LOOKBACK ENDOWMENT ASSURANCE

FOR MUTUAL FUND INVESTMENT

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ABSTRACT

In the insurance market, there are many products aimed at offering the investor protection against the intrinsic risks of investing; nevertheless, none of these attacks integrally one of their main concerns: the security of their family's patrimony on the event of their death. We design a product which offers this protection; being an endowment assurance that pays the difference between the maximum value attained by a fund and its value at the moment of payment. The product is priced using both the Black and Scholes model and Monte Carlo simulation for a series of European lookback puts. We conclude the product should be sold under the net single premium calculated by the latter approach.

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INTRODUCTION.

There are many products in the insurance market that aim at offering the investor a protection against the intrinsic risks of investing. Nevertheless, none of these products attacks integrally one of the investors' main concerns: the security of their family's patrimony on the event of their death. Mutual funds, since their creation, have been preferred by conservative investors since they reduce significantly the risk born by investment in the stock market, however, these funds are not exempt from suffering a substantial loss. Insurance companies offer several life insurance products whose sum assured is a function of an investment fund's value. With that in mind, in this dissertation a product with a broader protection is proposed: an endowment assurance that is bought when an investment in a fund is made, which guarantees that, if the policyholder's death occurs within the specified period, the beneficiaries will receive, as the fund's value, the maximum value attained by the fund during the contract's life. If the policyholder survives the endowment period he or she will receive the same benefit.

The valuation of the product outlined here is based on life insurance valuation techniques, because it is the moment of death (or survival) which triggers payment. However, the fundamental difference with a traditional life assurance product, as regards calculations, is that the benefit is not of a deterministic nature, but rather a contingent one, since it depends on the fund's return. To find the present value of the benefit a derivative product will be used: the European lookback put[1].

In the first chapter we present a brief summary of the theoretical framework under which this product will operate; in particular, reference is made to the premium and reserve valuation methods for traditional life insurance, as well as option pricing methods.

Chapter II deals with the lookback put, which will have a substantial input into the product's valuation. We derive the option's pricing formula under the Black and Scholes model and compare this to the prices obtained from Monte Carlo simulation. We also review the implications of discrete hedging.

[1] A lookback put is an option whose payoff depends on the maximum value reached by the underlying asset during a certain period. Its payoff is $S_{max} - S_T$ where S_T is the price of the underlying asset at maturity and S_{max} is the maximum value attained by the underlying during the interval $(0, T)$. See [8] §19.8, [5] and [6].

In Chapter III, the product is explicitly defined and priced; first using the Black and Scholes formula for the lookback put and then by Monte Carlo simulation. Chapter IV illustrates the product's operation, and these results are followed by the conclusions, discussing the product's viability.

I.- THEORETICAL FRAMEWORK.

i) Life insurance premium and reserve valuation.

There are four basic ingredients in a life insurance premium's calculation: death probabilities, interest rate, sum assured and expenses. The first three are used in the net premium calculation, which measures only the value of future claims but does not contemplate the insurance company's profit margin. The net premium plus an expenses charge form the gross premium, which is the one paid by the policyholder, nevertheless in this paper only net premiums will be used.

Mortality is measured, traditionally, through mortality tables. These tables let the insurance company predict the number of deaths that will occur in a group of people of a certain age.

Let us consider a person aged x, denoted *(x)*, and let's denote his future lifetime by *T(x)*. Hence, $x+T_x$ will be the person's age at death. The future lifetime T_x is a random variable with cumulative probability distribution function

$$G(t) = P[T_x \le t], \quad t \ge 0 . \qquad \text{(I.i.1)}$$

The function $G_x(t)$ represents the probability that *(x)* dies within a period of t years, for any fixed t. We suppose G to be known, continuous and having probability density $g_x(t)=G_x'(t)$. Then, we can write

$$g_x(t)\, dt = P[t < T_x < t + dt] , \qquad \text{(I.i.2)}$$

the probability that *(x)*'s death occurs in the infinitesimal interval from t to $t+dt$. In consonance with widespread actuarial notation, ${}_tq_x$ denotes the probability that *(x)* dies in a t-year period, thus

$${}_tq_x = G_x(t) . \qquad \text{(I.i.3)}$$

Similarly,

$${}_tp_x = 1 - G_x(t) \qquad \text{(I.i.4)}$$

denotes the probability that *(x)* survives at least t years.

The force of mortality of *(x)* at age $x+t$ is defined as

$$\begin{aligned}\mu_{x+t} &= \frac{g_x(t)}{1-G_x(t)} \\ &= -\frac{d}{dt}\ln\left(1-G_x(t)\right) \qquad \text{(I.i.5)} \\ &= -\frac{d}{dt}\ln\left({}_tp_x\right).\end{aligned}$$

From (I.i.2) and (I.i.5) another expression for the probability of death between t and $t+dt$ is derived

$$P[\,t < T_x < t + dt\,] = {}_tp_x\,\mu_{x+t}\,dt\,. \qquad \text{(I.i.6)}$$

Starting from T_x, we define the discrete random variable $K_x=[T_x]$, the number of complete years lived by (x), or, the curtate future lifetime of (x). The probability density function of K_x is given by

$$P[\,K_x= k\,] = P[\,k \le T_x < k+1\,] = {}_kp_x\, q_{x+k}\,, \qquad k=0,1,2,\ldots \qquad \text{(I.i.7)}$$

Unfortunately, although many attempts have been made[2], it has not been possible to find a universally valid analytical distribution of T_x. However, by adopting an adequate mortality table, a probability distribution can be built for the future lifetime. The yearly death probabilities (q_x) in the table define completely a distribution for K. To find the distribution of T by interpolation, the uniform distribution of deaths assumption (UDD)[3] can be used for intermediate ages $x+u$ (x an integer and $0<u<1$), *i.e.*

$${}_uq_x = u\, q_x, \qquad \text{(I.i.8)}$$

$${}_up_x = 1 - u\, q_x \text{ , and} \qquad \text{(I.i.9)}$$

$$\mu_{x+u} = \frac{q_x}{1-uq_x}\,. \qquad \text{(I.i.10)}$$

Under a life assurance contract the benefit consists of a payment: the sum assured. The present value of the payment is denoted by Z (which is a random variable itself). Following Bowers [2], we shall use a constant interest rate i to calculate the present value; the expected present value of the payment, E[Z], is the contract's net single premium. When the sum assured is one unit payable at the end of the year of death, if this happens within

[2] De Moivre(1724) assumes that T follows a uniform distribution between ages 0 and ω. Gompertz (1824) assumes that the mortality force grows exponentially. Makeham (1860) adds a constant to Gompertz's model. Weibull (1939) suggest that μ grows as a power of t. See [2].

[3] There are other assumptions like constant force of mortality over each unit interval.

the first n years, or else at the end of the n-th year, it is called an endowment assurance and the random variable Z is

$$Z = \begin{cases} v^{K+1}, & K = 0,1,\dots,n-1 \\ v^n, & K = n, n+1, \cdots \end{cases} \tag{I.i.11}$$

where

$$v^j = (1+i)^{-j}, \tag{I.i.12}$$

In this case, the net single premium is denoted by $A_{x:\overline{n}|}$ and is given by

$$A_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}\, {}_k p_x . q_{x+k} + v^n\, {}_n p_x . \tag{I.i.13}$$

It has been assumed that the sum assured will be paid at the end of the year of death, which does not reflect the real practice. Now it will be assumed that the sum assured will be paid instantly upon death, *i.e.*, at time T. Z becomes, then,

$$Z = \begin{cases} v^T, & 0 < T < n \\ v^n, & T \geq n \end{cases}, \tag{I.i.14}$$

and the net single premium (denoted by $\overline{A}_{x:\overline{n}|}$) is calculated as

$$\overline{A}_{x:\overline{n}|} = \int_0^n v^t\, {}_t p_x \mu_{x+t} dt + \int_n^\infty v^n\, {}_t p_x \mu_{x+t} dt . \tag{I.i.15}$$

A practical approximation can be derived under the uniform distribution of deaths in a year assumption for ${}_u q_x$ seen in (I.i.8), (I.i.9) and (I.i.10):

$$\overline{A}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}\, {}_k p_x\, q_{x+k} \int_0^1 (1+i)^u\, du + v^n\, {}_n p_x . \tag{I.i.16}$$

In practise there are very few insurance policies sold under a net single premium (e.g. one-year term assurance). To calculate the level annual premium that will be paid through the assurance's duration (or a shorter period) the following equivalence is used:

$$\ddot{a}_{x:\overline{h}|} \cdot P(\overline{A}) = \overline{A} \tag{I.i.17}$$

where $P(\overline{A})$ is the net level premium (the premium that the policyholder will pay annually) and $\ddot{a}_{x:\overline{h}|}$ is an annuity bought at age x that pays one unit at the beginning of each year, during h years, as long as the person is alive. Its value can be calculated as:

$$\ddot{a}_{x:\overline{h}|} = \sum_{k=0}^{h-1} v^k \cdot {}_k p_x . \tag{I.i.18}$$

Given that through a single premium a protection for many years is being bought the insurance company must create a reserve, for each policy, equivalent to their outstanding liabilities towards the policyholder. There are several methodologies for the reserve calculation: the retrospective method involves comparing the payments received with the protection already provided. The prospective method subtracts the value of the remaining payments to that of the remaining protection. Fackler's method gives the reserve at the end of year k as a function of the reserve up to year k-1. In this document the prospective method will be used because, given the product's characteristics, is the most suitable.
Under a level premium scheme, the reserve at the end of year k (${}_kV$) is given by

$$ {}_kV\left(\bar{A}_{x:\overline{n}|}\right) = \begin{cases} \bar{A}_{x+k:\overline{n-k}|} - P\left(\bar{A}_{x:\overline{n}|}\right)\ddot{a}_{x+k:\overline{n-k}|}, & k < n \\ 1, & k = n. \end{cases} \tag{I.i.19} $$

ii) Option pricing.

In the last few decades, derivative products have acquired great importance in financial markets. In general, derivatives are financial instruments whose value depends on the value of an underlying asset, which could be oil, currency or stock. The main derivative products are futures, options and swaps.[4]
There are two main kinds of options. A put gives the buyer the right to sell the underlying asset at a certain price for a given period of time. A call gives the buyer the right to buy the underlying asset for a specified price over a given period of time. The price in the contract is called the exercise price and the date is known as expiry or maturity. If the option can be exercised at any time between inception and maturity it is said to be an American option, while, if it can only be exercised on the expiry date it is said to be a European option.
It is important to point out that the holding of an option, as opposed to a future, only implies the right to buy or sell but not the obligation to do so. While futures are contracts that require no payment at inception, options have a non-zero price.
In this dissertation, we shall be most interested in a class of options known as path dependent options. These are derivative products whose payoff function depends on the

[4] For more on derivatives see [8].

behaviour of the underlying asset throughout the whole life of the option and not only on its final price. There are several such options, for example:

- Asian options which depend on an average of the price of the asset,
- barrier options which depend on the asset reaching a certain level, and
- lookback options which depend on a maximum or minimum price, among others.

As it has been said, the amount of the benefit of the proposed insurance is random. To estimate its value, in Chapter III, a lookback put will be used, since the value of the product's premium is precisely the expected value of the present value of the difference between the maximum attained by the fund's price and its price at the moment of claim.

It is clear that the price of an option is affected by the price of the underlying asset at inception, the risk-free interest rate, the volatility of the underlying asset's price, the exercise price, the term to maturity and the kind of option. However, this is not enough to value the premium that should be paid for them; some assumptions on the market are necessary. The most important of these assumptions is the no-arbitrage assumption, *i.e.*, that the market only allows risk-free arbitrage opportunities to arise for instants, compensating itself immediately. Furthermore, we must assume that the stocks are infinitely divisible, that there are no transaction costs and that it is possible to invest and borrow at the same risk-free interest rate.

There are several methodologies for pricing options, in this paper both the analytic solution under the Black and Scholes model and Monte Carlo simulation will be used.

II.- THE LOOKBACK PUT.

Plain vanilla options are traded actively and their prices and volatility smiles quoted on a regular basis, but other products have been created on over the counter operations to satisfy specific needs. Among these non-standard products, we find the lookback options, which are a path dependent option that pays a function of the maximum or minimum asset price reached during the life of the option. There are various types of lookback options, but for the aims of this paper, we'll study the European style lookback put. This option pays at maturity (T), the difference between the maximum price reached by the asset (S^*_T) and the price of the asset at that time (S_T), and to price it two approaches will be studied: Black and Scholes method and Monte Carlo simulation.

i) The Black and Scholes model

In 1973, Fischer Black, Myron Scholes and Robert Merton developed what is now known as the Black and Scholes model to price options[5]. Under the Black and Scholes model, we will assume the economy to consist of two assets: a zero-coupon bond B_T maturing at time T and the security of price S_t.

Let $\tilde{W} = \{\tilde{W}_t\}_{t \geq 0}$ be a P-Brownian motion, and $\mathfrak{F} = \{\mathfrak{F}_t\}_{t \geq 0}$ be the filtration generated by $\tilde{W}$.

We will model S_t as a stochastic process on a fitted probability space $\left(\Omega, \mathfrak{F}_T, \{\mathfrak{F}_t\}_{0 \leq t \leq T}, P\right)$, where P is the real world probability measure, S_t follows:

$$dS_t = mS_t dt + sS_t d\tilde{W}_t \qquad \text{(II.i.1)}$$

and B_t follows

$$dB_t = rB_t dt \qquad \text{(II.i.2)}$$

where r is the risk-less rate of return, m is the security's expected return and s its volatility. The market is said to be arbitrage-free if there exists a probability measure Q (the risk-neutral measure) equivalent to P under which the process $S_t e^{-rt}$ is a martingale. The market is said to be complete if and only if this measure Q is unique.

Applying Girsanov's theorem shows that under Q the market is complete and arbitrage-free if and only if S_t follows the stochastic differential equation

$$dS_t = rS_t dt + sS_t dW_t \qquad \text{(II.i.3)}$$

where W_t is a Q-Brownian motion.

A contingent claim (e.g. an option) is defined as any $\mathfrak{F}_t$-measurable non-negative random variable X. If the market is complete, every contingent claim can be replicated (and, therefore, hedged) with the bond and the security. This implies that the price at time t of any contingent claim X, $\Pi_X(t)$, is the expectation of its discounted payoff under Q; this is

$$\Pi_t^X = \mathrm{E}^Q\left[e^{-r(T-t)} X \middle| \mathfrak{F}_t \right]. \qquad \text{(II.i.4)}$$

Applying Itô's Lemma to the process ln S_t, and solving the resulting stochastic differential equation we have

$$S_t = S_0 \exp\left\{ \left(r - \frac{1}{2} s^2 \right) t + s W_t \right\}. \qquad \text{(II.i.5)}$$

Let

$$S_t^* = \max\{S_s; 0 \le s \le t\}, \qquad \text{(II.i.6)}$$

$$Z_t = \ln\left(\frac{S_t}{S_0} \right) = \left(r - \frac{1}{2} s^2 \right) t + s W_t \text{, and} \qquad \text{(II.i.7)}$$

$$Y_t = \ln\left(\frac{S_t^*}{S_0} \right) = \max\{Z_s; 0 \le s \le t\}. \qquad \text{(II.i.8)}$$

In the case of a lookback put, the payoff is given by

$$X = \left(S_T^* - S_T \right)_+ = S_T^* - S_T , \qquad \text{(II.i.9)}$$

since $S_T^* \ge S_T$ almost surely. Thus its price at time t is

$$\Pi_t^X = e^{-r(T-t)} \mathrm{E}^Q\left[S_T^* \mid \mathfrak{F}_t \right] - S_t . \qquad \text{(II.i.10)}$$

[5] See [1] and [10] for the complete derivation of the model.

Equation (II.i.10) tells us that to price a lookback put we only need to price a security that pays the realised maximum and subtract the stock price at time t. So, at time 0, equation (II.i.10) becomes

$$\begin{aligned}\Pi_0^X &= e^{-rT}\mathrm{E}^Q\left[S_T^*\right]-S_0 \\ &= e^{-rT}\mathrm{E}^Q\left[S_0\exp\left\{\ln\left(\frac{S_T^*}{S_0}\right)\right\}\right]-S_0\ , \\ &= e^{-rT}S_0\mathrm{E}^Q\left[\exp\{Y_T\}\right]-S_0\end{aligned} \qquad \text{(II.i.11)}$$

where Y_T is given as in (II.i.8).

Let Z_t be a standard Brownian motion, and $Y_t=sup\{Z_s;0<s<t\}$. Define the joint distribution function

$$F_t(z,y)=P(Z_t\le z,Y_t\le y). \qquad \text{(II.i.12)}$$

Since $Z_0=0$, it only makes sense to calculate $F(z,y)$ for $y>0$ and $z<y$. We note that:

$$\begin{aligned}F_t(z,y) &= P(Z_t\le z)-P(Z_t\le z,Y_t>y) \\ &= \Phi\left(z/\sqrt{t}\right)-P(Z_t\le z,Y_t>y)\end{aligned} \qquad \text{(II.i.13)}$$

where $\Phi(\bullet)$ denotes the N(0,1) cumulative distribution function. Let T be a stopping time such that T is the first t at which $Z_t=y$, and define $Z^*_t=Z_{T+t}-Z_T$. By the strong Markov property Z^* is a Brownian motion with $Z^*_0=0$ and independent of the path followed to reach Z_T [6]. From here, it is clear that

$$\begin{aligned}P(Z_t\le z,Y_t>y) &= P\left(T<t,Z^*_{t-T}\le z-y\right) \\ &= P\left(T<t,Z^*_{t-T}\ge y-z\right);\end{aligned} \qquad \text{(II.i.14)}$$

by construction $Z^*_{t-T}=Z_t-y$, thus

$$P(Z_t\le z,Y_t>y)=P(Z_t\le z-2y)=\Phi\left((z-2y)/\sqrt{t}\right). \qquad \text{(II.i.15)}$$

Combining (II.i.13) and (II.i.15) and differentiating with respect to z, allows us to write the density function g_t

$$g_t(z,y)=P(Z_t\in dz,Y_t\le y)=\left[f\left(z/\sqrt{t}\right)-f\left((z-2y)/\sqrt{t}\right)\right]t^{-1/2}, \qquad \text{(II.i.16)}$$

[6] See [7] on the Strong Markov Property, the Change of Measure Theorem and the Reflection Principle.

Where $f(\bullet)$ denotes the N(0,1) probability density function. Now, by fixing t>0, letting $m = r - \frac{1}{2}s^2$, setting the Radon-Nikodym derivative as

$$V_t = \exp\left\{ mZ_t - \frac{1}{2}m^2 t \right\} \tag{II.i.17}$$

and defining a new probability measure P_t^* by taking $dP_t^* = V_t dP$, the process $\left\{ \widetilde{Z}_s ; 0 \le s \le t \right\}$ is a Brownian motion with drift $r - s^2/2$ and volatility[7] σ under P_t^*.

Then

$$\begin{aligned} P^*\left(\widetilde{Z}_t \le z, Y_t \le y\right) &= \mathrm{E}^{P_t^*}\left[\mathrm{I}_{\{\widetilde{Z}_t \le z, Y_t \le y\}} \right] \\ &= \mathrm{E}^{P}\left[V_t \cdot \mathrm{I}_{\{Z_t \le z, Y_t \le y\}} \right] \\ &= \int_{-\infty}^{z} e^{mx - m^2 t/2} P\left(Z_t \in dx, Y_t \le y\right) \\ &= \int_{-\infty}^{z} e^{mx - m^2 t/2} g_t(x, y) dx \end{aligned} \tag{II.i.18}$$

differentiating with respect to z gives

$$P_t^*\left(Z_t \in dz, Y_t \le y\right) = e^{mz - m^2 t/2} g_t\left(z, y\right) dz\ . \tag{II.i.19}$$

A straightforward rescaling extends this result to any value of s, yielding that for any arbitrary values of m and s

$$\begin{aligned} P\left(Z_t \in dz, Y_t \le y\right) &= f_t\left(z, y\right) dz \\ &= \left(1/s\right) \exp\left\{ mz/s^2 - m^2 t/2s^2 \right\} g_t\left(z/s, y/s\right) dz \end{aligned} \tag{II.i.20}$$

Now, $F_t(z,y)$ for arbitrary m and s is given by

---

[7] Remember Z_t is a standard Brownian motion so σ=1 here, but the result will hold in general and will be used later.

$$
\begin{aligned}
F_t(z,y) &= \int_{-\infty}^{y} f_t(x,y)\,dx \\
&= \int_{-\infty}^{y} \frac{1}{s\sqrt{t}}\exp\left\{\frac{mx}{s^2}-\frac{m^2t}{2s^2}\right\}\left[f\left(\frac{x}{s\sqrt{t}}\right)-f\left(\frac{x-2y}{s\sqrt{t}}\right)\right]dx \\
&= \int_{-\infty}^{0} \frac{e^{-\frac{m^2t}{2s^2}}}{s\sqrt{2pt}}\left[\exp\left\{-\frac{(x+y)^2-2mt(x+y)}{2s^2t}\right\}-\exp\left\{-\frac{(x-y)^2-2mt(x+y)}{2s^2t}\right\}\right]dx \\
&= \int_{-\infty}^{0} \frac{1}{s\sqrt{2pt}}\left[\exp\left\{-\frac{1}{2}\left(\frac{x+y-mt}{s\sqrt{t}}\right)^2\right\}-\exp\left\{-\frac{1}{2}\left(\frac{x-y-mt}{s\sqrt{t}}\right)^2\right\}e^{\frac{2my}{s^2}}\right]dx \\
&= \Phi\left(\frac{y-mt}{s\sqrt{t}}\right)-e^{\frac{2my}{s^2}}\Phi\left(\frac{-y-mt}{s\sqrt{t}}\right)
\end{aligned}
\qquad \text{(II.i.21)}
$$

which, by differentiating with respect to y, yields

$$
f_t(y)=\frac{1}{s\sqrt{t}}f\left(\frac{y-mt}{s\sqrt{t}}\right)+\frac{1}{s\sqrt{t}}e^{\frac{2my}{s^2}}f\left(\frac{-y-mt}{s\sqrt{t}}\right)-\frac{2m}{s^2}e^{\frac{2my}{s^2}}\Phi\left(\frac{-y-mt}{s\sqrt{t}}\right). \qquad \text{(II.i.22)}
$$

This formula for the distribution of Y_T, allows us to price the lookback put. Equation (II.i.11) becomes

$$
\begin{aligned}
\Pi_0^X &= e^{-rT}S_0\int_0^{\infty} e^{y_T} f\left(y_T\right)dy_T - S_0 \\
&= e^{-rT}S_0\int_0^{\infty} e^{y}\frac{1}{s\sqrt{T}}f\left(\frac{y-mT}{s\sqrt{T}}\right)dy \\
&\quad + e^{-rT}S_0\int_0^{\infty} e^{y}\frac{1}{s\sqrt{T}}e^{\frac{2my}{s^2}}f\left(\frac{-y-mT}{s\sqrt{T}}\right)dy \\
&\quad - e^{-rT}S_0\int_0^{\infty} e^{y}\frac{2m}{s^2}e^{\frac{2my}{s^2}}\Phi\left(\frac{-y-mT}{s\sqrt{T}}\right)dy - S_0 \\
&= e^{-rT}S_0\int_0^{\infty}\frac{1}{\sqrt{2p}}\exp\left\{y-\frac{1}{2}\left(\frac{y-mT}{s\sqrt{T}}\right)^2\right\}\frac{1}{s\sqrt{T}}dy \\
&\quad + e^{-rT}S_0\int_0^{\infty}\frac{1}{\sqrt{2p}}\exp\left\{y+\frac{2my}{s^2}-\frac{1}{2}\left(\frac{-y-mT}{s\sqrt{T}}\right)^2\right\}\frac{1}{s\sqrt{T}}dy \\
&\quad - e^{-rT}S_0\int_0^{\infty}\frac{2m}{s^2}e^{y+\frac{2my}{s^2}}\int_{-\infty}^{b(y)}\frac{1}{\sqrt{2p}}e^{-\frac{1}{2}x^2}dx\,dy - S_0
\end{aligned}
\qquad \text{(II.i.23)}
$$

where $b(y) = \dfrac{-y - mT}{s\sqrt{T}}$ and $m = r - s/2$. The first two terms can be evaluated by completing squares in the exponent and a change of variable, to get standard normal cumulative distribution functions. The third term requires first a change in the order of integration. These integrals yield:

$$\begin{aligned}\Pi_0^X &= S_0 e^{-rT}\left[\Phi\left(-d + s\sqrt{T}\right) - \frac{s^2}{2r}\Phi\left(d - \frac{2r}{s}\sqrt{T}\right)\right] \\ &\quad - S_t\left[\Phi(-d) - \frac{s^2}{2r}\Phi(d)\right] \\ &= -S_t\Phi(-d) + e^{-rT}S_0\Phi\left(-d + s\sqrt{T}\right) \\ &\quad + e^{-rT}\frac{s^2}{2r}S_0\left[e^{rT}\Phi(d) - \Phi\left(d - \frac{2r}{s}\sqrt{T}\right)\right]\end{aligned} \qquad \text{(II.i.24)}$$

where

$$d = \left(r + \frac{s^2}{2}\right)\left(\frac{\sqrt{T}}{s^2}\right).$$

In general, at any time *0<t<T*,

$$\begin{aligned}\Pi_t^X &= S_t^* e^{-r(T-t)}\left[\Phi\left(-d + s\sqrt{T-t}\right) - \left(\frac{S_t^*}{S_t}\right)^{\frac{2r}{s^2}-1}\frac{s^2}{2r}\Phi\left(d - \frac{2r}{s}\sqrt{T-t}\right)\right] \\ &\quad - S_t\left[\Phi(-d) - \frac{s^2}{2r}\Phi(d)\right] \\ &= -S_t\Phi(-d) + e^{-r(T-t)}S_t^*\Phi\left(-d + s\sqrt{T-t}\right) \\ &\quad + e^{-r(T-t)}\frac{s^2}{2r}S_t\left[e^{r(T-t)}\Phi(d) - \left(\frac{S_t^*}{S_t}\right)^{2r/s^2}\Phi\left(d - \frac{2r}{s}\sqrt{T-t}\right)\right]\end{aligned} \qquad \text{(II.i.25)}$$

where

$$d = \left(\ln\left(\frac{S_t}{S_t^*}\right) + \left(r + \frac{s^2}{2}\right)(T-t)\right)\left(s^2\sqrt{T-t}\right)^{-1}. \qquad \text{(II.i.26)}$$

It is clear that (II.i.25) meets its terminal conditions, that is

$$\Pi_T^X = S_T^* - S_T \tag{II.i.27}$$

and

$$\Pi_t^X \Big|_{S_t=0} = S_t^* e^{-r(T-t)} \tag{II.i.28}$$

ii) Monte Carlo simulation

The first references to Monte Carlo methods date back to the 18th century, when Comte de Buffon calculated the value of π by repeatedly throwing a needle into a ruled plane[8], nevertheless the name was first given to these methods during the 1940's by scientists working in the development of nuclear weapons in Los Alamos. The essence of the method is inventing a game of chance whose behaviour and outcome help study interesting phenomena.

The risk neutral valuation result will be used to price the lookback put. The expected payoff in a risk-neutral world is calculated using a sampling procedure, and then discounted at the risk free interest rate.

First we will simulate the asset price dynamics, under the risk neutral measure, in order to price a lookback put where the asset's movement is not continuous and where the maximum is only updated at these points. The asset price will follow a discrete version of equation (II.i.5) i.e.

$$S_{t+\Delta t} = S_t \exp\left\{\left(r - \frac{s}{2}\right)\Delta t + se\sqrt{\Delta t}\right\} \tag{II.ii.1}$$

where e is a random draw from the N(0,1) distribution[9]. Then the lookback's payoff is calculated. These steps are repeated several times and the mean of these sampled payoffs is discounted at the risk-free rate of interest r, yielding the price of the derivative.

[8] See [4].
[9] See [3], [9], [11] and [12] on the computational aspects of Monte Carlo simulation.

iii) Comparison

Applying formula (II.i.25) with S_0 = 100, S_0^*= 100, σ = 0.10, a risk-free interest rate of 4% and time to maturity $T = \{1/12, 2/12, \cdots, 60/12\}$, we get the values for the 60 lookback puts that will be needed later, and that can be seen in Table II.iii.1.

Table II.iii.1.- Price of a European lookback put under Black and Scholes

i	Π(i/12)	i	Π(i/12)	i	Π(i/12)	i	Π(i/12)
1	1.7842	16	7.2540	31	9.7570	46	11.3784
2	2.5471	17	7.4644	32	9.8862	47	11.4662
3	3.1369	18	7.6667	33	10.0119	48	11.5519
4	3.6353	19	7.8616	34	10.1342	49	11.6356
5	4.0740	20	8.0494	35	10.2533	50	11.7173
6	4.4696	21	8.2308	36	10.3692	51	11.7971
7	4.8319	22	8.4060	37	10.4821	52	11.8750
8	5.1675	23	8.5754	38	10.5920	53	11.9511
9	5.4807	24	8.7395	39	10.6992	54	12.0254
10	5.7750	25	8.8983	40	10.8037	55	12.0980
11	6.0529	26	9.0523	41	10.9056	56	12.1690
12	6.3163	27	9.2017	42	11.0049	57	12.2383
13	6.5670	28	9.3466	43	11.1017	58	12.3060
14	6.8061	29	9.4874	44	11.1962	59	12.3721
15	7.0348	30	9.6241	45	11.2884	60	12.4367

Source: own calculations

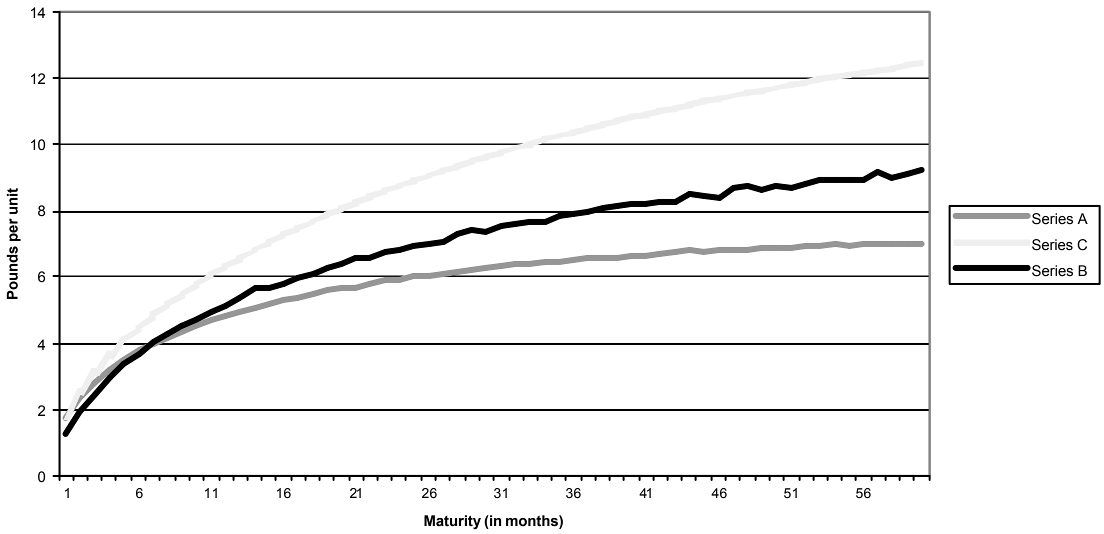
On the other hand, by Monte Carlo simulation (on 100,000 iterations), for the same data and $Dt=1/12$, we obtain the prices shown in Table II.iii.2. From Exhibit II.iii.1 it is clear that the prices arrived at by Monte Carlo simulation tend to the analytic counterpart as Dt tends to zero, for a sufficiently high number of iterations. Series A shows the values of Table II.iii.2 while series B shows the values arrived at by Monte Carlo simulation with Dt =1/24, and series C shows the prices under Black and Scholes.

Table II.iii.2.- Price of a European lookback put by Monte Carlo simulation

i	Π(i/12)	i	Π(i/12)	i	Π(i/12)	i	Π(i/12)
1	1.7477	16	5.2745	31	6.3234	46	6.8401
2	2.3797	17	5.385	32	6.3579	47	6.8043
3	2.8213	18	5.5041	33	6.3749	48	6.8117
4	3.2163	19	5.5837	34	6.4555	49	6.8713
5	3.4846	20	5.6549	35	6.4719	50	6.8749
6	3.7613	21	5.7133	36	6.5135	51	6.8824
7	3.9789	22	5.8098	37	6.5662	52	6.9331
8	4.1828	23	5.8906	38	6.5751	53	6.9201
9	4.3589	24	5.9296	39	6.6058	54	6.9737
10	4.5247	25	6.0054	40	6.6246	55	6.9364
11	4.6991	26	6.0389	41	6.6637	56	6.9927
12	4.8432	27	6.1304	42	6.6806	57	6.973
13	4.967	28	6.1553	43	6.7399	58	6.9737
14	5.0765	29	6.2159	44	6.7991	59	6.9876
15	5.1962	30	6.2964	45	6.7479	60	6.9982

Source: own calculations

Exhibit II.iii.1.- Price of a Lookback Put



Source: own calculations

iv) Hedging and Value at Risk.

Anyone writing an option is faced with the problem of managing its risk; when the option is a standard one, the problem could in theory be easily solved by taking the opposite position, but when the option is tailored to specific needs this can be more difficult.

A way to hedge an option is given by creating a delta neutral position[10], this is a portfolio whose value is immune to changes in the underlying's price, which comprises the stock, bond and the underlying itself. From equation (II.i.10) it is clear that to hedge the lookback put we only need to hedge *M*, since S_T can be hedged by the same stock. To hedge the option we will need to hold a short position in p_1 units of the stock, where $p_1(t) = \partial \Pi_t^X / \partial S_t$. Below are some preliminary results:

$$\frac{\partial}{\partial x}\Phi(x) = \frac{1}{\sqrt{2p}} e^{-\frac{1}{2}(x)^2} dx = f(x)\,dx \qquad \text{(II.iv.1)}$$

and

$$\frac{\partial}{\partial S_t} d = \left(S_t s \sqrt{T-t}\right)^{-1}, \qquad \text{(II.iv.2)}$$

where *d* is given by (II.i.26). Hence p_1 is given by

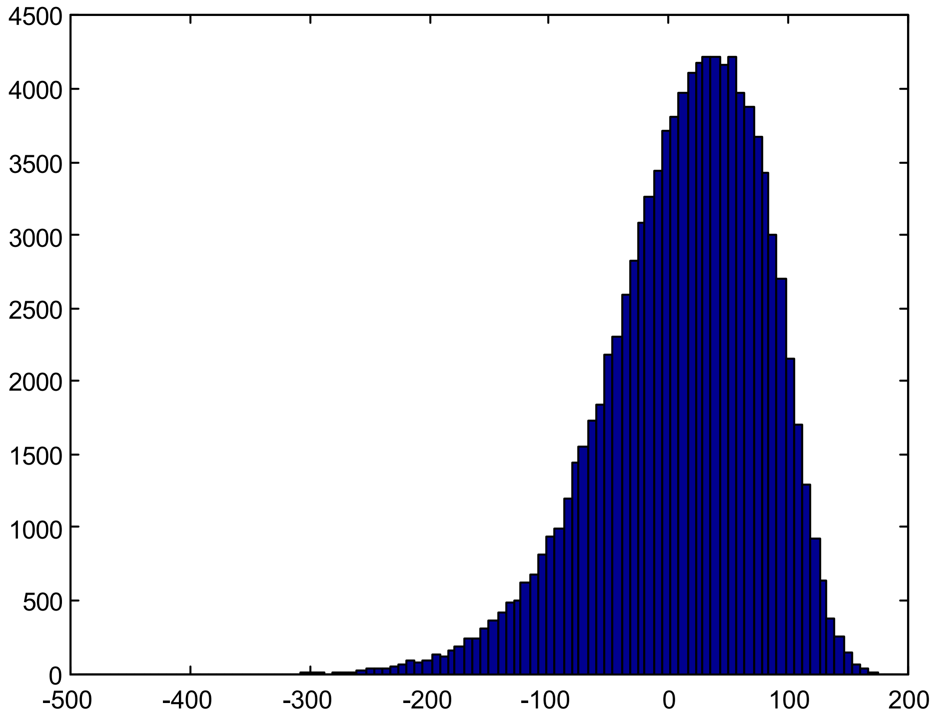
$$\begin{aligned}
p_1(t) &= -\Phi(-d) + \frac{1}{s\sqrt{T-t}} f(-d) + \frac{e^{-r(T-t)}}{s\sqrt{T-t}} \frac{S_t^*}{S_t} f\left(-d + s\sqrt{T-t}\right) \\
&\quad + e^{-r(T-t)} \frac{s^2}{2r}\left[e^{r(T-t)}\Phi(d) - \left(\frac{S_t}{S_t^*}\right)^{-\frac{2r}{s^2}} \Phi\left(d - \frac{2r}{s}\sqrt{T-t}\right)\right] \\
&\quad + \frac{s}{2r\sqrt{T-t}} f(d) - e^{-r(T-t)} \frac{s}{2r\sqrt{T-t}} \left(\frac{S_t}{S_t^*}\right)^{-\frac{2r}{s^2}} f\left(d - \frac{2r}{s}\sqrt{T-t}\right) \\
&\quad + e^{-r(T-t)} \Phi\left(d - \frac{2r}{s}\sqrt{T-t}\right)\left(\frac{S_t}{S_t^*}\right)^{-\frac{2r}{s^2}} + 1 \\
&= \left(\frac{2r + s^2}{2r}\right)\Phi(d) + \frac{2r + s^2}{2rs\sqrt{T-t}} f(d) + \frac{e^{-r(T-t)}}{s\sqrt{T-t}} \frac{S_t^*}{S_t} f\left(-d + s\sqrt{T-t}\right) \\
&\quad - e^{-r(T-t)} \left(\frac{S_t}{S_t^*}\right)^{-\frac{2r}{s^2}} \left[\left(\frac{s^2 - 2r}{2r}\right)\Phi\left(d - \frac{2r}{s}\sqrt{T-t}\right)\right] \\
&\quad + e^{-r(T-t)} \left(\frac{S_t}{S_t^*}\right)^{-\frac{2r}{s^2}} \left[\frac{s}{2r\sqrt{T-t}} f\left(d - \frac{2r}{s}\sqrt{T-t}\right)\right] \qquad \text{(II.iv.3)}
\end{aligned}$$

It is important to note that since the delta changes, the portfolio only remains delta neutral for a very short period of time, hence needing rebalancing. If it were possible to rebalance

[10] See [4] §14.4 on Delta hedging.

this portfolio continuously (and obviously, disregarding transaction costs), the risk would be completely removed. Unfortunately, this is not a realistic assumption and when the rebalancing only occurs at certain points there could be a loss in the end. The Value at Risk of writing a five-year-lookback put on the stock that has been used throughout the dissertation was calculated as the 95-percentile of the final loss distribution found by simulating 100,000 paths of the stock and rebalancing the portfolio only semi-annually. This VaR was found to be £201.57, which is quite high considering the original price of the stock is only £100, and the distribution of the profit can be seen in Exhibit II.iv.1. This is something that should not be disregarded if the product were to be sold, but it is outside the scope of this dissertation.

Exhibit II.iv.1.- Distribution of the profit arising from discrete hedging.



Source: own calculations

III.- THE PRODUCT: SAFINVEST

i) Product definition.

In the case of Safinvest, the random event triggering the payment is the death of the policyholder. If he/she dies within the policy's vigour (in this case 10 years) or, else, if he/she survives the benefit is paid. However, this is not the only random component in the product, since the amount is also non-deterministic. The sum assured is given by the difference between the maximum value achieved by the investment fund (during the policy's vigour) and its value at the moment of the claim (death or survival). Formally, the random variable T (as defined in § 1.i) is the one determining the distribution of Z (*idem*).

$$Z = \begin{cases} f(T), & 0 < T_x < 10 \\ f(10), & T_x \geq 10 \end{cases} \qquad \text{(III.i.1)}$$

where *f (T)* is the present value of $S_{max} - S_T$,[11] in agreement with the notation used in §II.i.

This insurance policy must be bought after investing a certain amount in an investment fund and the premium will be a function of the policyholder's age, the fund's volatility, and the amount invested. The benefit will be calculated (and if possible paid) in the fund's redemption date immediately after the death and proof of uninterrupted possession of the fund's stock throughout the policy's vigour will be required.

ii) Product pricing.

To price the product the payment random variable function must be defined. The premium will then be the expected value of this random variable. Assume the assurance is bought on a fund where there are no restrictions in the time of redemption, if the policyholder dies at any time t, the corresponding payment would be the difference between the maximum value reached by the fund since the policy's inception and its value at that time, *i.e.* $S_{max} - S_t$ (it should not be forgotten that this difference is also to be calculated if the policyholder survives the endowment period). Considering interest under continuous compounding, the present value of this payment, *f (t)*, is:

---

[11] Explained with more detail in §2.iii.

$$f(t) = e^{-rt}\left(S_{\max} - S_t\right). \qquad \text{(III.ii.1)}$$

Since $S_{\max} \geq S_t$, $\forall t$, *f(t)* is never negative and, therefore, can be written as

$$f(t) = e^{-rt} \max\left\{S_{\max} - S_t, 0\right\}. \qquad \text{(III.ii.2)}$$

Then the benefit's present value random variable is

$$Z = \begin{cases} f(t), & 0 < t < 10 \\ f(10), & t \geq 10 \end{cases}. \qquad \text{(III.ii.3)}$$

Nevertheless *f(t)* is a random variable that depends on the path followed by the stock's prices, and hence the expected value of *Z* is

$$\begin{aligned} E[Z] &= E\left[E\left[Z(T) \mid T\right]\right] \\ &= \int_0^{10} E\left[f(t)\right] {}_t p_x \mu_{x+t} dt + \int_{10}^{\infty} E\left[f(10)\right] {}_t p_x \mu_{x+t} dt. \end{aligned} \qquad \text{(III.ii.4)}$$

It can be seen that

$$E\left[f(t)\right] = E\left[e^{-rt} \max\left\{S_{\max} - S_t, 0\right\}\right] = \Pi_0(t) \qquad \text{(III.ii.5)}$$

is the price at time *0* of a European lookback put with maturity t years, this allows the expected value of Z[12] to be expressed as

$$E[Z] = \int_0^{10} \Pi_0(t)\, {}_t p_x \mu_{x+t} dt + \int_{10}^{\infty} \Pi_0(10)\, {}_t p_x \mu_{x+t} dt. \qquad \text{(III.ii.6)}$$

This expression is not practical, since it would be impossible to find a fund in which positions could be redeemed constantly, so, now consider a fund with possible redemption dates n times per year. This leads us to rewrite equation (III.ii.3) as

$$Z = \begin{cases} f\left(k + \frac{1}{n}\right), & K = 0, 1/n, \cdots, 10 - 1/n \\ f(10), & K = 10, 10 + 1/n, \cdots \end{cases}, \qquad \text{(III.ii.7)}$$

extending the definition of *K*, given in *(I.i.7)*, to periods of length *1/n* years.[13] Therefore, the new expression for the expected value of Z is

[12] Using the notation of §1.i.

[13] If the distribution of this new random variable K is not known it is very easily constructed, under UDD, from a standard mortality table.

$$E[Z] = \sum_{k\in[0,10)} \Pi_0 \left(k+\frac{1}{n}\right) {}_k p_x q_{x+k} + \sum_{k>10} \Pi_0 (10)\, {}_k p_x q_{x+k}$$
$$= \sum_{k\in[0,10)} \Pi_0 \left(k+\frac{1}{n}\right) {}_k p_x q_{x+k} + \Pi_0 (10)\, {}_{10} p_x \quad . \qquad \text{(III.ii.8)}$$

This form of the expected value of the random variable Z will be the one used to evaluate the product's net single premium. To obtain the level annual premium, P, the process would be the one described in §I.i, dividing the net single premium by a contingent 10 year annuity, yielding:

$$P = \frac{\sum_{k\in[0,10)} \Pi_0 \left(k+\frac{1}{n}\right) {}_k p_x q_{x+k} + \Pi_0 (10)\, {}_{10} p_x}{\ddot{a}_{x:\overline{10}|}} . \qquad \text{(III.ii.9)}$$

As stated in the Chapter I, to evaluate the reserves the prospective method will be used, *i.e.*, the terminal reserve in the t-th policy year (${}_tV$) is defined as the actuarial present value of the company's remaining liabilities minus the premiums still to be paid; however, by selling the product through a single premium, as in this case, the second term is zero. The company's remaining liabilities can be calculated using the following expression:

$$E[{}_tL] = \sum_{k\in[0,10-t)} \Pi_t^* \left(k+\frac{1}{n}\right) {}_k p_{x+t} q_{x+t+k} + \Pi_t^* (10-t)\, {}_{10-t} p_{x+t} , \qquad \text{(III.ii.10)}$$

where

$$\Pi^*_t (k) = E\left[e^{-rt} \max\left\{S^*_{\max} - S_k, 0\right\}\right] \qquad \text{(III.ii.11)}$$

and

$$S^*_{\max} = \max_{0<i<k} \{S_i\}. \qquad \text{(III.ii.12)}$$

Formula (III.ii.10) is an insurance with the same characteristics as the original except for three variations: t years less of term, t years more of age and that the lookback put will have a barrier, *i.e.*, it won't pay the maximum attained during the (new) insurance's vigour (S_{max}), but the maximum attained since time 0 (S^*_{max}).

If it were needed to calculate the reserves under the level premium scheme the following formula could be used:

$$E[{}_tL] = \sum_{k\in[0,10-t)} \Pi_t^* \left(k+\frac{1}{n}\right) {}_k p_{x+t} q_{x+t+k} + \Pi_t^* (10-t)\, {}_{10-t} p_{x+t} - P(A_x) \cdot \ddot{a}_{x+t:\overline{n-t}|} , \qquad \text{(III.ii.13)}$$

using, again, formulae (III.ii.11) and (III.ii.12).

Using these formulae, in the next chapter, a practical application of the product will be analysed to determine the product's viability.

IV.- APPLICATION.

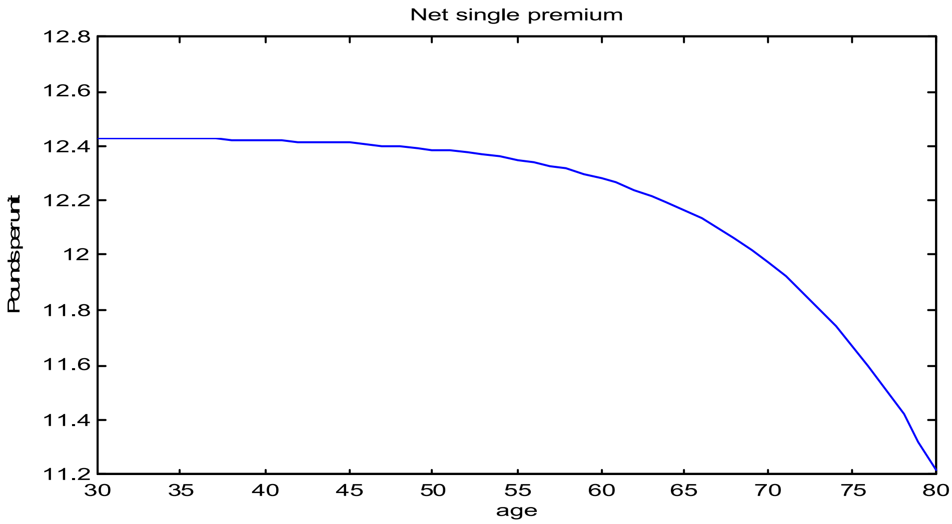
Consider a portfolio of 10,000 investors with actuarial average age 50 years, who wish to invest in 1,000,000 units of stock of a fund, each, and buy the protection that Safinvest would offer. To exemplify the behaviour of the product, the endowment period has been shortened to 5 years and the benefit's payment (as well as the calculation of the same) will be made at the end of the month of death of the policyholder. Suppose the fund's stock value is £100 and it has a volatility of 10% p.a. (the exercise could be repeated for any values of S_0 and s).

i) Product pricing.

The first step is to price the series of European lookback puts that represent the expected value of each of the 60 possible payments. This was done previously, and the results obtained, can be seen in Table II.iii.1. Using the ultimate Assured Lives Mortality Table AM92 and the UDD assumption, the net single premium for the product (at age 50) was determined to be equal to £12.3884 per unit of stock, by formula (III.ii.8).

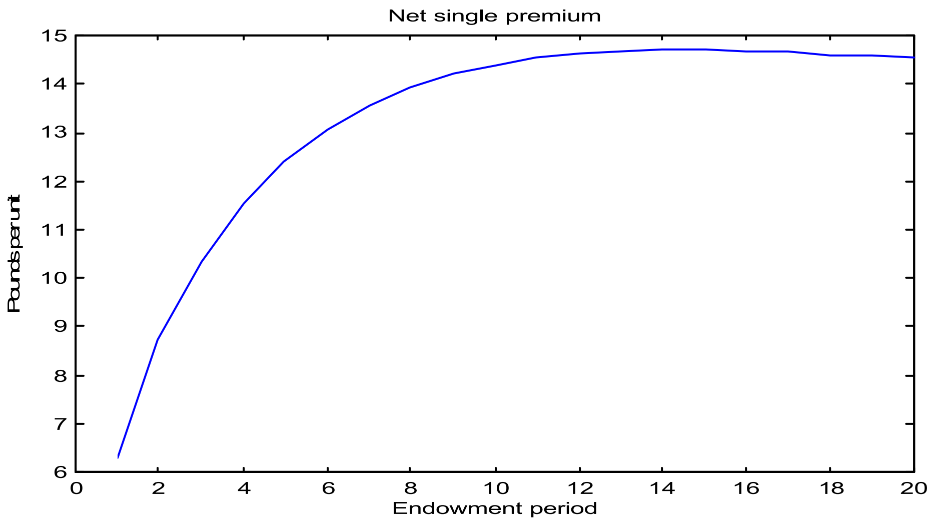
It is noteworthy that the price for the product decreases with age (as opposed to any regular product) since in this case the present value of the future payments is larger the farther we look. This can be appreciated in Exhibit IV.i.1, where the premium is shown as a function of age. As expected, as the endowment period grows, so does the premium (*ceteris paribus*), at least for the first few years, until the effect of the longer endowment period is offset by the higher early death probabilities, yielding a lower net premium, as can be seen in Exhibit IV.i.2

Exhibit IV.i.1.- Safinvest's net single premium as a function of age.



Source: own calculations

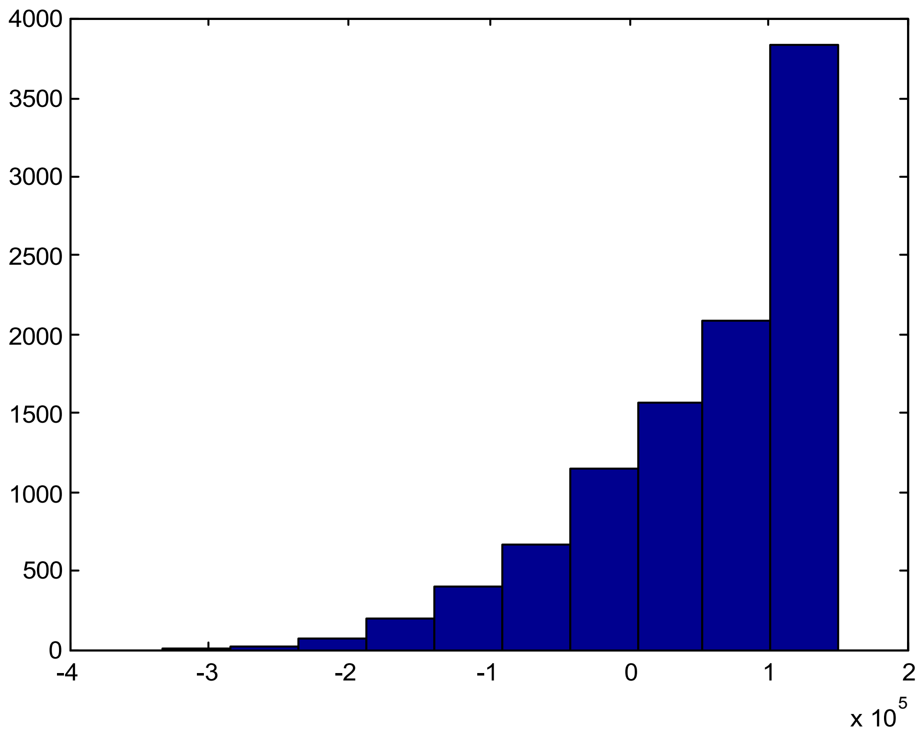
Exhibit IV.i.2.- Net single premium as a function of the endowment period's length.



Source: own calculations

A simulation was carried out to study the behaviour of the product. With this aim, a surrender value of 90% of the reserve was assumed along with withdrawal rates of 3.5% in the first year, 3% in the second year and 2.5% in years three and four. Once again, the Am92 tables were used in conjunction with the UDD assumption to simulate the death process. In the first case, the 10,000 policies were sold under a net single premium scheme, for a premium of £12.3884 per unit of stock. The result was an expected discounted profit of £45,654m, i.e., 36% of the premiums. In the second case, the product was sold under a level premium scheme, *ceteris paribus*, the expected discounted loss was £50,950m. In the third scenario the product was sold under a net single premium, but this time the premium was calculated using the Monte Carlo method to price the lookback puts, hence the premium was £8.6283 per unit; and the expected discounted profit was £5,157m, or 5.98% of the premium. In Exhibit IV.i.3 a histogram of the final profit under this third scenario can be seen.

Exhibit IV.i.3.- Distribution of the product's final profit



Source: own calculations

If the product were priced using the analytic price for the lookback puts we would be charging the policyholders for a protectio n we are not providing, i.e. we would be charging for a product that has continuous updates in the maximum, while only paying a discretely updated benefit. This is what causes the disproportionate profit in the first scenario; a profit of roughly 6% of the premium is more adequate, as in the third case.

The second scenario shows how selling the product under a level premium scheme can cause the company to incur in losses. Given the characteristics of the product (specially the lack of monotony of the benefit) the reserve could (and in many cases will) be negative, which is unacceptable; in these cases, the surrender value was set to zero, but the loss had to be absorbed by the insurance company.
Having reviewed the behaviour of the product, we will proceed to the conclusions in the next chapter.

V.- CONCLUSIONS.

In this dissertation, the usefulness of derivative products was clearly shown in a completely different environment to that of direct investment in futures and options. From the combination of purely financial products with the existing insurance models interesting products can be developed to cover the needs, of certain market segments, that simple products cannot attack. It is clear that Safinvest's protection is much wider than that offered by the traditional products.

In Chapter I the formulae for life insurance valuation, which were later used to price Safinvest's net premiums, were derived as the expectation of the benefit's present value random variable.

Deriving a distribution for the maximum of a Brownian motion allowed us to apply Black and Scholes model to a series of European style lookback puts, which were found to be the present value of the product's benefit. These theoretical results, found in Chapter II, were useful also when considering the hedging issues faced by the insurance company. Monte Carlo methods were used extensively throughout this dissertation and proved very useful in all instances; in particular, when pricing exotic options (at least in the case of options where exercise does not depend on the path followed by the asset) the method converges neatly to the analytic solution and allows specific features of the option to be embedded in its valuation.

The simulations in the Chapter IV lead us to conclude that the product should not only be sold under a single premium scheme but, furthermore, that the premium should be calculated by Monte Carlo simulation.

Obviously, this dissertation does not exhaust the topic: the development of a new insurance product is a multidisciplinary task and here only some of the technical aspects were covered, nevertheless we do not underestimate the fact that in order to launch this product to the market the rest of the aspects (e.g. legal, marketing, systems) should be studied.

Finally, we would like to highlight the new possibilities this kind of product could open to pension funds (public and private), educational insurance, and other areas that the industry has not explored fully yet.

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