas after one, two and three years (100,000 simulations)

Figure 6.6 shows the resulting distribution of the total aggregate claim amount after one, two and three years, and Table 6.6 shows some elements of descriptive statistics. We point out that we can make the same comments as in the case of the Gaussian copula. The total mean is more or less the same as in the case of the Gaussian copula, the standard deviation is slightly bigger and the skewness is bigger, because the upper tail dependence of the Gumbel copula is higher. Hence, the maximum is bigger in the case of the Gumbel copula rather than in the case of the Gaussian copula.



**Figure 6.6:** Simulated distribution of the total aggregate claim amount given by the Gumbel copulas after one, two and three years (100,000 simulations)

**Table 6.6:** Descriptive statistics of the simulated total aggregate claim amount given by the Gumbel copulas after one, two and three years (amounts in millions)

$X_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	73.74	9.19	4.7043	47.64	68.05	72.58	77.97	573.34
$t = 2$	77.47	9.37	3.2831	51.66	71.53	76.31	81.88	402.64
$t = 3$	81.33	9.83	3.5019	54.42	75.13	80.13	85.96	405.33

## 6.2 Market risk

In this section we extend the analysis of the market risk of the previous chapter to the case of the multi-line insurance company. As a result, the risk reserve is found to be:

$$\tilde{U}_t = (1 + \tilde{j}_t) \cdot \tilde{U}_{t-1} + \tilde{j}_t \cdot \delta \cdot \pi_{t-1} + \tilde{j}_t \cdot \sum_{k=1}^{t-1} \varphi \cdot P_k$$

where:

$$\delta = \frac{\delta^{MTPL} \cdot \pi_0^{MTPL} + \delta^{MOD} \cdot \pi_0^{MOD} + \delta^{GTPL} \cdot \pi_0^{GTPL}}{\pi_0} = 187.67\%$$

and:

$$\pi_t = \pi_t^{MTPL} + \pi_t^{MOD} + \pi_t^{GTPL}$$

moreover:

$$\varphi = \frac{\varphi^{MTPL} \cdot P_0^{MTPL} + \varphi^{MOD} \cdot P_0^{MOD} + \varphi^{GTPL} \cdot P_0^{GTPL}}{P_0} = 5.02\% \quad (6.2)$$

and:

$$P_t = P_t^{MTPL} + P_t^{MOD} + P_t^{GTPL} \quad (6.3)$$

The equations of the risk reserve ratio and annual net cash flows are equal to the equations in section 5.3, even though the values of the parameters are different. The annual net cash flows are once again deterministic, hence Table 6.7 shows the resulting value after one, two and three years.

**Table 6.7:** Value of the annual net cash flows after one, two and three years

$F_1$	$F_2$	$F_3$
13,201,154	13,869,133	14,570,911

As a result, the initial claims reserve is found to be:

$$L_0 = 187,675,000$$

and the initial asset value of the portfolio is found to be:

$$A_0 = U_0 + L_0 = 212,675,000$$

We point out that in this case the portfolio optimization would give us another annual rate of return. Nevertheless, we are now interested in the analysis of the difference with the results of the previous chapter, hence we keep the asset allocation of the previous chapter and the rate that we have previously obtained. Therefore, the initial value of the stock portfolio is found to be:

$$A_0^S = 15\% \cdot A_0 = 31,901,250$$

and the initial value of the bond portfolio is found to be:

$$A_0^B = 85\% \cdot A_0 = 180,773,750$$

In conclusion, Tables 6.8 and 6.9 show the new initial value of each bond and stock investment.

**Table 6.8:** Initial values of the bond investments

$A_0^{B_1}$	$A_0^{B_2}$	$A_0^{B_3}$	$A_0^{B_5}$	$A_0^{B_{10}}$
72,309,500	45,193,438	27,116,063	18,077,375	18,077,375

**Table 6.9:** Initial values of the stock investments

**(a)** Vasicek model used

$A_0^{S_1}$	$A_0^{S_2}$	$A_0^{S_3}$
21,054,825	7,018,275	3,828,150

**(b)** Cox-Ingersoll-Ross model used

$A_0^{S_1}$	$A_0^{S_2}$	$A_0^{S_3}$
21,692,850	7,018,275	3,190,125

Figure 6.7 shows the simulated risk reserve ratio over a period of three years. Figure 6.8 shows the resulting distribution of the risk reserve ratio after



one, two and three years, and Table 6.10 shows some elements of descriptive statistics. We point out that we can make the same comments as in the case of the single-line insurance company, because the annual rate of return is the same. However, the claims reserve, safety loadings and expense loadings are now bigger. As a result, the mean and standard deviation are bigger than in the case of the single-line insurance company.

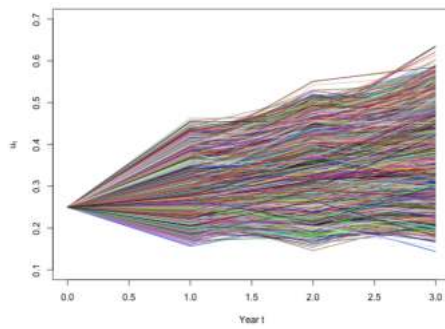
### **6.2.1 Capital requirements according to the model**

We are now able to calculate the capital requirements over a period of one, two or three years. Once again, we compute the result for each investment category and we determine the overall diversification benefit.

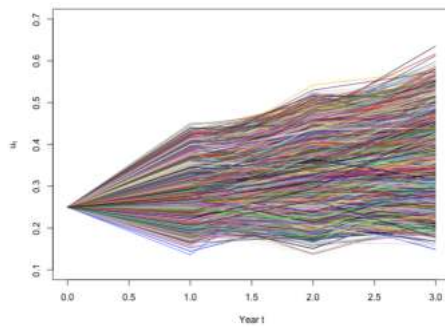
Tables 6.11 and 6.12 show the minimum Risk-Based Capital of each stock and bond investment as a percentage of the initial gross premium amount over a period of one, two and three years, according to equations (5.5) and (5.6). We point out that we we can make the same comments as in the case of the single-line insurance company. However, the minimum Risk-Based Capital of the stock investments and the minimum Risk-Based Capital of the bond investments are usually higher than in the case of the single-line insurance company. This is because the claims reserve and safety loadings are now bigger, so that also the invested resources are now bigger and the risk raises. Despite this, the low-risk investments produce a reduction of capital requirements. Indeed, the increase in the risk is more than compensated by the increase in the expected profit, due to the investment of higher claims reserve and safety loadings.

Table 6.13 shows the overall minimum Risk-Based Capital as a percentage of the initial gross premium amount over a period of one, two and three years. We point out that the overall minimum Risk-Based Capital decreases over time. It is higher over a period of one and two years and lower over a period of three years than in the case of the single-line insurance company.

Table 6.14 shows the Degree of Diversification of the risk reserve over a period of one, two and three years, according to equation (5.7). Once again, we can make the same comments as in the case of the single-line insurance

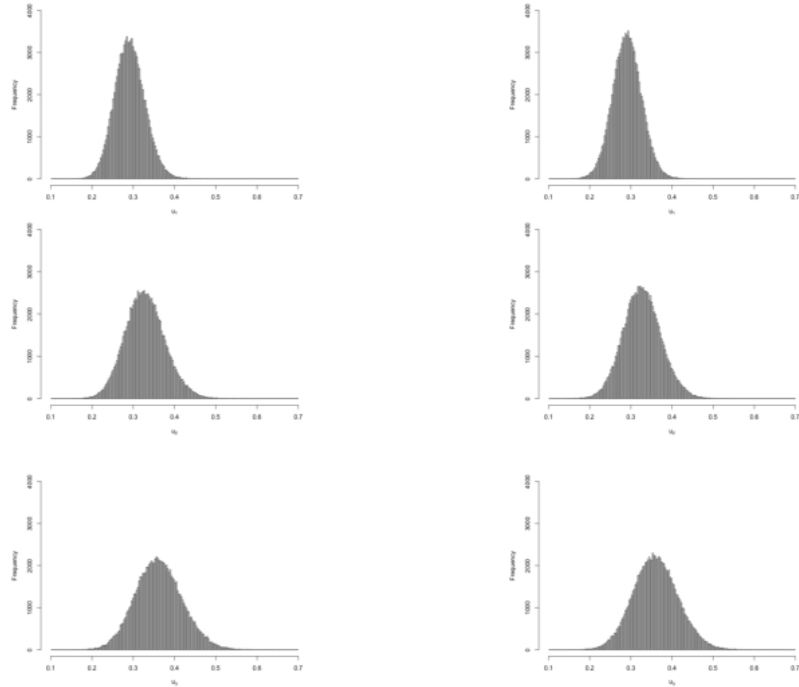


(a) Vasicek model used



(b) Cox-Ingersoll-Ross model used

**Figure 6.7:** Samples of 100,000 possible trajectories of the risk reserve ratio over a period of three years



(a) Vasicek model used

(b) Cox-Ingersoll-Ross model used

**Figure 6.8:** Simulated distribution of the risk reserve ratio after one, two and three years

**Table 6.10:** Descriptive statistics of the simulated risk reserve ratio after one, two and three years

(a) Vasicek model used

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	29.15%	3.64%	0.1766	15.65%	26.65%	29.05%	31.52%	46.27%
$t = 2$	32.86%	4.77%	0.1933	14.62%	29.59%	32.71%	35.95%	55.12%
$t = 3$	36.21%	5.59%	0.1755	14.38%	32.34%	36.04%	39.88%	63.57%

(b) Cox-Ingersoll-Ross model used

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	29.10%	3.52%	0.0720	13.57%	26.71%	29.06%	31.42%	45.15%
$t = 2$	32.73%	4.59%	0.1208	13.63%	29.63%	32.64%	35.74%	54.20%
$t = 3$	36.04%	5.41%	0.1364	14.89%	32.33%	35.91%	39.60%	63.55%

**Table 6.11:** Ratios of minimum Risk-Based Capital of the stock investments and initial gross premium amount over a period of one, two and three years

**(a)** Vasicek model used

$u_{RBC}^{S_h}(0, t)$	$h = 1$	$h = 2$	$h = 3$	Total
$t = 1$	4.18%	1.93%	1.33%	7.44%
$t = 2$	3.97%	1.89%	1.29%	7.15%
$t = 3$	3.80%	1.82%	1.25%	6.86%

**(b)** Cox-Ingersoll-Ross model used

$u_{RBC}^{S_h}(0, t)$	$h = 1$	$h = 2$	$h = 3$	Total
$t = 1$	4.30%	1.93%	1.11%	7.35%
$t = 2$	4.10%	1.89%	1.07%	7.06%
$t = 3$	3.91%	1.82%	1.04%	6.78%

**Table 6.12:** Ratios of minimum Risk-Based Capital of the bond investments and initial gross premium amount over a period of one, two and three years

**(a)** Vasicek model used

$u_{RBC}^{B_i}(0, t)$	$i = 1$	$i = 2$	$i = 3$	$i = 5$	$i = 10$	Total
$t = 1$	-1.25%	0.03%	0.37%	0.56%	0.85%	0.55%
$t = 2$	-0.50%	-0.08%	0.19%	0.38%	0.66%	0.65%
$t = 3$	-0.95%	-0.43%	-0.06%	0.20%	0.48%	-0.76%

**(b)** Cox-Ingersoll-Ross model used

$u_{RBC}^{B_i}(0, t)$	$i = 1$	$i = 2$	$i = 3$	$i = 5$	$i = 10$	Total
$t = 1$	-1.25%	0.16%	0.51%	0.69%	1.01%	1.11%
$t = 2$	-0.49%	-0.05%	0.26%	0.47%	0.78%	0.96%
$t = 3$	-0.96%	-0.45%	-0.04%	0.24%	0.54%	-0.67%

company. However, the Degree of Diversification is always higher than in the case of the single-line insurance company. We remember that the annual rate of return is obtained through the portfolio optimization procedure of the single-line insurance company. As a consequence, now we could have

**Table 6.13:** Ratio of minimum Risk-Based Capital and initial gross premium amount over a period of one, two and three years

**(a)** Vasicek model used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}^{market}(0, t)$	4.22%	2.48%	0.49%

**(b)** Cox-Ingersoll-Ross model used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}^{market}(0, t)$	4.40%	2.60%	0.43%

probably found a better asset allocation that would have further increased the diversification effect.

**Table 6.14:** Degree of Diversification of the risk reserve over a period of one, two and three years

**(a)** Vasicek model used

	$DoD_{RBC}^{market}(0, t)$
$t = 1$	47.18%
$t = 2$	68.22%
$t = 3$	92.01%

**(b)** Cox-Ingersoll-Ross model used

	$DoD_{RBC}^{market}(0, t)$
$t = 1$	48.02%
$t = 2$	67.55%
$t = 3$	92.97%

## 6.2.2 Capital requirements according to the standard formula

We can compare the results according to our model with the results according to the Solvency II standard formula.

Tables 6.15 and 6.16 show the Solvency Capital Requirement for equity

risk, calculated on a single stock investment according to equation (5.8), and the Solvency Capital Requirement for interest rate risk, calculated on a single bond investment according to equation (5.9), as percentages of the initial gross premium amount. We point out that we can make the same comments as in the case of the single-line insurance company. However, the Solvency Capital Requirements are bigger than in the case of the single-line insurance company, because the invested resources are now bigger.

**Table 6.15:** Ratios of Solvency Capital Requirement of the stock investments and initial gross premium amount

**(a)** Vasicek model used

	$h = 1$	$h = 2$	$h = 3$	Total
$u_{SCR}^{S_h}$	8.21%	2.74%	1.49%	12.44%

**(b)** Cox-Ingersoll-Ross model used

	$h = 1$	$h = 2$	$h = 3$	Total
$u_{SCR}^{S_h}$	8.46%	2.74%	1.24%	12.44%

**Table 6.16:** Ratios of Solvency Capital Requirement of the bond investments and initial gross premium amount

	$i = 1$	$i = 2$	$i = 3$	$i = 5$	$i = 10$	Total
$u_{SCR}^{B_i}$	0.88%	1.03%	0.82%	0.77%	1.26%	4.77%

The Solvency Capital Requirement for equity risk and the Solvency Capital Requirement for interest rate risk are described by equations (5.10) and (5.11). The Solvency Capital Requirement for equity risk is the same if we consider the initial values of the stock investments found by using the Vasicek model or Cox-Ingersoll-Ross model. As a result, also the Solvency Capital Requirement for market risk is the same in both the cases.

We point out that the equity risk and interest rate risk are the only sources of market risk we have. Hence, using equation (5.12), the Solvency Capital Requirement for market risk as a percentage of the initial gross premium

amount is found to be:

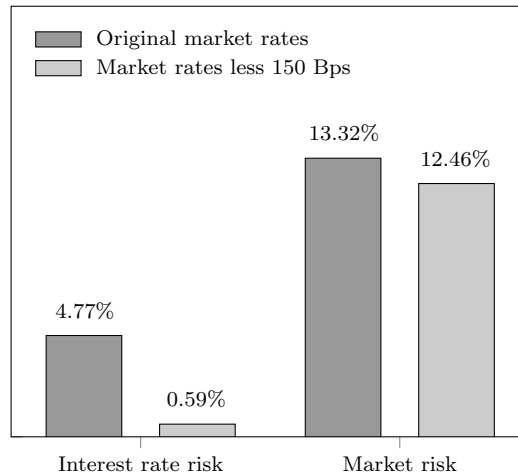
$$u_{SCR}^{market} = \frac{SCR_{market}}{\pi_0} = 13.32\%$$

Hence, the Degree of Diversification of the risk reserve, calculated on the basis of the Solvency Capital Requirement, is found to be:

$$DoD_{SCR}^{market} = \frac{\sum_{h=1}^3 u_{SCR}^{S_h} + \sum_{\{i=1,2,3,5,10\}} u_{SCR}^{B_i} - u_{SCR}^{market}}{\sum_{h=1}^3 u_{SCR}^{S_h} + \sum_{\{i=1,2,3,5,10\}} u_{SCR}^{B_i}} = 22.57\%$$

We point out that we can make the same comments as in the case of the single-line insurance company. However, the Solvency Capital Requirement is bigger than in the case of the single-line insurance company and the Degree of Diversification is the same as in the case of the single-line insurance company.

Figure 6.9 shows a sensitivity analysis on the Solvency Capital Requirement for interest rate risk and on the Solvency Capital Requirement for market risk. We compare the results above with the results obtained after having applied a parallel negative shift of 150 Bps in the interest rate curve. We point out that, after the shift, the Solvency Capital Requirement for interest rate risk is found to be significantly smaller. As a result, there is a non-negligible reduction in the Solvency Capital Requirement for market risk.



**Figure 6.9:** Sensitivity analysis on the ratios of Solvency Capital Requirements and initial gross premium amount

### 6.3 Non-life premium risk

In this section we extend the analysis of the non-life premium risk of the previous chapter to the case of the multi-line insurance company. As a result, the risk reserve is found to be:

$$\tilde{U}_t = \tilde{U}_{t-1} + \left[ (1 + \varphi) \cdot P_t - \tilde{X}_t \right]$$

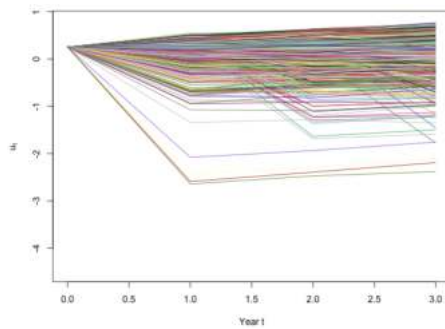
where the total aggregate claim amount, safety loading coefficient and risk premium amount are described by equations (6.1), (6.2) and (6.3).

The equation of the risk reserve ratio is equal to the equation in section 5.4, even though the values of the parameters are different.

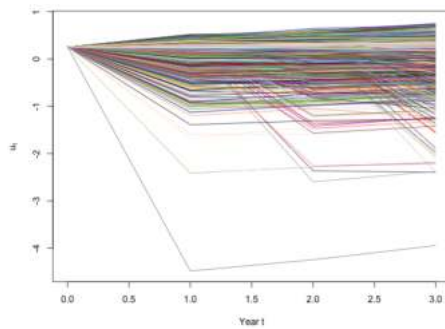
We now consider the total aggregate claim amounts given by the collective risk model. Figure 6.10 shows the simulated risk reserve ratio over a period of three years. Figure 6.11 shows the resulting distribution of the risk reserve ratio after one, two and three years, and Table 6.17 shows some elements of descriptive statistics. We point out that we can make the same comments as in the case of the single-line insurance company. However, the mean increases over time, because the underwriting result is now big enough to compensate the increase in the gross premium amount. The multi-line insurance company has a different risk-return profile from the single-line insurance company, so that the risk of rapid growth is now avoided. The mean and standard deviation are bigger than in the case of the single-line insurance company. Moreover, the mean in the case of the Gumbel copula is similar to the case of the Gaussian copula and the standard deviation is bigger in the case of the Gumbel copula rather than in the case of the Gaussian copula, because the upper tail dependence of the Gumbel copula is higher.

Figure 6.12 shows the resulting distribution of the annual net cash flows after one, two and three years, according to equation (1.7), and Table 6.18 shows some elements of descriptive statistics. We point out that we can make the same comments as in the case of the single-line insurance company. However, the mean, standard deviation and absolute value of the skewness are higher than in the case of the single-line insurance company.



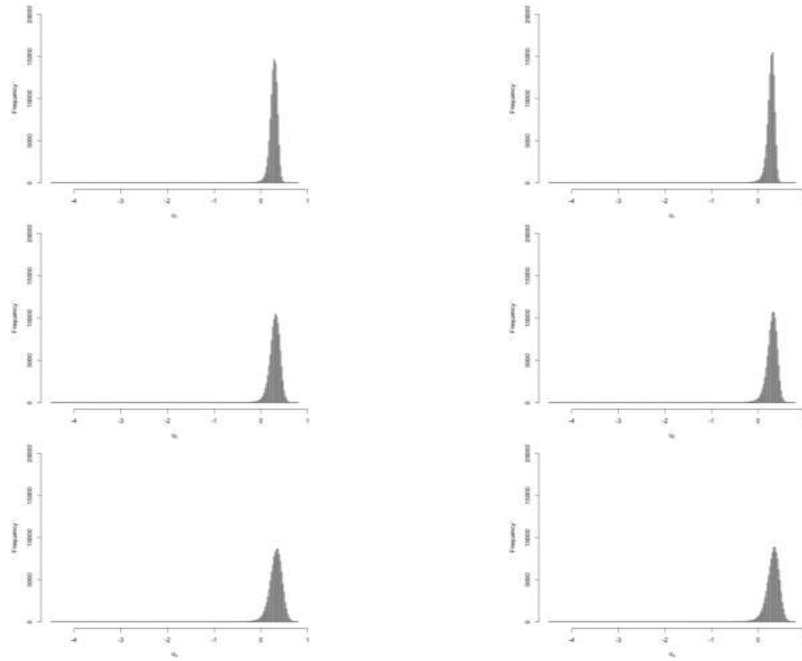


**(a)** Gaussian copula used



**(b)** Gumbel copula used

**Figure 6.10:** Samples of 100,000 possible trajectories of the risk reserve ratio over a period of three years



(a) Gaussian copula used

(b) Gumbel copula used

**Figure 6.11:** Simulated distribution of the risk reserve ratio after one, two and three years

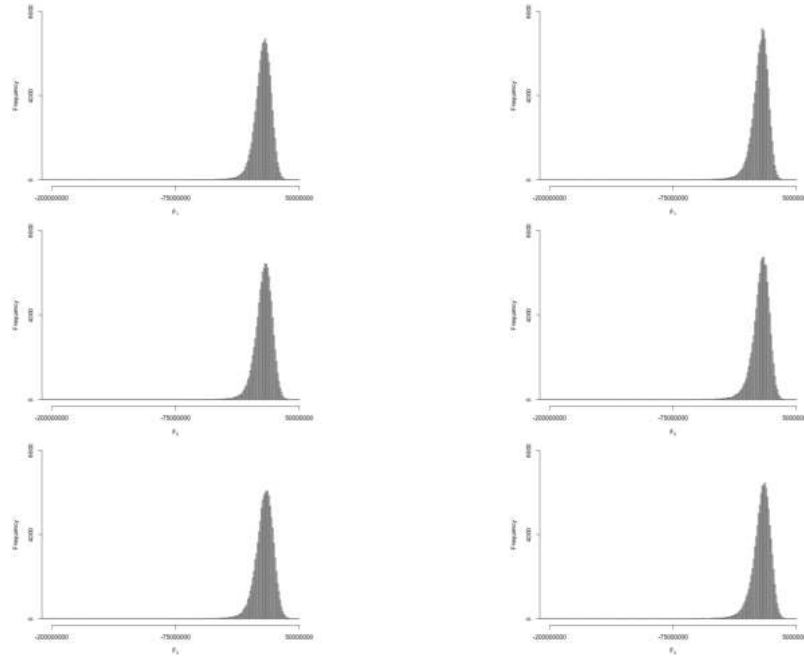
**Table 6.17:** Descriptive statistics of the simulated risk reserve ratio after one, two and three years

(a) Gaussian copula used

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	27.32%	8.37%	-2.9481	-264.25%	22.98%	28.07%	32.73%	54.90%
$t = 2$	29.58%	11.41%	-1.7611	-247.27%	23.34%	30.50%	37.13%	63.69%
$t = 3$	31.69%	13.56%	-1.2773	-238.17%	23.98%	32.76%	40.70%	76.30%

(b) Gumbel copula used

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	27.34%	8.75%	-4.7043	-448.20%	23.31%	28.44%	32.76%	52.18%
$t = 2$	29.56%	11.88%	-2.7949	-424.45%	23.58%	30.87%	37.23%	64.74%
$t = 3$	31.73%	14.09%	-2.1630	-394.56%	24.30%	33.14%	40.94%	74.71%



(a) Gaussian copula used

(b) Gumbel copula used

**Figure 6.12:** Simulated distribution of the annual net cash flows after one, two and three years (100,000 simulations)

**Table 6.18:** Descriptive statistics of the simulated annual net cash flows after one, two and three years (amounts in millions)

(a) Gaussian copula used

$F_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	13.20	8.79	-2.9481	-293.12	8.64	13.99	18.88	42.17
$t = 2$	13.93	9.01	-2.0912	-206.32	9.19	14.80	19.81	44.13
$t = 3$	14.58	9.40	-1.7415	-207.49	9.57	15.47	20.77	43.29

(b) Gumbel copula used

$F_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	13.21	9.19	-4.7043	-486.38	8.99	14.38	18.91	39.32
$t = 2$	13.88	9.37	-3.2831	-311.28	9.48	15.05	19.82	39.70
$t = 3$	14.65	9.83	-3.5019	-309.35	10.02	15.85	20.85	41.56

### 6.3.1 Capital requirements according to the model

We are now able to calculate the capital requirements over a period of one, two or three years.

Table 6.19 shows the minimum Risk-Based Capital of each line of business as a percentage of the initial gross premium amount over a period of one, two and three years. We point out that the minimum Risk-Based Capital of each line of business increases over time, except for the MOD line of business. Moreover, the capital requirement of the GTPL line of business is the biggest, i.e. the GTPL line of business is the riskiest business, and the capital requirement of the MOD line of business is the smallest, i.e. the MOD line of business is the less risky one.

**Table 6.19:** Ratios of minimum Risk-Based Capital of the lines of business and initial gross premium amount over a period of one, two and three years

**(a)** Vasicek model used

$u_{RBC}^{LoB}(0, t)$	$LoB = MTPL$	$LoB = MOD$	$LoB = GTPL$	Total
$t = 1$	13.17%	0.40%	20.36%	33.94%
$t = 2$	17.94%	-0.64%	27.68%	44.98%
$t = 3$	21.76%	-2.00%	32.78%	52.54%

**(b)** Cox-Ingersoll-Ross model used

$u_{RBC}^{LoB}(0, t)$	$LoB = MTPL$	$LoB = MOD$	$LoB = GTPL$	Total
$t = 1$	13.17%	0.40%	20.36%	33.94%
$t = 2$	17.94%	-0.64%	27.69%	44.99%
$t = 3$	21.77%	-2.00%	32.80%	52.56%

Table 6.20 shows the overall minimum Risk-Based Capital as a percentage of the initial gross premium amount over a period of one, two and three years. We point out that we can make the same comments as in the case of the single-line insurance company. However, the minimum Risk-Based Capital is always bigger than in the case of the single-line insurance company, because the distribution of the total aggregate claim amount is riskier. Moreover, the capital requirements are bigger in the case of the Gumbel copula rather than

in the case of the Gaussian copula, because the upper tail dependence of the Gumbel copula is higher.

**Table 6.20:** Ratio of minimum Risk-Based Capital and initial gross premium amount over a period of one, two and three years

**(a)** Vasicek model and Gaussian copula used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}^{non-life}(0, t)$	27.99%	34.99%	40.34%

**(b)** Vasicek model and Gumbel copula used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}^{non-life}(0, t)$	31.40%	39.85%	43.98%

**(c)** Cox-Ingersoll-Ross model and Gaussian copula used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}^{non-life}(0, t)$	27.99%	35.00%	40.35%

**(d)** Cox-Ingersoll-Ross model and Gumbel copula used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}^{non-life}(0, t)$	31.40%	39.86%	44.00%

The Degree of Diversification of the risk reserve, calculated on the basis of the minimum Risk-Based Capital, is given by:

$$DoD_{RBC}^{non-life}(0, t) = \frac{u_{RBC}^{MTPL}(0, t) + u_{RBC}^{MOD}(0, t) + u_{RBC}^{GTPL}(0, t) - u_{RBC}^{non-life}(0, t)}{u_{RBC}^{MTPL}(0, t) + u_{RBC}^{MOD}(0, t) + u_{RBC}^{GTPL}(0, t)}$$

Table 6.21 shows the Degree of Diversification of the risk reserve over a period of one, two and three years. The Degree of Diversification is not extremely high, but it is significant. It is always lower in the case of the Gumbel copula rather than in the case of the Gaussian copula, because the diversification effect of the Gumbel copula is worse.

The differences between Tables 6.19a and 6.19b, 6.20a and 6.20c, 6.20b and 6.20d, 6.21a and 6.21c, or 6.21b and 6.21d are only due to the different theoretical rates used for the minimum Risk-Based Capital calculation.

**Table 6.21:** Degree of Diversification of the risk reserve over a period of one, two and three years

**(a)** Vasicek model and Gaussian copula used

	$DoD_{RBC}^{non-life}(0, t)$
$t = 1$	17.51%
$t = 2$	22.20%
$t = 3$	23.22%

**(b)** Vasicek model and Gumbel copula used

	$DoD_{RBC}^{non-life}(0, t)$
$t = 1$	7.47%
$t = 2$	11.40%
$t = 3$	16.30%

**(c)** Cox-Ingersoll-Ross model and Gaussian copula used

	$DoD_{RBC}^{non-life}(0, t)$
$t = 1$	17.52%
$t = 2$	22.21%
$t = 3$	23.23%

**(d)** Cox-Ingersoll-Ross model and Gumbel copula used

	$DoD_{RBC}^{non-life}(0, t)$
$t = 1$	7.48%
$t = 2$	11.41%
$t = 3$	16.30%

### 6.3.2 Capital requirements according to the standard formula

We can compare the results according to our model with the results according to the Solvency II standard formula.

Now, not only does the insurance company work in the MTPL line of business, but also in the MOD and GTPL lines of business. Table 6.22 shows the Solvency Capital Requirement for non-life premium and reserve risk, calculated on a single line of business according to equation (2.1), as a percentage of the initial gross premium amount. Even though the GTPL line of business is the riskiest business, the capital requirement of the MTPL line of business is the biggest, because the volume measure of the MTPL line of business is the highest. On the other hand, the capital requirement of the MOD line of business is the smallest. Furthermore, the capital requirement of the MOD line of business is bigger in the case of the standard formula rather than in our model, but the capital requirement of the MTPL line of business is smaller in the case of the standard formula rather than in our model, and the capital requirement of the GTPL line of business is much smaller in the case of the standard formula rather than in our model. This is because the standard formula was calibrated on different distributions than the distributions of our model and because we have different assumptions.

**Table 6.22:** Ratios of Solvency Capital Requirement of the lines of business and initial gross premium amount

	$LoB = MTPL$	$LoB = MOD$	$LoB = GTPL$	Total
$u_{SCR}^{LoB}$	15.76%	6.30%	11.03%	33.09%

The standard deviation for non-life premium and reserve risk is found to be:

$$\sigma_{nl} = 8.5\%$$

The volume measure for non-life premium and reserve risk is found to be:

$$V_{nl} = \pi_1 = 105,060,000$$

Using equation (2.1), the Solvency Capital Requirement for non-life premium and reserve risk as a percentage of the initial gross premium amount is found to be:

$$u_{SCR}^{nl\ prem\ res} = \frac{SCR_{nl\ prem\ res}}{\pi_0} = 26.79\%$$

We point out that the capital requirement is smaller in the case of the standard formula rather than in our model. This is because the capital requirement of the GTPL line of business is much smaller in the case of the standard formula rather than in our model and because the standard formula is based on a different aggregation procedure than the aggregation through the copula functions. Moreover, the Solvency Capital Requirement is smaller than in the case of the single-line insurance company, since the standard deviation for non-life premium and reserve risk is smaller as well, because of a diversification effect between the lines of business.

Furthermore, we point out that the non-life premium risk is the only source of non-life underwriting risk we have. Hence, the Solvency Capital Requirement for non-life underwriting risk is found to be the same, namely:

$$u_{SCR}^{non-life} = \frac{SCR_{non-life}}{\pi_0} = 26.79\%$$

In conclusion, the Degree of Diversification of the risk reserve, calculated on the basis of the Solvency Capital Requirement, is found to be:

$$DoD_{SCR}^{non-life} = \frac{u_{SCR}^{MTPL} + u_{SCR}^{MOD} + u_{SCR}^{GTPL} - u_{SCR}^{non-life}}{u_{SCR}^{MTPL} + u_{SCR}^{MOD} + u_{SCR}^{GTPL}} = 19.05\%$$

The Degree of Diversification is significant, because the correlation coefficients of the lines of business are not so high. It is slightly bigger than in the case of the Gaussian copula and it is significantly bigger than in the case of the Gumbel copula.



## 6.4 Market and non-life premium risk

In this section we extend the analysis of the market and non-life premium risk of the previous chapter to the case of the multi-line insurance company, either according to the integrated model or stand-alone model. Once again, we keep the annual rate of return that we have previously obtained.

### 6.4.1 Integrated model

We now deal with the integrated model, considering equations (1.4) and (1.15) with the parameter and amount values described in this chapter.

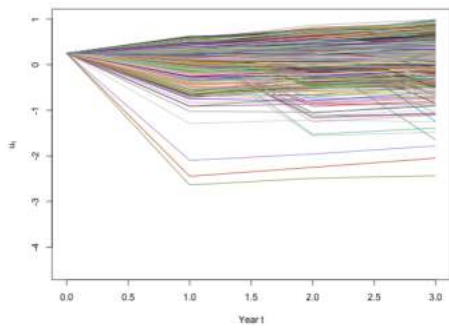
Figure 6.13 shows the simulated risk reserve ratio over a period of three years. Figure 6.14 shows the resulting distribution of the risk reserve ratio after one, two and three years, and Table 6.23 shows some elements of descriptive statistics. We point out that we can make the same comments as in the case of the single-line insurance company. However, the mean and standard deviation are bigger than in the case of the single-line insurance company. Moreover, the mean in the case of the Gumbel copula is similar to the case of the Gaussian copula and the standard deviation is bigger in the case of the Gumbel copula rather than in the case of the Gaussian copula, because the upper tail dependence of the Gumbel copula is higher.

The annual net cash flows are the same as in presence of the non-life premium risk only.

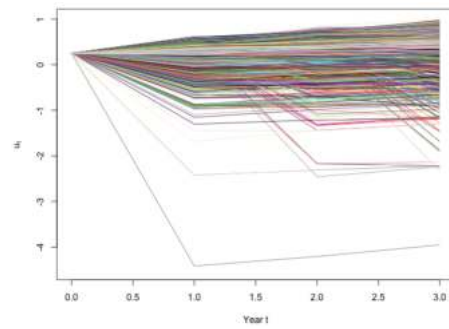
### 6.4.2 Capital requirements according to the integrated model

We are now able to calculate the capital requirements over a period of one, two or three years.

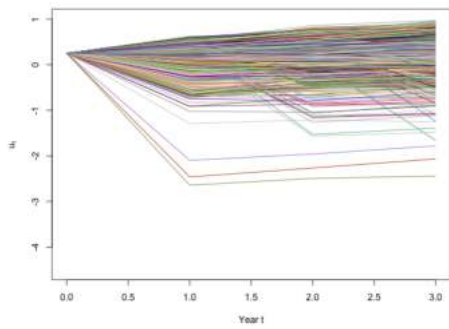
Table 6.24 shows the minimum Risk-Based Capital as a percentage of the initial gross premium amount over a period of one, two and three years. We point out that we can make the same comments as in the case of the single-line insurance company. However, in Table 6.24c, the minimum Risk-Based Capital is smaller over a period of one and two years and bigger over a period



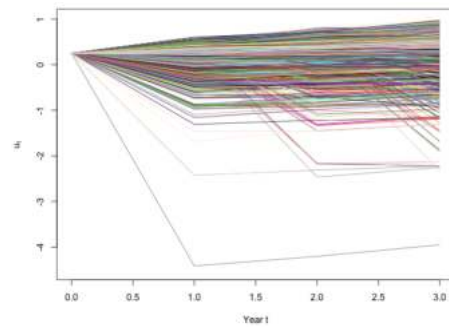
(a) Vasicek model and Gaussian copula used



(b) Vasicek model and Gumbel copula used

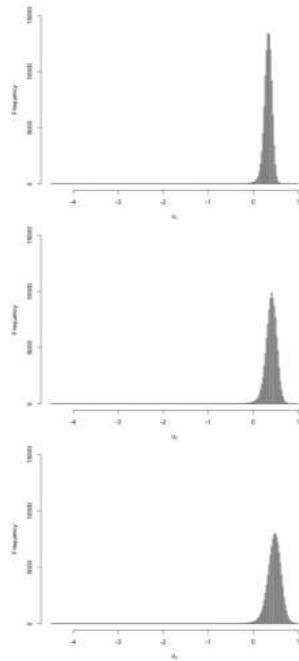


(c) Cox-Ingersoll-Ross model and Gaussian copula used

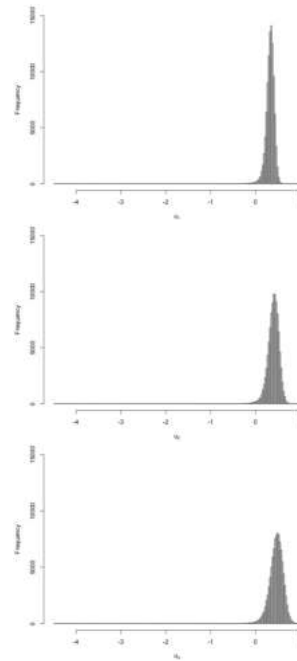


(d) Cox-Ingersoll-Ross model and Gumbel copula used

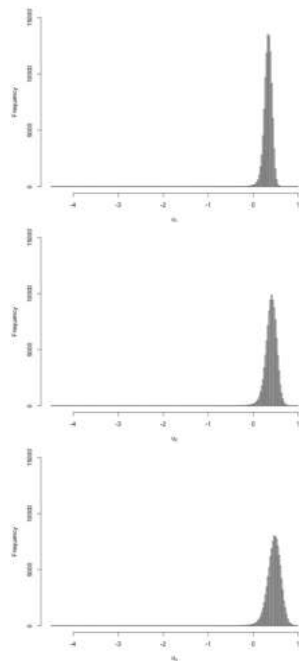
**Figure 6.13:** Samples of 100,000 possible trajectories of the risk reserve ratio over a period of three years



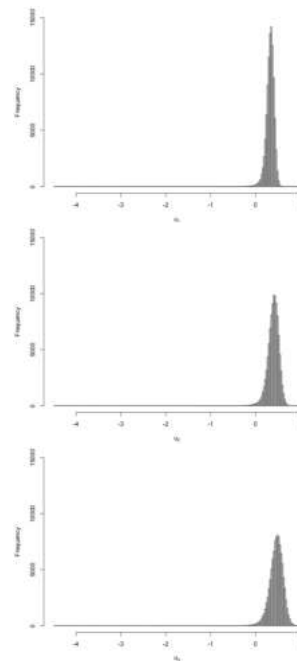
**(a)** Vasicek model and Gaussian copula used



**(b)** Vasicek model and Gumbel copula used



**(c)** Cox-Ingersoll-Ross model and Gaussian copula used



**(d)** Cox-Ingersoll-Ross model and Gumbel copula used

**Figure 6.14:** Simulated distribution of the risk reserve ratio after one, two and three years

**Table 6.23:** Descriptive statistics of the simulated risk reserve ratio after one, two and three years

**(a)** Vasicek model and Gaussian copula used

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	32.67%	9.12%	-2.2674	-263.93%	27.65%	33.30%	38.56%	63.46%
$t = 2$	39.79%	12.49%	-1.3734	-249.04%	32.62%	40.53%	48.04%	86.48%
$t = 3$	46.34%	14.97%	-0.9975	-243.46%	37.56%	47.16%	56.28%	98.64%

**(b)** Vasicek model and Gumbel copula used

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	32.69%	9.49%	-3.7059	-441.50%	27.90%	33.60%	38.63%	61.55%
$t = 2$	39.77%	12.94%	-2.2328	-420.77%	32.85%	40.87%	48.17%	84.30%
$t = 3$	46.37%	15.44%	-1.7362	-395.29%	37.81%	47.49%	56.47%	99.58%

**(c)** Cox-Ingersoll-Ross model and Gaussian copula used

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	32.62%	9.07%	-2.3172	-264.34%	27.64%	33.26%	38.49%	63.21%
$t = 2$	39.67%	12.42%	-1.4041	-249.46%	32.55%	40.43%	47.88%	85.76%
$t = 3$	46.17%	14.89%	-1.0184	-244.21%	37.45%	47.02%	56.06%	96.87%

**(d)** Cox-Ingersoll-Ross model and Gumbel copula used

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	32.64%	9.44%	-3.7663	-441.42%	27.89%	33.56%	38.55%	60.22%
$t = 2$	39.64%	12.87%	-2.2709	-420.52%	32.78%	40.76%	47.99%	82.56%
$t = 3$	46.21%	15.37%	-1.7625	-395.10%	37.71%	47.33%	56.27%	97.58%

of three years than in Table 6.24a. In Table 6.24d, the minimum Risk-Based Capital is smaller over a period of two years and bigger over a period of one and three years than in Table 6.24b. The minimum Risk-Based Capital is always bigger than in the case of the single-line insurance company, because the distribution of the total aggregate claim amount is riskier. Moreover, the capital requirements are bigger in the case of the Gumbel copula rather than in the case of the Gaussian copula, because the upper tail dependence of the Gumbel copula is higher.

**Table 6.24:** Ratio of minimum Risk-Based Capital and initial gross premium amount over a period of one, two and three years

**(a)** Vasicek model and Gaussian copula used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}(0, t)$	23.32%	25.98%	26.90%

**(b)** Vasicek model and Gumbel copula used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}(0, t)$	26.59%	30.55%	30.81%

**(c)** Cox-Ingersoll-Ross model and Gaussian copula used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}(0, t)$	23.32%	25.94%	27.16%

**(d)** Cox-Ingersoll-Ross model and Gumbel copula used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}(0, t)$	26.63%	30.53%	30.87%

The partial Degree of Diversification of the risk reserve is calculated according to equation (5.13) and the total Degree of Diversification of the risk reserve is given by:

$$DoD_{RBC}^{total}(0, t) = \frac{u\text{-sum}_{RBC}^{market}(0, t) + u\text{-sum}_{RBC}^{non-life}(0, t) - u_{RBC}(0, t)}{u\text{-sum}_{RBC}^{market}(0, t) + u\text{-sum}_{RBC}^{non-life}(0, t)}$$

where:

$$u\text{-sum}_{RBC}^{\text{market}}(0, t) = \sum_{h=1}^3 u_{RBC}^{S_h}(0, t) + \sum_{\{i=1,2,3,5,10\}} u_{RBC}^{B_i}(0, t)$$

and:

$$u\text{-sum}_{RBC}^{\text{non-life}}(0, t) = u_{RBC}^{MTPL}(0, t) + u_{RBC}^{MOD}(0, t) + u_{RBC}^{GTPL}(0, t)$$

Table 6.25 shows the partial and total Degrees of Diversification of the risk reserve over a period of one, two and three years. Once again, we can make the same comments as in the case of the single-line insurance company. However, in Table 6.25c (Table 6.25d), the partial and total Degrees of Diversification are higher over a period of one and two years and lower over a period of three years than in Table 6.25a (Table 6.25b). Almost always, the partial Degree of Diversification is higher in the case of the Gaussian copula rather than in the case of the single-line insurance company, but it is lower in the case of the Gumbel copula rather than in the case of the single-line insurance company. The total Degree of Diversification is always higher than in the case of the single-line insurance company. Moreover, the partial and total Degrees of Diversification are lower in the case of the Gumbel copula rather than in the case of the Gaussian copula, because the diversification effect of the Gumbel copula is worse.

### 6.4.3 Stand-alone model

We now deal with the stand-alone model, considering equations (5.14) and (5.15) with the parameter and amount values described in this chapter.

Figure 6.15 shows the simulated risk reserve ratio over a period of three years. Figure 6.16 shows the resulting distribution of the risk reserve ratio after one, two and three years, and Table 6.26 shows some elements of descriptive statistics. We point out that we can make the same comments as in the case of the integrated model for the multi-line insurance company and/or in the case of the single-line insurance company.

Figure 6.17 shows the QQ-plot of the integrated and stand-alone distributions of the risk reserve ratios after one, two and three years. Once again, we can make the same comments as in the case of the single-line insurance company.

**Table 6.25:** Degree of Diversification of the risk reserve over a period of one, two and three years

**(a)** Vasicek model and Gaussian copula used

	$DoD_{RBC}(0, t)$	$DoD_{RBC}^{total}(0, t)$
$t = 1$	27.60%	44.38%
$t = 2$	30.66%	50.77%
$t = 3$	34.12%	54.14%

**(b)** Vasicek model and Gumbel copula used

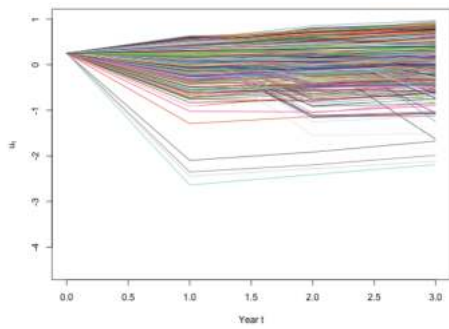
	$DoD_{RBC}(0, t)$	$DoD_{RBC}^{total}(0, t)$
$t = 1$	25.36%	36.58%
$t = 2$	27.82%	42.11%
$t = 3$	30.71%	47.46%

**(c)** Cox-Ingersoll-Ross model and Gaussian copula used

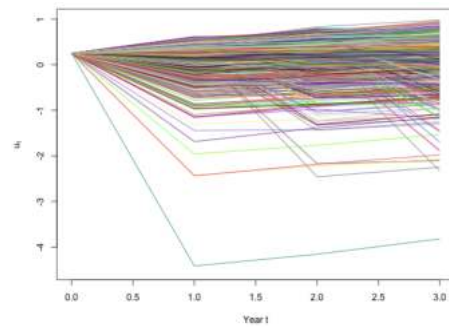
	$DoD_{RBC}(0, t)$	$DoD_{RBC}^{total}(0, t)$
$t = 1$	28.01%	45.00%
$t = 2$	31.01%	51.06%
$t = 3$	33.41%	53.72%

**(d)** Cox-Ingersoll-Ross model and Gumbel copula used

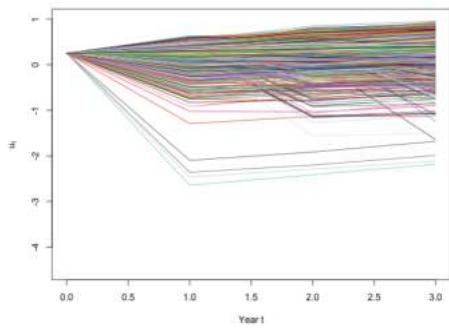
	$DoD_{RBC}(0, t)$	$DoD_{RBC}^{total}(0, t)$
$t = 1$	25.61%	37.19%
$t = 2$	28.10%	42.40%
$t = 3$	30.52%	47.39%



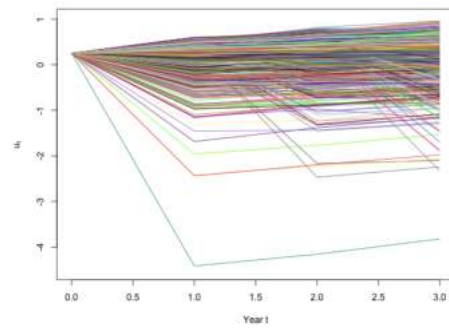
(a) Vasicek model and Gaussian copula used



(b) Vasicek model and Gumbel copula used



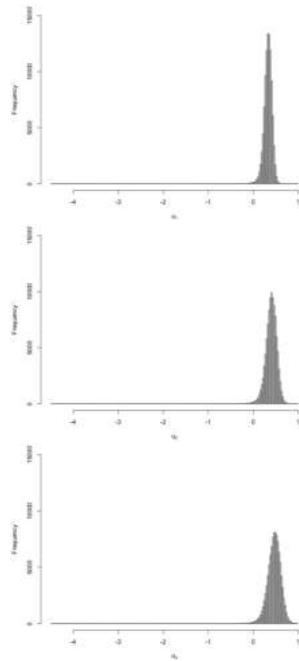
(c) Cox-Ingersoll-Ross model and Gaussian copula used



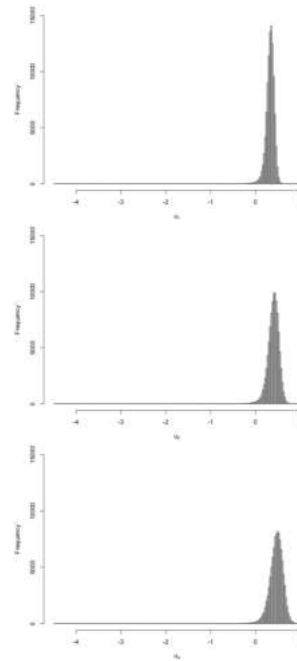
(d) Cox-Ingersoll-Ross model and Gumbel copula used

**Figure 6.15:** Samples of 100,000 possible trajectories of the risk reserve ratio over a period of three years

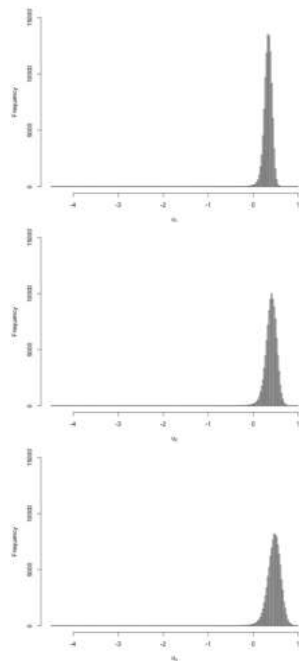




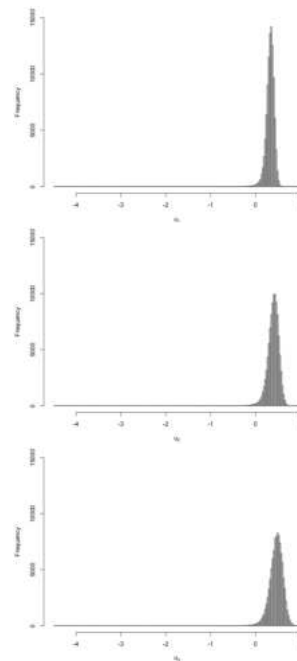
**(a)** Vasicek model and Gaussian copula used



**(b)** Vasicek model and Gumbel copula used



**(c)** Cox-Ingersoll-Ross model and Gaussian copula used



**(d)** Cox-Ingersoll-Ross model and Gumbel copula used

**Figure 6.16:** Simulated distribution of the risk reserve ratio after one, two and three years

**Table 6.26:** Descriptive statistics of the simulated risk reserve ratio after one, two and three years

**(a)** Vasicek model and Gaussian copula used

$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	32.67%	9.12%	-2.2674	-263.93%	27.65%	33.30%	38.56%	63.46%
$t = 2$	39.79%	12.37%	-1.3819	-241.11%	32.70%	40.55%	47.98%	85.41%
$t = 3$	46.34%	14.68%	-1.0039	-219.16%	37.72%	47.19%	56.10%	96.80%

**(b)** Vasicek model and Gumbel copula used

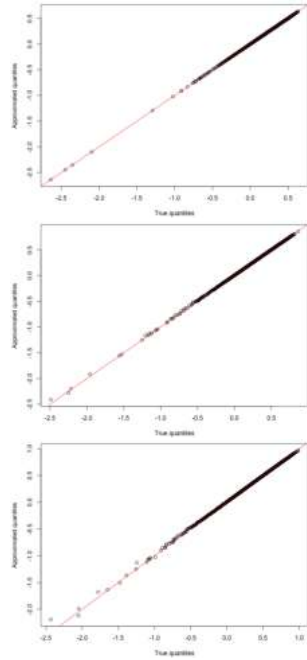
$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	32.69%	9.49%	-3.7059	-441.50%	27.90%	33.60%	38.63%	61.55%
$t = 2$	39.77%	12.80%	-2.2275	-415.71%	32.92%	40.87%	48.10%	82.92%
$t = 3$	46.38%	15.14%	-1.7423	-382.13%	37.98%	47.52%	56.29%	97.80%

**(c)** Cox-Ingersoll-Ross model and Gaussian copula used

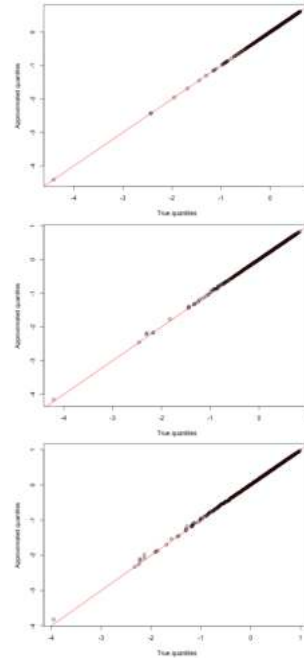
$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	32.62%	9.07%	-2.3172	-264.34%	27.64%	33.26%	38.49%	63.21%
$t = 2$	39.66%	12.30%	-1.4115	-241.43%	32.63%	40.44%	47.80%	84.72%
$t = 3$	46.17%	14.61%	-1.0211	-218.34%	37.62%	47.04%	55.89%	95.27%

**(d)** Cox-Ingersoll-Ross model and Gumbel copula used

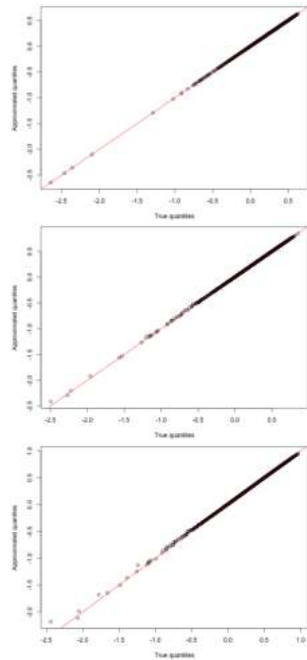
$u_t$	Mean	St.Dev.	Skew.	Min.	1 <sub>st</sub> Qu.	Median	3 <sub>rd</sub> Qu.	Max.
$t = 1$	32.64%	9.44%	-3.7663	-441.42%	27.89%	33.56%	38.55%	60.22%
$t = 2$	39.64%	12.74%	-2.2673	-415.80%	32.85%	40.78%	47.93%	81.27%
$t = 3$	46.21%	15.07%	-1.7659	-382.17%	37.89%	47.35%	56.08%	96.00%



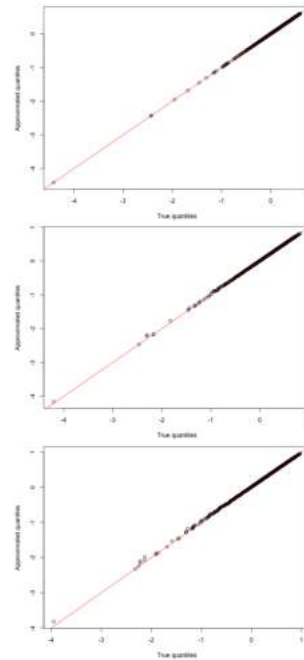
**(a)** Vasicek model and Gaussian copula used



**(b)** Vasicek model and Gumbel copula used



**(c)** Cox-Ingersoll-Ross model and Gaussian copula used



**(d)** Cox-Ingersoll-Ross model and Gumbel copula used

**Figure 6.17:** QQ-plot of the integrated and stand-alone distributions of the risk reserve ratios after one, two and three years

### 6.4.4 Capital requirements according to the stand-alone model

We are now able to calculate the capital requirements over a period of one, two or three years.

Table 6.27 shows the minimum Risk-Based Capital as a percentage of the initial gross premium amount over a period of one, two and three years. We point out that we can make the same comments as in the case of the integrated model for the multi-line insurance company and/or in the case of the single-line insurance company.

**Table 6.27:** Ratio of minimum Risk-Based Capital and initial gross premium amount over a period of one, two and three years

**(a)** Vasicek model and Gaussian copula used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}(0, t)$	23.32%	25.71%	26.19%

**(b)** Vasicek model and Gumbel copula used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}(0, t)$	26.59%	30.05%	29.83%

**(c)** Cox-Ingersoll-Ross model and Gaussian copula used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}(0, t)$	23.32%	25.63%	26.24%

**(d)** Cox-Ingersoll-Ross model and Gumbel copula used

	$t = 1$	$t = 2$	$t = 3$
$u_{RBC}(0, t)$	26.63%	30.05%	29.82%

Table 6.28 shows the partial and total Degrees of Diversification of the risk reserve over a period of one, two and three years. Once again, we can make the same comments as in the case of the integrated model for the multi-line insurance company and/or in the case of the single-line insurance company.

**Table 6.28:** Degree of Diversification of the risk reserve over a period of one, two and three years

**(a)** Vasicek model and Gaussian copula used

	$DoD_{RBC}(0, t)$	$DoD_{RBC}^{total}(0, t)$
$t = 1$	27.60%	44.38%
$t = 2$	31.38%	51.28%
$t = 3$	35.85%	55.34%

**(b)** Vasicek model and Gumbel copula used

	$DoD_{RBC}(0, t)$	$DoD_{RBC}^{total}(0, t)$
$t = 1$	25.36%	36.58%
$t = 2$	29.01%	43.06%
$t = 3$	32.91%	49.13%

**(c)** Cox-Ingersoll-Ross model and Gaussian copula used

	$DoD_{RBC}(0, t)$	$DoD_{RBC}^{total}(0, t)$
$t = 1$	28.01%	45.00%
$t = 2$	31.84%	51.65%
$t = 3$	35.65%	55.28%

**(d)** Cox-Ingersoll-Ross model and Gumbel copula used

	$DoD_{RBC}(0, t)$	$DoD_{RBC}^{total}(0, t)$
$t = 1$	25.61%	37.19%
$t = 2$	29.23%	43.31%
$t = 3$	32.88%	49.18%

### 6.4.5 Capital requirements according to the standard formula

We can compare the results according to our model with the results according to the Solvency II standard formula.

Using equation (5.16), the Solvency Capital Requirement for market and non-life underwriting risk as a percentage of the initial gross premium amount is found to be:

$$u_{SCR} = \frac{SCR}{\pi_0} = 32.77\%$$

Hence, the Degree of Diversification of the risk reserve, calculated on the basis of the Solvency Capital Requirement and on the level of risk modules, is found to be:

$$DoD_{SCR} = \frac{u_{SCR}^{market} + u_{SCR}^{non-life} - u_{SCR}}{u_{SCR}^{market} + u_{SCR}^{non-life}} = 18.31\%$$

and, the Degree of Diversification of the risk reserve, calculated on the basis of the Solvency Capital Requirement and on the level of risk sub-modules, is found to be:

$$DoD_{SCR}^{total} = \frac{u-sum_{SCR}^{market} + u-sum_{SCR}^{non-life} - u_{SCR}}{u-sum_{SCR}^{market} + u-sum_{SCR}^{non-life}} = 34.86\%$$

where:

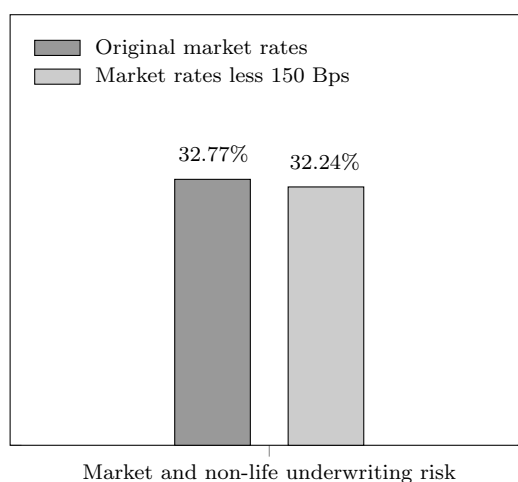
$$u-sum_{SCR}^{market} = \sum_{h=1}^3 u_{SCR}^{S_h} + \sum_{\{i=1,2,3,5,10\}} u_{SCR}^{B_i}$$

and:

$$u-sum_{SCR}^{non-life} = u_{SCR}^{MTPL} + u_{SCR}^{MOD} + u_{SCR}^{GTPL}$$

We point out that we can make the same comments as in the case of the single-line insurance company. However, the Solvency Capital Requirement is smaller than in the case of the single-line insurance company, and the partial and total Degrees of Diversification are bigger than in the case of the single-line insurance company.

Figure 6.18 shows a sensitivity analysis on the Solvency Capital Requirement for market and non-life underwriting risk. We compare the result above with the result obtained after having applied a parallel negative shift of 150 Bps in the interest rate curve. We point out that, after the shift, the Solvency Capital Requirement for market and non-life underwriting risk is found to be smaller, because the Solvency Capital Requirement for interest rate risk is found to be significantly smaller.



**Figure 6.18:** Sensitivity analysis on the ratio of Solvency Capital Requirement and initial gross premium amount

# Conclusion

Nowadays, risk management in insurance companies is more and more important. We showed that not only do insurance companies face underwriting risk, but also market risk, because the inversion of the production cycle implies that insurance companies have a lot of resources to be invested. Market risk is highly relevant in life insurance companies, because a lot of life insurance contracts are alternative forms of investments. Nevertheless, market risk is also quite relevant in non-life insurance companies. As a matter of fact, we showed that the resources produced by the insurance business are invested and consequently they create a risk. Going more into detail, we pointed out that insurance companies must care about equity and interest rate risks, that may affect both assets and liabilities. Moreover, we showed that the interest rate immunization is relevant to ensure that changes in interest rates do not affect the value of the portfolio.

In this thesis we described a way of modeling the distributions of the annual rate of return and aggregate claim amount, in order to calculate the capital requirements for market and non-life premium risk. We showed that the capital requirements, according to our model, are usually smaller than in the case of the standard formula. Firstly, this is because the standard formula was calibrated on riskier distributions than the distributions of our model and the correlation assumptions are more favourable in the case of our model rather than in the standard formula. Moreover, we applied a portfolio optimization strategy to minimize the capital requirements according to our model. We thus compensated the reduction of expected return with a reduction of capital requirements. Furthermore, the interest rate curve, according to our model, was estimated by Treasury rates in the United



States, assuming that the credit risk and currency risk were absent. As a consequence, the capital requirements for market risk, according to our model, should be bigger. In conclusion, the interest rate curve is higher than the EIOPA risk-free interest rate curve. Hence, we showed that the capital requirement for interest rate risk, according to the standard formula, becomes smaller if we apply a parallel negative shift of 150 Bps in the interest rate curve.

We also pointed out that it is useful to isolate the effect of a single source of risk, if we want to better understand this effect. As a result, we studied separately market risk and non-life premium risk. By doing so, we produced an approximation with respect to the integrated model, but we showed that it is quite small. We also showed that a riskier bond portfolio implies bigger capital requirements.

Finally, we described a way of modeling the distribution of the aggregate claim amount of a multi-line insurance company, in order to calculate the capital requirements for market and non-life premium risk. We used Gaussian copulas or Gumbel copulas to describe the dependence structure of the lines of business. We thus showed that the capital requirements for non-life premium risk, according to our model, are always bigger than in the case of the single-line insurance company, because the distribution of the total aggregate claim amount is riskier. Moreover, we showed that the capital requirements for non-life premium risk, according to our model, are bigger in the case of the Gumbel copula rather than in the case of the Gaussian copula, because the upper tail dependence of the Gumbel copula is higher. On the other hand, we showed that the capital requirement for non-life premium risk, according to the standard formula, is smaller than in the case of the single-line insurance company, because of a diversification effect between the lines of business.

Further studies can regard reinsurance, the calculation of the safety loading coefficient, or the analysis of other sources of risk. Reinsurance has a cost, but it can reduce risks and capital requirements. As a consequence, reinsurance can be combined with an increase in the riskiness of the investments, that produces an increase in the expected return. Reinsurance

can thus be used as an alternative portfolio optimization strategy. The safety loading coefficient can be linked to the volatility of the portfolio or amount of capital requirements. The higher the risk is, the higher the safety loading coefficient should be. In conclusion, we remember that our model can be accounted as a partial internal model, because we did not consider all the sources of risk. A full internal model shall consider all the sources of risk.

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