Sensitivity Analysis:

A weighting approach for ranking the order of model inputs

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Abstract

A new approach to sensitivity analysis is introduced in this paper to aid understanding of the nuances of model behavior. The main objective is to identify the most influential inputs in a risk model and rank the inputs based on the sensitivity levels that they exhibit. The framework is closely related to the approach taken by Pesenti et al. (2018).

The methodology adopted consists of two steps to obtain the sensitivities of the inputs. First, different scenarios (states of the world) are weighted using a change of probability measure. Different sets of weights are derived from each input, such that adverse scenarios of that input are assigned higher emphasis. In that sense, each set of weights produces a stress to the distribution of an input. The weights are derived using the normal transform, which has been defined in the context of distortion risk measures. Second, the change in the distribution of the output is quantified, for such a stress applied to each input. The sensitivity to an input, is assessed by comparing standard tail risk measures of the output, such as Value-at-Risk and Expected Shortfall, before and after stressing that input. Further, Monte-Carlo simulation is used to quantify the response of the model when the value of a certain input is varied.

The approach taken in this paper results in computational ease as the calculations can be performed on a single set of simulated input/output scenarios. In addition, it is a coherent approach since all inputs are varied in a consistent way. We use the same distortion function for obtaining all the different sets of weights and each set of weights produced exhibit similar characteristics i.e., higher valued outcomes are given higher emphasis and lower valued outcomes, lower emphasis.

A numerical example is presented to elucidate the effectiveness of the proposed approach.

Keywords: Sensitivity analysis, Value-at-Risk, Expected Shortfall, Distortion Risk Measures, Normal Transform.

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1 Introduction

Managing the risk that an institution faces in the short and long run is foremost the greatest responsibility of a risk manager. As part of the solvency II requirements, it is mandatory for the insurance companies to keep aside enough capital so that the largest loss can be dealt with at a 99.5% confidence level. Companies are in charge of their *own risk and solvency assessment* (ORSA), rather than following a perspective approach to risk quantification and management. Depending on the risk profiles of the insurance businesses, the capital requirements would change and therefore building their own internal models to assess the risk leads to more appropriate calculations (Ralph, 2016).

Numerous factors must be taken into account for model building. Due to the increasing complexities that arise in running a business, complicated models are being integrated into insurance operations. Specifically for an insurance company, all the business lines have to incorporated into the model along with factors such as inflation, interest rate, and expected premium growth. These numerous variables in a model makes it impossible for the analyst to base any judgments on mere intuition and therefore reliance is placed on various simulation techniques. By quantifying portfolio risk and by performing sensitivity analysis, it becomes tractable to pinpoint a particular scenario or a combination of events as catastrophic to the company's performance. It is therefore essential that the analysts understand the model behavior before proposing any changes in the business strategy.

Sensitivity analysis helps aid understanding the relationships between the inputs and output and is an essential component of model building. By running the model several times, it allows us to identify the set of inputs which have the greatest impact on the model output. The inputs which affect the output significantly are said to be highly sensitive. It therefore allows an analyst to focus more on the highly sensitive inputs rather than on inputs which exhibit low sensitivities. Due to process being time-consuming as well as expensive, *reverse* *sensitivity testing* introduced by Pesenti et al. (2018) is an alternative method as it requires only a single set of input/output scenarios.

The structure of the paper is as follows: we discuss risk measures in section 2 placing emphasis on certain risk measures such as Value-at-Risk, Expected Shortfall and Distortion Risk Measures. In section 3, we provide an introduction to sensitivity analysis and the various methods available in the literature. Recent methods such as reverse sensitivity analysis method is acknowledged. In section 4, we propose a new sensitivity analysis method to analyze and compare the sensitivities of the inputs of the model. We use the concept of distortion risk measures to increase the severity of the assumptions considered in the model. We then use Monte-Carlo simulation as a tool to investigate the model behavior. Finally, we set out three variations of the model for the reader to appreciate the usefulness of the method.

2 Risk Measures

2.1 Introduction

The term *risk* in a layman's context indicates any kind of uncertainty that he/she may be facing. Where decisions concerning money have to be taken, there is a natural tendency to quantify the risk involved in the decision. Extending the concept of risk to an insurance company, the process of quantifying risk that it faces is an important task for identifying and to differentiate between business lines which perform well against those that perform poorly. One can view risk measures as a tool which allows companies to control and limit the amount of risk they are willing to take (McNeil et al., 2015).

According to Denuit et al. (2005), a risk measure is a "functional ρ mapping a risk X to a non-negative real number $\rho(X)$, possibly infinite, representing the extra cash which has to be added to X to make it acceptable" to an internal or external controller.

The above definition implies that the interpretation and analysis of the risk inherent in an investment is based on a real value which is obtained as the output. The most obvious relationship between $\rho(X)$ and the risk is a direct relationship ie., as $\rho(X)$ increases, the riskiness of the portfolio under consideration increases. In principle, we can view the underlying concept of risk measures as being analogous to the premium calculations in the insurance sector and this was studied extensively by Bühlmann (2007) and Goovaerts et al. (2001). The basic idea of such a comparability emerges from the viewpoint that the premium calculation reflects the minimum amount that an insurer is willing to accept in order to bear the risk of proceeding with an insurance contract (Tsanakas and Desli, 2003). Therefore, the insurer would charge a higher premium if there is a higher risk involved in the contract. As aptly put, "both insurance premium and the price of a financial product can be regarded as a measure of risk involved in the financial transaction between the buyer of the product and the seller in the market (Goovaerts et al., 2001). For a more comprehensive disquisition, see Denuit et al. (2005) and Goovaerts et al. (2001)).

Quantifying financial risks plays an important role in safeguarding the interests and to ensure smooth functioning of the company. According to Denuit et al. (2005), there are two key approaches to decision making under risk and are as follows:

1. Classical Expected Utility Theory, which was axiomatized by von Neumann and Morgenstern in 1947 (for further discussion, see Kaas et al. (2008, chapter 1)).
2. Yaari's (1987) Dual Theory for Choice under risk.

Yaari's Dual Theory of Choice was proposed as a complementary approach to the Expected Utility Theory in 1987. There were two main reasons behind the motivation for the Dual Theory. The first being that under the utility theory, *risk aversion* and *diminishing marginal utility of wealth* are in many respects alike. Risk aversion is the basic characteristic exhibited by an agent. Here, an agent would tend to avoid those deals which involves greater uncertainty. Diminishing marginal utility implies that the agents' total wealth relates directly to the size of the losses he is willing to accept. Therefore, higher the amount of wealth that an agent possesses, the higher the level of risk that he is willing to accept. The second reason behind the motivation as stated by Yaari is that certain behavior patterns that are inconsistent, such as those observed in Allais (1953) and Kahneman-Tversky (1979) can be explained by the dual theory. Allais questions the "independence axiom" and from experimental evidence, we note that some people prefer certain choices of investment which do not seem rational when Expected Utility theory is considered. Kahneman-Tversky in their seminal paper *Prospect Theory: An analysis of Decision under Risk* argue that people tend to change their behavior patterns depending on how the problem is presented to them. For instance, if there is a greater emphasis on "gains", individuals are "risk averse" but when emphasis is placed on "losses", individuals tend to be more risk-seeking. Further, individuals prefer to choose those situations where an outcome is certain as opposed to those where a probability is attached to the outcome amount, even if the latter is an equally good or a better deal (Kahneman and Tversky, 2013). Hence, the inconsistencies that arise while considering Expected Utility Theory are enormous and therefore Dual Theory might be a good alternative.

Dual theory itself generates certain paradoxes which can be resolved by the utlity theory. Thus according to Yaari, the two theories resolve each others paradoxes. Attempts have been made by Tsanakas and Desli (2003) to combine the aforementioned theories to obtain a new risk measure, *Distorted Exponential Premium Principle*, which possesses properties from both theories.

2.2 Properties of Risk Measures

There are a few fundamental and important properties of risk measures that are useful to know, but there exists no such set of properties that an ideal risk measure should satisfy under all circumstances. It should be left to the discretion of the analyst to decide whether a particular metric which they would like to apply, must satisfy a particular property or not depending on the situation. A few properties are mentioned below and some of them are simple and self-explanatory.

Let X, Y be the random variables (r.v. for short) denoting the losses.

1. No-Ripoff

$$\text{For bounded r.v. } X, \rho(X) \leq \max(X) = F_X^{-1}(1) \tag{1}$$

   The no-ripoff property implies that it is inefficient for an institution to hold more money than the maximum possible loss as this would guarantee 100% solvency. Holding more capital than what is required is uneconomical as this can be invested in other instruments such as bonds and shares to obtain a higher value.

2. Non-Negative Loading

$$\rho(X) \geq E(X) \tag{2}$$

   The least amount of capital that must be kept aside by an institution must be equal to at least the loss expected to incur during a given time period. This is because we assume that the rough estimate of the predictive loss is equal to the mean of the past losses.

3. Translation Invariance

$$\rho(X + c) = \rho(X) + c \tag{3}$$

   As we have defined X to be a random variable representing the losses, this implies that if the precise amount of a loss c is known in advance then it is equivalent to adding the deterministic cash amount c to the position in order to comply with the definition of a risk measure mentioned above (Denuit et al., 2005).

4. Constancy

$$\rho(c) = c \tag{4}$$

We had specified c to be a deterministic loss and hence for a company to remain solvent, we would need to set aside an amount c. This is not the case that we generally expect to observe in the real world situations but, if we broaden X to denote liabilities, then we can expect to have some deterministic liabilities in the future (Denuit et al., 2005). This property does not take into consideration the interest rate.

5. Sub-additivity

$$\rho(X + Y) \leq \rho(X) + \rho(Y) \text{ for all r.v.s } X \text{ and } Y \tag{5}$$

The central idea of sub-additivity was summed up by Artzner et al. (1999) that *a merger does not create extra risk*. This introduces the effect of diversification and for sub-additive risk measures, the resultant effect is positive (Denuit et al., 2005). Diversification effect is defined by McNeil et al. (2015) as the "difference between the sum of the risk measures of stand-alone risks and the risk measure of all risks taken together."

Sub-additivity is a much debated concept in respect to the necessity of a risk measure satisfying the property as we will see further on that Value-at-Risk, which is a commonly used risk measure, does not satisfy the property. However, there have been attempts to justify its necessity as according to McNeil et al. (2015), a financial institution can reduce its capital requirements by breaking it up into various subsidiaries if a non-subadditive risk measure is used. Artzner et al. (1999) and Wang et al. (1997) also claim that risk measures would satisfy sub-additivity for any dependence structure, as risks that exhibit positive dependence also provide diversification to a certain level (Tsanakas and Desli, 2003).

6. Co-monotonic Additivity

Firstly, co-monotonicity is defined as the following by Denuit et al. (2005):

A random vector $\boldsymbol{X}$ is co-monotonic if and only if there exist a r.v. Z and non-decreasing functions $t_1, t_2, ..., t_n$, such that :

$$\boldsymbol{X} =_d (t_1(Z), t_2(Z), ..., t_n(Z))^t \tag{6}$$

$(X_1^c, X_2^c, ..., X_n^c)$ denotes a co-monotonic random vector and for any two r.v.s X and Y if $F_X \equiv F_Y$, then $X =_d Y$(Denuit et al., 2005).

Co-monotonic r.v.s are also referred to as *undiversifiable r.v.s.* It implies that the variables are increasing functions of one another. A more precise definition can be found in McNeil et al. (2015).

The property is stated as follows :

$$\rho(X + Y) = \rho(X) + \rho(Y) \text{ for all r.v.s } X \text{ and } Y \tag{7}$$

From the above, we can notice that the combined risk can never decrease with respect to their stand-alone risks. This can be viewed as a special case of the sub-additivity property where there is no effect of diversification. Therefore, it is a *no-hedge condition* (Denuit et al., 2005).

7. Positive Homogeneity

$$\rho(cX) = c\rho(X), \text{ for all random variables } X \text{ and } c \in \mathbb{R}_+ \tag{8}$$

This is a similar concept to the one seen above and it was proposed by Artzner et al. (1999). From the equation, it is clear that if we increase the portfolio size, then the capital requirements would also increase proportionally. However, it does fail to take into consideration the extra increase in the liquidity risk that the agent would face if he increases his portfolio size.

By increasing the magnitude of the portfolio, the agent is now faced with a higher risk of experiencing larger losses which in turn makes it harder to cope up with the losses (Tsanakas and Desli, 2003).

8. Monotonicity

$$\text{If } X \leq Y \text{ almost surely} \Rightarrow \rho(X) \leq \rho(Y) \tag{9}$$

If loss Y is at least equal to or greater than loss X in most cases, then it is rational to allocate proportionally more capital for Y than X to deal with the losses appropriately. In other words, the greater the returns of a portfolio, the less risky it is considered to be.

9. Law of Invariance / Objectivity

$$X =_d Y \Rightarrow \rho(X) = \rho(Y) \tag{10}$$

This property implies that a risk measure is 'objective' if ρ(X) depends on X only through its cumulative distribution function $F_X(x)$.
Objectivity plays an important role for the applicability of a risk measure. If a measure is observed to satisfy this property, then the risk measure can be estimated using the empirical data (Denuit et al., 2005). Therefore, it is a desirable property as the riskiness of X can be measured through $F_X(x)$.

10. Convexity

Convexity is a more general property and we see below that the risk measure is sub-additive when the property of convexity is combined with positive homogeneity.

$$\rho(\lambda X + (1-\lambda)Y \leq \lambda\rho(X) + (1-\lambda)\rho(Y) \tag{11}$$

Further, a risk measure is classified as a *coherent risk measure* if it exhibits four properties: translation invariance, positive homogeneous, sub-additivity and monotonicity (Artzner et al., 1999). However according to

Danielsson et al. (2005), coherence is not an appealing mathematical property from a financial point because it is a restrictive condition and alternative risk measures which are coherent can not be widely applied due to the complexities that arise while back-testing. Priority must be given to economic justification rather than mathematical tractability (Dhaene et al., 2003).

2.3 Some Important Risk Measures

Risk Measures can be classified and studied in a number of ways and we note that the classification need not be exclusive. In other words, there may be instances where a risk measure may be classified under more than one category. Without dwelling deep into the different classifications that exist, we would like to specifically look at Value-at-Risk, Expected Shortfall, Distortion Risk Measures and their important properties.

2.3.1 Value at Risk

According to Denuit et al. (2005),

$$\mathrm{VaR}_p(X) = F_X^{-1}(p), \tag{12}$$

for a given risk X and $p \in (0,1)$ and where,

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} | F_X(x) \geq p\} \tag{13}$$

p is also referred to as the confidence level.

The origins of VaR can be traced back to the early twentieth century where actuaries used the concept of VaR implicitly for calculating the initial reserve in classical ruin problems. The measure was used by the financial institutions as a means of forecasting and aggregating risks but, as firms started performing more complex operations, it became increasingly more important to understand the interactions between various variables in context (Dowd and Blake, 2006). The interpretation of VaR as given in McNeil et al. (2015) is that it is the "maximum loss that is not exceeded with a given high probability." It is clear from the definition, that VaR does not take into account the tail of the distribution. In other words, it is blind to the shape of the distribution beyond the value p and therefore it is indifferent to the magnitude of the damage beyond the agreed threshold p. Hence, it is advantageous to use a measure which can cope with dependencies and also takes into account the tail of the distribution (Goovaerts et al., 2001). One such measure which takes into account the tail of the distribution is the Expected Shortfall (also called as the Tail Value-at-Risk) which is discussed in section 2.3.2. Furthermore, as VaR is a quantile based risk measure, the user must be aware of the existent discontinuities and intervals in the distribution function as this can have a significant effect on the output.

According to Acerbi and Tasche (2002), VaR is in fact not suitable to be labeled as a risk measure as it does not satisfy the property of sub-additivity. They place a more stringent emphasis on the need for a measure to be coherent and even though there may exist alternative means of defining coherence, they stress on the importance of sub-additivity as a mandatory requirement. They argue that it is easy to show that different portfolios with varying levels of risk can have the same VaR. One such example can be found in Danielsson et al. (2005). To summarize the flaws of VaR, we state the two reasons from Artzner et al. (1999) to reject VaR as a measure of risk :

(a) value at risk does not behave nicely with respect to the addition of risks, even independent ones, thereby creating severe aggregation problems.

(b) the use of value at risk does not encourage and, indeed, sometimes prohibits diversification because value at risk does not take into account the economic consequences of the events, the probabilities of which it controls.

Despite its limitations and weaknesses, VaR is the most commonly used risk metric today by banks and regulators and its importance in Basel II and Solvency II is likely to remain unchanged. This may be because it is a convenient measure to estimate and implement and also according to Danielsson et al. (2005), subadditivity violations are a serious concern only if the tail of the distribution is super fat (ie., 1st moment does not exist). We note here for the benefit of the reader that the tail index measures the thickness of the tail and for most asset classes, the tail index is observed to be between 3 and 5 (Danielsson et al., 2005), which is not considered as super fat. Another important practical advantage of VaR is that back-testing VaR estimates are easier than other types of estimates.

2.3.2 Expected Shortfall

The definition given by Denuit et al. (2005) is as follows :

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_p(X)dq, \tag{14}$$

for a given risk X and a probability level $p \in (0,1)$.

Corresponding changes in the formula can be made if we consider a discrete loss distribution rather than a continuous one.

The interpretation given by Denuit et al. (2005) for the expected shortfall (ES for short) is that it is the "average loss in the worst $100(1-p)\%$ cases" or simply the average of the losses exceeding the VaR at the confidence level p.

An alternative representation of ES when $F_X(x)$ is continuous is as follows:

$$\mathrm{ES}_p(X) = E(X|X > \mathrm{VaR}_p(X)) \tag{15}$$

ES is also referred to as the tail conditional expectation, worst conditional expectation, conditional VaR and so on in other literature papers. These inconsistencies arise as different authors use different terminology for the same definition or otherwise the definitions vary slightly. However, all the mathematical formulas of the various alternatives lead to same results if a continuous loss distribution is considered, while differences emerge when we either use a discrete loss distribution or a distribution that has discontinuities (Acerbi and Tasche (2002); Dowd and Blake (2006); McNeil et al. (2015)).

ES at p can also be understood as an adjusted value of VaR at a confidence level greater than p. From the above, it is clear that ES is at least equal to or greater than VaR (Rockafellar et al., 2000). Hence, ES is better suited to measure the tail as compared to VaR and unlike VaR, it is always sub-additive. We note here that ES is a coherent risk measure as it satisfies all the four required properties needed for it to be classified as a coherent risk measure whereas, VaR is not coherent in general as it fails to fulfill sub-additivity in some cases.

A table is presented below which shows the properties satisfied by the two risk measures - VaR and ES. We use check marks to denote if a particular property is satisfied, whereas a cross denotes that the particular property is not satisfied by the corresponding risk measure at all times.

Property	Value-at-Risk	Expected Shortfall
No-Ripoff	✓	✓
Non-Negative Loading	✗	✓
Translation Invariance	✓	✓
Constancy	✓	✓
Sub-additivity	✗	✓
Co-monotonic Additivity	✓	✓
Positive Homogeneity	✓	✓
Monotonicity	✓	✓
Objectivity	✓	✓
Convexity	✗	✓

2.4 Distortion Risk Measures

We have earlier seen the motivation behind the dual theory of choice under risk. In Yaari (1987), he mentions that his proposed theory requires independence with respect to direct mixing of payments of risky prospects and not the independence with respect to probability mixtures of risky prospects.

Distortion risk measures (DRM for short) are widely applied for calculating insurance premiums, risk management and in importance sampling for the Monte Carlo simulations (McLeish and Reesor, 2003). ES for example is a coherent DRM while VaR is a DRM but not coherent.

2.4.1 Definition and Interpretation

Dowd and Blake (2006) define *distortion risk measure* as the expected loss under a transformation of the cumulative distribution function.

$$\rho_g(X) = -\int_{-\infty}^{0} (1 - g(\overline{F}_X(x)))dx + \int_{0}^{\infty} g(\overline{F}_X(x))dx \tag{16}$$

When $X \geq 0$, we get

$$\rho_g(X) = \int_{0}^{\infty} g(\overline{F}_X(x))dx, \tag{17}$$

where, $\overline{F}_X(x) = P(X > x)$ and g is a *distribution function* such that g : [0,1] $\rightarrow$ [0,1] satisfying $g(0) = 0$ and $g(1) = 1$. g must be increasing and right-continuous (McLeish and Reesor, 2003).

$F_X(x)$ is a tail function and for g as defined above, $g(\overline{F}_X(x))$ is also a tail function.

ρ_g is thus the expectation with respect to the distorted tail function $g(\overline{F}_X)$ and in particular if $g(s) = s$, $\rho_g(Y) = E(Y)$.

g is referred to as the *distortion function* as it distorts $\overline{F}_X(x)$ before the generalized expected value is calculated (Denuit et al., 2005).

The above function ρ possesses the following properties: no-ripoff, positive homogeneity, translation invariance, monotonicity, co-monotonic additivity.

ρ is coherent if and only if g is concave (refer to the appendix for the proof).

If g is differentiable,

$$\begin{aligned}
\rho_g(X) &= \int_0^\infty g(\overline{F}_X(x))dx \\
&= [xg(\overline{F}_X(x)]_0^\infty - \int_0^\infty xg\prime(\overline{F}_X(x))\frac{d}{dx}\overline{F}_X(x)dx \\
&= \int_0^\infty xg\prime(\overline{F}_X(x))dF_X(x) \\
&= E[Xg\prime(\overline{F}_X(X))] \\
&= E[F_X^{-1}(U)g\prime(1-U)] \qquad\qquad \text{where, } U \sim U(0,1).
\end{aligned}$$

We notice that $g\prime(\overline{F}_X(x))$ is a re-weighting of the loss distribution and for concave g, higher weights are assigned to larger values of X (Tsanakas and Desli, 2003). Therefore, the distorted risk measure is in fact a weighted average of percentiles.

We can understand the inherent nature of the agent in terms of the risk he is willing to accept by observing the behavior of the distortion and this is done by looking at the difference between the original and the distorted preferences. A risk averse agent would tend to be more cautious while estimating the probabilities to take as little risk as possible. In such a case, it is natural and prudent for the agent to overestimate the extreme events and therefore the tail probabilities. According to Denuit et al. (2005), for a risk averse agent the following would always hold:

$$g(q) \geq q \implies g(\overline{F}_X(x)) \geq \overline{F}_X(x) \text{ , } x \epsilon \mathbb{R}.$$

From the above, we can say that $\rho_g(X) \geq E(X)$.
For a risk neutral agent, the distortion and the original preferences would match at each point and therefore, $g(q) = q \implies \rho_g(X) = E(X)$.

2.4.2 Examples

1. We now demonstrate that ES is a DRM:

   ES is obtained as a DRM using the distortion function

   $$g(q) = \begin{cases} \frac{q-\alpha}{1-\alpha}, & \text{if q} \geq \alpha \\ 0, & \text{if } q < \alpha \end{cases}$$

   ES is coherent since $g(q)$ is concave.

   The distortion of ES is obtained by adjusting one parameter and therefore, they are also referred to as the "one parameter distortion functions." ES is also used in Basel III for determining minimum capital requirements to ensure capital adequacy (Schumacher, 2018).

2. Normal Transform Risk Measure:

   A very important concept is the Normal Transform risk measure introduced by Wang (2000), which will be used in later sections of this paper as well. The distortion function is given by

   $$g(q) = \Phi(\Phi^{-1}(q) + \lambda) \tag{18}$$

   Φ represents the cumulative distribution function of a standard normal distribution and $q \in [0, 1]$. We refer to the risk measures derived from the given distortion function as normal transform risk measures (Denuit et al., 2005).

   As noted in section 2.3.1, VaR is just a point on the distribution whereas, ES considers the tail of the distribution beyond VaR. The normal transform on the other hand from equation (18), takes into consideration the entire distribution of the porftolio and in that sense, it may be regarded as a superior measure to ES and VaR (Dowd and Blake, 2006).

3. VaR can be obtained using the piecewise function :

   $$g(q) = \begin{cases} 1 & \text{if } q \geq \alpha \\ 0 & \text{if } q < \alpha \end{cases}$$

   VaR is not coherent as $g(q)$ is not concave.

3 Sensitivity Analysis

3.1 Literature Review

3.1.1 Why Sensitivity Analysis is used?

Saltelli (2002) defines sensitivity analysis as the study of how the uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in its inputs. An insurance company needs to take a rigorous approach before amending its existing business strategy, as any change has to be validated to confirm that it enhances the value of the business. Sensitivity analysis is an important tool for substantiating such changes. Another reason for the extensive use of sensitivity analysis by the insurance companies is because an analyst can not foresee the variations in the outputs by mere intuition as the models used are complicated (Borgonovo and Plischke, 2016).

Sensitivity analysis is a flexible approach where each input in the model can be altered and by performing simulations, the corresponding output can be obtained. Therefore, the user is able to detect the response of the model in terms of magnitude of the impact and the direction of the change (Borgonovo and Plischke, 2016). Consequently, comparisons of the output obtained before and after the change in the value of an input can be used as a litmus test before taking a decision (Pritchard et al., 2014). Altering other parameter values one at a time in a similar way would result in many outputs and the procedure can be repeated for each input in the model. Referred to as the One-at-a-time (OAT) method, it is the simplest method that can be applied on a model. Various analyses can be performed on the different outputs obtained to understand the vulnerability of the model and conclusions can be made by ranking the degree of the influence of each input (Tsanakas and Millossovich, 2016).

Sensitivity analysis is thus used when the values of the input parameters are uncertain and using VaR and ES when carrying out sensitivity analysis is particularly helpful for decision making (Tsanakas and Millossovich, 2016). Though sensitivity analysis can be applied for both quantitative and qualitative assessments (Pritchard et al., 2014), in this paper we will focus on the quantitative assessments.

3.1.2 Local and Global Measures

Sensitivity analysis methods can be broadly classified into local and global methods (Borgonovo and Plischke, 2016). Local methods are put into practice when probability distributions are not specified to the inputs (deterministic framework) and when the user is required to make analysis around "a point of interest in the model input space" (Borgonovo and Plischke, 2016). The local includes methods such as Tornado diagrams, One-way sensitivity functions, Differentiation-based methods and Scenario decomposition through finite change sensitivity indices while methods such as sequential bifurcation, Morris method, Variance-based, Moment-Independent are examples of global measures.

A global measure "reflects the model behavior over the whole of the input range" (Tsanakas and Millossovich, 2016). " Both local and global measures are useful depending on the information available to the analyst. Variance-Based methods are used for apportioning the uncertainty of an output to different inputs and the interactions between them (Marzban, 2013). Here, the output variance is split up into two parts as the 'explained variance' resulting from the dependence on the inputs and the 'residual variance' (Borgonovo and Plischke, 2016). Moment-Independent methods can be used where the model inputs are allowed to vary thus making it possible to obtain the unconditional output densities. By using this method, the sensitivities can be assessed by considering entire distribution (Borgonovo and Plischke, 2016). For further discussion about the different types of sensitivity methods, refer to Borgonovo and Plischke (2016).

3.1.3 Reverse Sensitivity Testing

In addition to the existing local and global methods, Pesenti et al. (2018) have introduced a new sensitivity analysis method, referred to as the *Reverse Sensitivity Testing*, in an attempt to determine the most influential factor which drives the model, as it could be the basis under which a model can fall apart. It is a global measure as it is used when probability distributions of inputs are known.

The application of reverse sensitivity testing can bring about a few advantages. Firstly, it is computationally less arduous, as Monte-Carlo simulations can be performed on a single set of input/output scenarios. Secondly, factor prioritisation, which is the process of identifying and ranking the most influential factors, is based on the changes in the output distribution rather than considering an output state. Furthermore, an input is proclaimed to be highly sensitive from the changes observed in its distribution. Therefore, the one with the most noticeable change is termed to be the most sensitive. The user should be able to decide whether the highly sensitive input(s) can 'break the model'.

We state the three steps required in order to perform the analysis presented by Pesenti et al. (2018).

1. An output stress is specified, corresponding to an increase in the risk measure(s) of the output.

   The objective of deriving the stresses is to intensify the problem faced by the agent and therefore obtaining larger losses. The risk measure VaR was used to obtain the stress and by doing so there may arise a situation where there are inadequate resources to meet the regulatory requirements. This would allow to test the model under a more severe set of conditions.

2. A stressed probability measure is derived by minimizing the Kullback-Leibler divergence with respect to the baseline probability, under constraints generated by the stress on the output risk measure.

Due to the manner in which the stresses are derived, they have obtained a constant weight which is applied to claims below a certain value and a larger weight which is applied to claims above a certain value to derive the distortion probability distributions.

3. changes in the distributions of input factors are evaluated under the stressed conditions to assess the sensitivities. An input is said to be highly sensitive if there is a considerable change in its distribution under the stressed conditions.

3.2 A New Sensitivity Analysis Method

3.2.1 Motivation

A modified approach of the reverse sensitivity testing is presented in this paper. Pesenti et al. (2018) focused on distorting the output in order to observe the changes in the inputs. However, in this paper, importance is given to the input to output relationship. In other words, we distort the inputs to observe changes in the output. As a consequence of the methodology, distorting one input would lead to the distortion of other correlated inputs of the model.

For the steps mentioned in section 3.1.3, we use different techniques for our modified version.

In the first step, instead of minimizing the Kullback-Leibler divergence, we use the normal transform risk measure to obtain the stresses on the inputs. Pesenti et al. (2018) derived a Radon-Nikodym density corresponding to weights on particular states of the world. The weights obtained are piecewise constant and therefore, a low value is used to stress the low values of Y and a higher emphasis is applied to higher values of Y. However, by using a normal transform, we obtain a continuously increasing weights to distort the output Y. This perhaps is more suitable as it is effective to obtain greater distortions for Y.

For the second step, we use the risk measures VaR and ES as a means of comparison between the baseline and the stressed models. We calculate the percentage increase in VaR and ES for the stressed models with respect to the baseline to analyze the sensitivity levels of the input factors.
Another possibility is to alternatively use mean and standard deviation to get a quick glance of the model behavior.

3.2.2 Setup and Notation

Let X_1, X_2,..., X_n denote n random variables representing the input factors of the model under consideration and we indicate these n input factors by a vector $\boldsymbol{X} = (X_1, X_2, \dots X_n)$. Let ψ be the aggregation function that maps the inputs to an output. We write, $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$. We apply the function ψ on the inputs to obtain the one-dimensional random output $Y = \psi(\boldsymbol{X})$.

Let X_i follow a distribution function denoted by F_i, $X_i \sim F_i$, and let Y follow a distribution denoted by H, $Y \sim H$.

Let $\mathcal{P}$ denote the set of all the probability measures on a measurable space $(\Omega, \mathcal{A})$. We denote the baseline model as $(\boldsymbol{X}, \psi, P)$ with baseline probability measure $P \in \mathcal{P}$.

Define the random variable W_i by $W_i = g'(1 - U_i)$, where $U_i \sim Uniform(0, 1)$ such that W_i is co-monotonic to X_i. g is the normal transform given by
$g(q) = \Phi(\Phi^{-1}(q) + \lambda)$.

Define a new measure Q_i by $\frac{dQ_i}{dP} = W_i$. This is the Radon-Nikodym density that operates as weights on different scenarios. Thus, we can write the expectation of a random variable, Z, under the new probability measure Q_i as follows:

$$E^{Q_i}(Z) = E(W_i Z)$$

The main objective of this paper is to observe the distortions caused to the distribution Y using the stresses obtained from the normal transform risk measure. Therefore, in the plots and tables that we produce in section 4.1.2, we aim to compare the distribution function and risk measures (VaR and ES) of the output Y under the baseline model (P) with the distribution function and risk measures under the new probability measure Q_i. Hence, a significant change in the output distribution under the new measure Q_i would indicate that the output is particularly sensitive to the input X_i.

We derive $g'(q)$ as follows:

$$\begin{aligned} g'(q) &= \frac{d}{dq}\Phi(\Phi^{-1}(q)+\lambda) \\ &= \Phi'(\Phi^{-1}(q)+\lambda)\frac{d}{dq}(\Phi^{-1}(q)+\lambda) \\ &= \Phi'(\Phi^{-1}(q)+\lambda)\frac{d}{dq}(\Phi^{-1}(q)) \\ &= \Phi'(\Phi^{-1}(q)+\lambda)\frac{1}{\Phi'(\Phi^{-1}(q))} \end{aligned}$$

Evaluating it at $(1-U_i)$, we arrive at the following which is used in our example to generate the weights for the distortion.

$$W_i = g_p'(1-U_i) = \Phi'(\Phi^{-1}(1-U_i)+\lambda)\frac{1}{\Phi'(\Phi^{-1}(1-U_i))}; i = 1, 2, ..., n \qquad (19)$$

We denote $Y \sim H^{Q_i}$ under the new measure Q_i. Any distribution function can be written as an expectation using the indicator function, we can write the new distorted distributions obtained under Q_i as follows:

$$H^{Q_i}(t) = Q_i(Y \leq t) = E^{Q_i}(\mathbf{1}_{\{Y\leq t\}}) = E(\mathbf{1}_{\{Y\leq t\}}W_i)$$

It holds that: $\mathrm{VaR}_p^{Q_i}(Y)$ is the inverse of H^{Q_i} at p, that is, the value such that $H^{Q_i}(\mathrm{VaR}_p^{Q_i}(Y)) = p$.

Thus, a change in the measure is a simple method to put different weights for different outcomes.

For Expected Shortfall we have:

$$
\begin{aligned}
\mathrm{ES}_p^{Q_i}(Y) &= E^{Q_i}(Y|Y > \mathrm{VaR}_p^{Q_i}(Y)) \\
&= \frac{1}{Q_i(Y > \mathrm{VaR}_p^{Q_i}(Y))} E^{Q_i}\left(Y \cdot \mathbf{1}_{\{Y > \mathrm{VaR}_p^{Q_i}(Y)\}}\right) \\
&= \frac{1}{1-p} E\left(W_i \cdot Y \cdot \mathbf{1}_{\{Y > \mathrm{VaR}_p^{Q_i}(Y)\}}\right) \qquad (20)
\end{aligned}
$$

Percentage increase in VaR and ES for each input is calculated for each distortion case with respect to the baseline model. This value is assigned to be the sensitivity of the input.

3.2.3 Implementation in a Monte-Carlo Setting

As mentioned above, H is the distribution of Y under P and H^{Q_i} is the distribution of Y under measure Q_i, where $W_i = \frac{dQ_i}{dP}$. Let us for simplicity assume that both distributions H and H^{Q_i} are continuous. Assume that we have an iid simulated sample from Y obtained under P (in other words, drawn from the distribution H) and denote that sample by $y^{(1)}, \ldots, y^{(m)}$, where m is the number of simulations performed. Furthermore, we have corresponding samples from the Radon-Nikodym density W_i; these are denoted by $w_i^{(1)}, \ldots, w_i^{(m)}$. We are interested in calculating $\mathrm{VaR}_p^{Q_i}(Y)$ and $\mathrm{ES}_p^{Q_i}(Y)$ from that simulated sample.

We can estimate H^{Q_i} and $\mathrm{VaR}_p^{Q_i}(Y))$ by

$$
\widehat{H^{Q_i}}(t) = \frac{1}{m}\sum_{k=1}^{m} w_i^{(k)} \mathbf{1}_{\{y^{(k)} \leq t\}}
$$

$$
\widehat{\mathrm{VaR}_p^{Q_i}}(Y) = \inf\{t \in R : H^{Q_i}(t) \geq p\}
$$

The above expression for $\widehat{\mathrm{VaR}_p^{Q_i}}(Y)$ reflects the fact that the empirical distribution $\widehat{H^{Q_i}}(t)$ is actually discrete. Now, turning attention to the Expected Shortfall, we can estimate equation (20) by

$$
\widehat{\mathrm{ES}_p^{Q_i}}(Y) = \frac{1}{m(1-p)} \sum_{k=1}^{m} w_i^{(k)} y^{(k)} \mathbf{1}_{\{y^{(k)} > \widehat{VaR_p^{Q_i}}(Y)\}}
$$

4 Examples

4.1 Benchmark Model

4.1.1 Model Specification

Consider an insurance portfolio consisting of the input factors X_1, X_2, X_3 and X_4 with the same specifications as in Pesenti et al. (2018). X_1 represents a set of claims that follow a Log-normal distribution with mean 150 and standard deviation 35 and X_2 represents the set of claims that follow a Gamma distribution with mean 200 and standard deviation 20. We do not in this paper discuss the validity of making such assumptions.

Define $L = X_1X_3 + X_2X_3$, where X_3 is the inflation factor and it follows a Log-normal distribution with mean 1.05 and standard deviation 0.02. We make a further assumption that X_1, X_2 and X_3 are independent random variables.

The model takes into consideration the reinsurance. Let X_4 model the proportion of the amount that is lost by the insurer due to the failure of payment by the re-insurer. Assume that X_4 follows a Beta distribution with mean 0.1 and standard deviation 0.2. We assume that X_4 is related to L through a Gaussian copula and has a correlation of 0.6.

We can now represent the total liability (Y) as

$$Y = L - (1 - X_4)\min\{(L-d)_+, l\} \tag{21}$$

where, d is the deductible and l is the limit. Assign $d = 380$ and $l = 30$.

4.1.2 Analysis and Results

As discussed in section 3.2.1, the weights derived from the normal transform risk measure are used to obtain the stressed models. We assume three different values for λ thus, generating three sets of weights or three sets of distorted probability measures for each random variable separately in the model considered.
We use the values: $\lambda = 0.5$, 1.0 and 1.5.

In figure 1, we plot samples of the random variables $W_1, \ldots, W_4$ against $X_1, \ldots, X_4$ respectively.

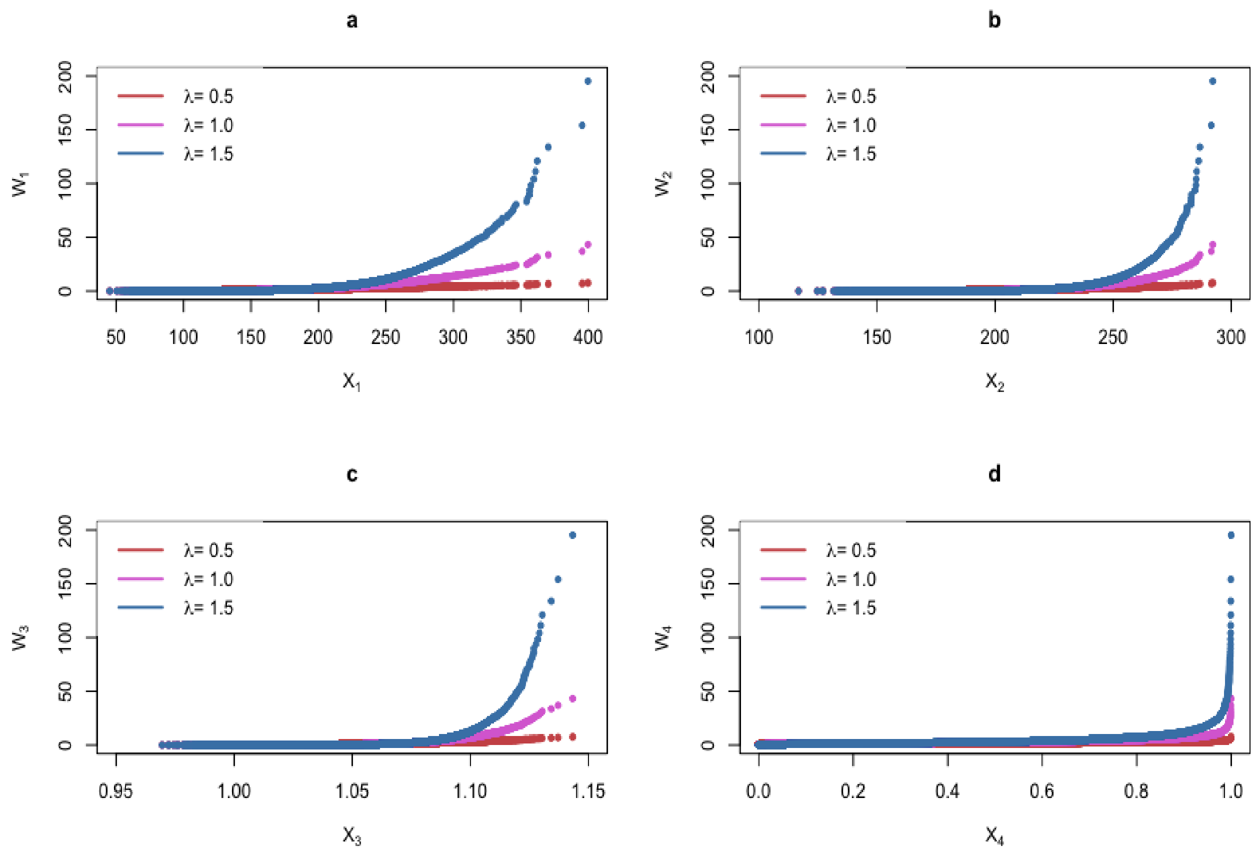
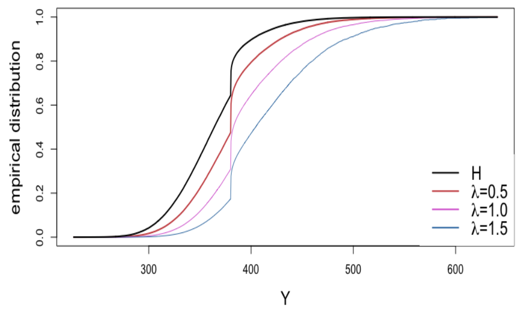


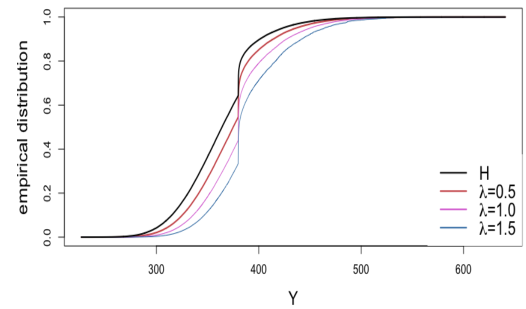
Figure 1: Samples of W_1, W_2, W_3, W_4 against X_1, X_2, X_3, X_4 respectively, for $\lambda = 0.5, 1, 1.5$.

From figure 1(a), we note that for claims below 200 units the weights, W_1, do not vary significantly for different λ values. However, for claims greater than 200 units, heavier weights are assigned as λ increases. Similar observations can be made from figure 1(b), 1(c) and 1(d). Hence, it is reasonable to say that a higher λ focuses more on the higher valued claims rather than lower valued claims. By increasing the λ, we are intensifying the severity of the conditions assumed and hence, we can presume that a highly risk averse agent would prefer assigning a higher λ value to carry out the sensitivity analysis.

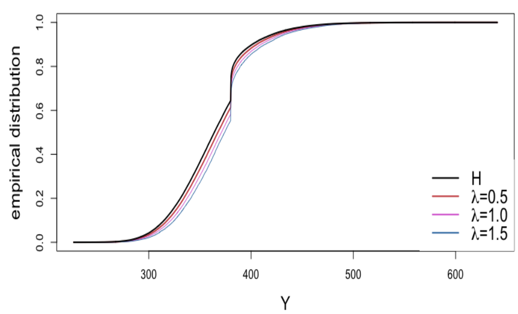
(a) Distortions of Y obtained using W_1



(b) Distortions of Y obtained using W_2



(c) Distortions of Y obtained using W_3



(d) Distortions of Y obtained using W_4

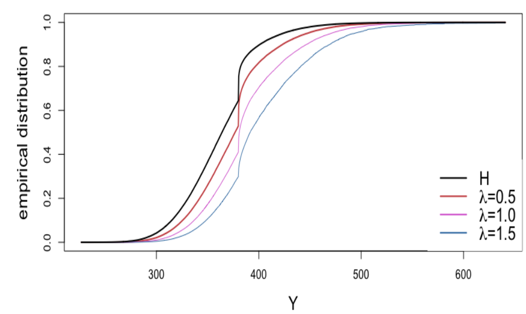


Figure 2: Distorted probability distributions of Y obtained using W_1, W_2, W_3, W_4 for $\lambda = 0.5, 1.0, 1.5$

Figure 2 shows $H^{Q_i}(Y)$ using W_i, i =1,2,3,4 for the λ values alongside the baseline probability distribution, H. As the distortions for each λ value in figure 2(a) and 2(d) are more pronounced than in figures 2(b) and 2(c), X_1 and X_4 are the most sensitive inputs for Y. This is validated from the values obtained in Tables 1-4 as the tables show the percentage increase in VaR and ES with respect to the VaR calculated for the benchmark model for Y at different confidence intervals.

The r.v.s X_1 and X_2 are the claims modeled by a Log-normal and a Gamma distribution respectively. As Log-normal is heavier tailed than a Gamma distribution, the percentage increase in VaR and ES of Y obtained for different values of λ in Tables 1 and 2 vary substantially. For instance in Table 1, the percentage increase in VaR at $p = 0.99$ for $\lambda = 1.5$ is 24.08 whereas, the corresponding value in Table 2 is only 8.38. Further, the percentage increase in VaR and ES at each confidence level in Table 2 is lower than the corresponding value in Table 1. This is attributable to the fact that a higher λ impacts the Log-normal to a greater extent than the Gamma distribution. Hence, we say that X_1 is more sensitive than X_2. Comparing the tables in a similar manner results in the ordering of inputs' sensitivities as X_1, X_4, X_2 and X_3.

Table 1: Percentage increase in VaR_p and ES_p for H^{Q_1} with respect to the baseline model, H.

	% increase in VaR_p				% increase in ES_p			
	p =0.75	p=0.85	p= 0.95	p=0.99	p =0.75	p=0.85	p= 0.95	p=0.99
$\lambda = 0.5$	3.06	6.35	6.94	7.51	6.15	6.92	7.11	7.68
$\lambda = 1.0$	10.11	13.75	14.46	15.48	13.72	14.63	15.31	16.59
$\lambda = 1.5$	17.92	21.74	23.56	24.08	22.06	23.23	23.92	25.05

Table 2: Percentage increase in VaR_p and ES_p for H^{Q_2} with respect to the baseline model, H.

	% increase in VaR_p				% increase in ES_p			
	p =0.75	p=0.85	p= 0.95	p=0.99	p =0.75	p=0.85	p= 0.95	p=0.99
$\lambda = 0.5$	0.90	2.87	3.10	2.73	2.63	3.04	2.88	2.53
$\lambda = 1.0$	3.28	6.34	6.33	5.43	5.78	6.25	5.84	5.03
$\lambda = 1.5$	6.86	9.71	9.64	8.38	9.19	9.56	8.78	7.23

Table 3: Percentage increase in VaR_p and ES_p for H^{Q_3} with respect to the baseline model, H.

	% increase in VaR_p				% increase in ES_p			
	p =0.75	p=0.85	p= 0.95	p=0.99	p =0.75	p=0.85	p= 0.95	p=0.99
$\lambda = 0.5$	0.15	0.88	1.07	1.04	0.86	1.06	1.11	0.93
$\lambda = 1.0$	0.42	1.85	2.07	2.01	1.74	2.08	2.08	1.77
$\lambda = 1.5$	0.88	2.68	2.97	2.93	2.57	2.99	2.93	2.28

Table 4: Percentage increase in VaR_p and ES_p for H^{Q_4} with respect to the baseline model, H.

	% increase in VaR_p				% increase in ES_p			
	p =0.75	p=0.85	p= 0.95	p=0.99	p =0.75	p=0.85	p= 0.95	p=0.99
$\lambda = 0.5$	2.19	4.94	5.58	5.43	4.77	5.44	5.42	5.46
$\lambda = 1.0$	7.23	10.85	11.10	10.75	10.46	11.16	11.10	12.57
$\lambda = 1.5$	13.13	16.66	16.43	18.02	16.35	17.03	17.53	23.43

While our emphasis is on demonstrating how a stress in X_i affects the distribution of Y, it is also the case that a stress on X_i modifies the distributions of other inputs, if these are correlated with X_i. We exhibit these distorted probability distributions of the input factors obtained when H is stressed by W_1 in figure 3. The probability distributions for X_2 and X_3 do not change with λ as X_1, X_2 and X_3 are assumed to be independent in section 4.1.1. We do however see distortions in X_1 and X_4, as a change in X_1 has a cascading effect on X_4. Therefore, the distortions of the output (Y) shown in figure 2 occur due to the changes observed in X_1 and X_4 in figure 3.

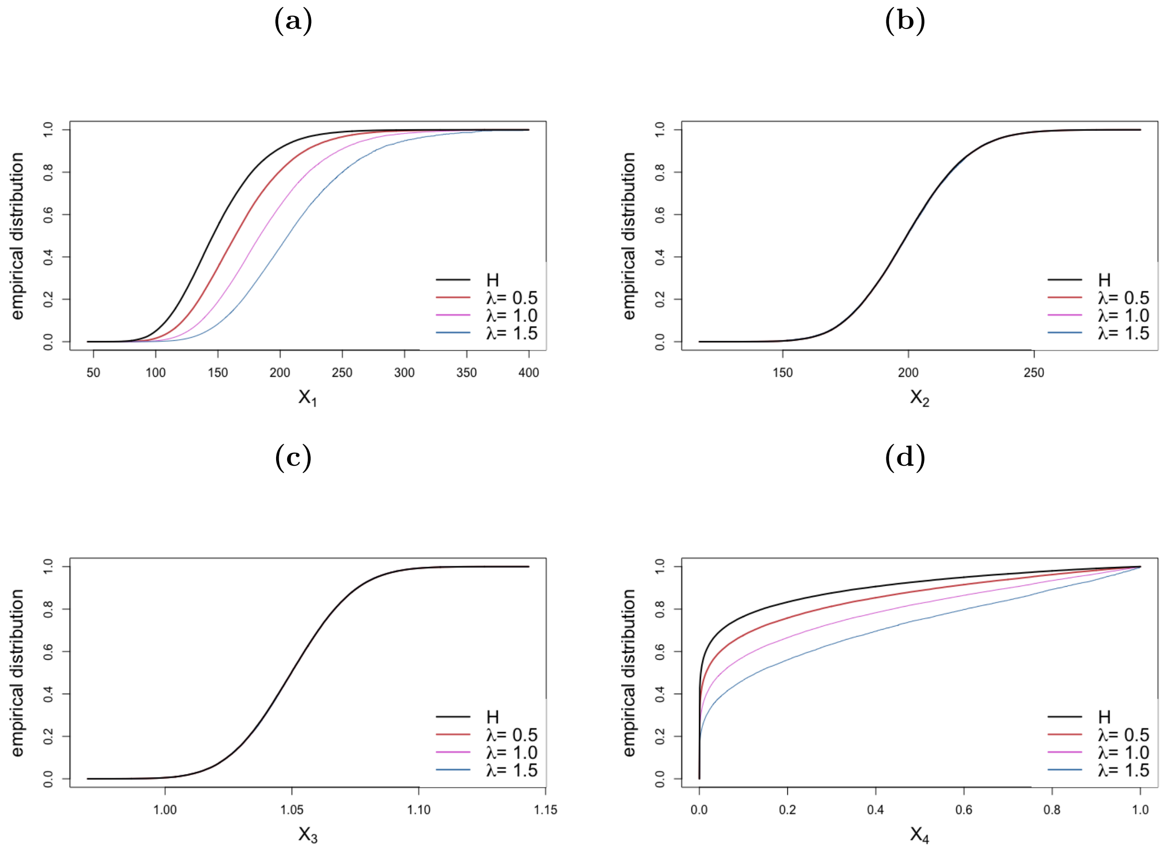


Figure 3: Distorted probability distributions for X_1, X_2, X_3, X_4 obtained using W_1 for $\lambda = 0.5, 1.0, 1.5$

Similar distortions can be derived when weights W_2 and W_3 are used. Since, the inferences drawn are analogous to the inferences made when weights W_1 were used, we do not provide any more details on this.

In Figure 4, we show the stressed distributions of the input factors obtained when weights W_4 are used. As X_4 is related to L, it affects all the probability distributions of all the other inputs. The cascading effect on the inputs due to the stress in X_4 is seen clearly here.

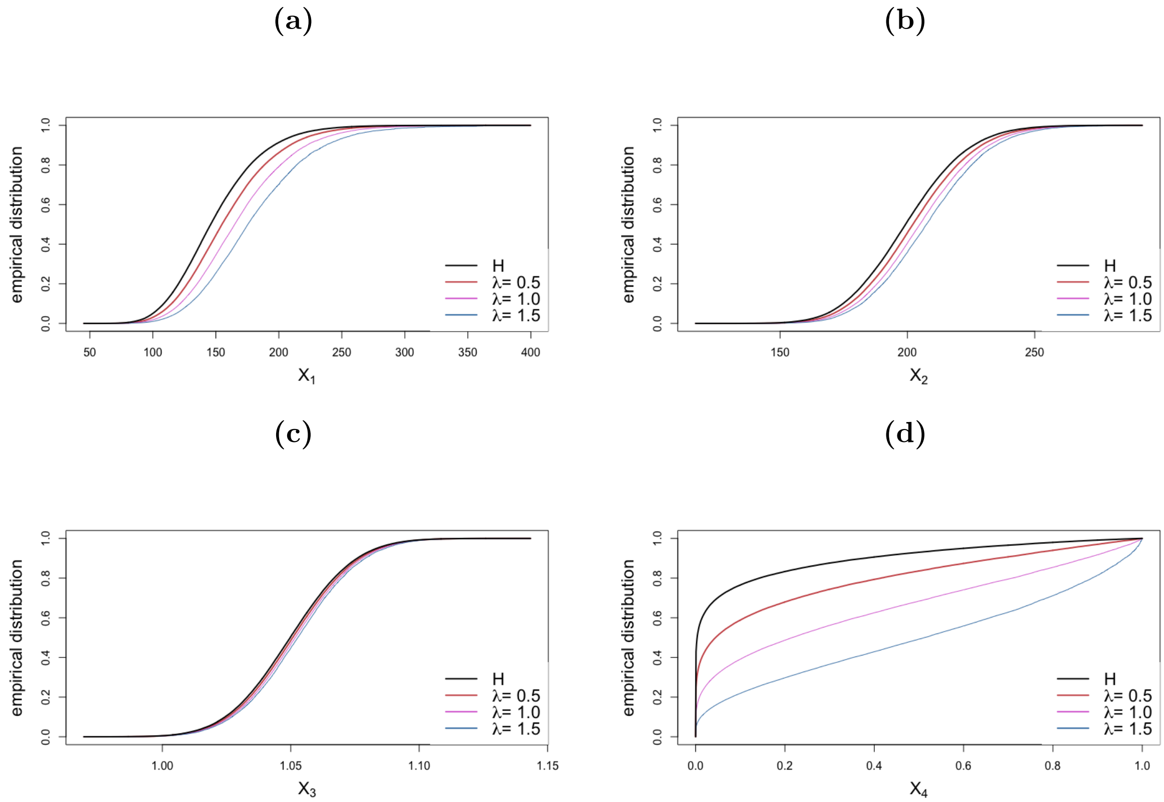


Figure 4: Distorted probability distributions for X_1, X_2, X_3, X_4 obtained using W_4 for $\lambda = 0.5, 1.0, 1.5$

4.2 Variations of the example

In this section, we vary one of the factors of the model and observe the changes in the model behavior. We are particularly interested in inspecting the changes in the order of the sensitivities of the inputs. We look at three specific variations:

1. When correlation between X_1, X_2, X_3 and X_4 is varied between 0 and 1.
2. When d is varied between 300 and 600.
3. (a) When claims modeled by X_2 follow an Inverse Gamma distribution.

   (b) We also vary the standard deviation of X_2 from 10 to 100 units.

Using Monte-Carlo simulation, we measure the percentage increase in the ES when the distribution of Y is distorted using $\lambda = 1$, with respect to the ES calculated for the benchmark model. The percentage increase in ES is taken to be the sensitivity of the input. The sensitivities of the inputs are ordered for $p = 0.95$.

Variation 1

We vary the correlation between the inputs from the smallest plausible value (0) to the highest possible value (1). When the correlation is 0, we notice that X_1 has the highest sensitivity followed by X_2. X_3 and X_4 exhibit very little sensitivities. As the correlation increases, X_1, X_2 and X_3 show almost no variation in their sensitivity levels. X_4 is the only input whose sensitivity changes as the correlation changes. The sensitivity of X_4 increases as the correlation between the inputs increases and its sensitivity level is approximately that of X_1 for high values of correlation.
Figure 5 shows the sensitivity level of the inputs when the correlation between X_1, X_2, X_3 and X_4 increases from 0 to 1. From figure 5, we observe that X_4 starts to become more prevalent than X_2 when the correlation is greater than 0.2 approximately.

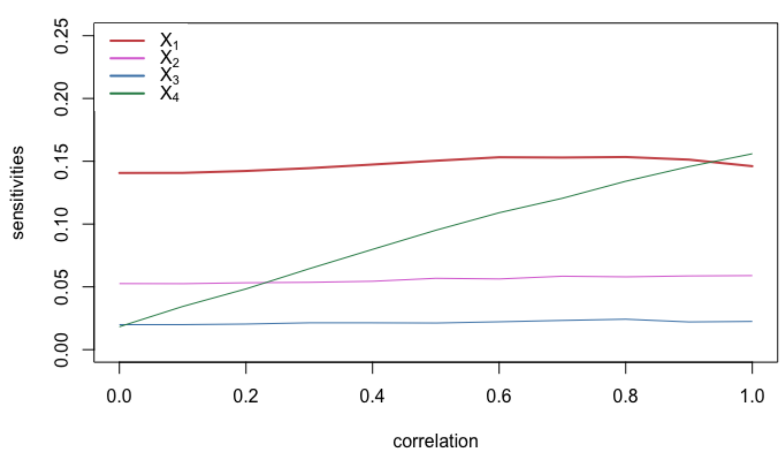


Figure 5: Sensitivity levels of X_1, X_2, X_3, X_4 when the correlation between inputs change from 0 to 1 for $p = 0.95$.

Variation 2

We vary the value of d from 300 to 600 units to examine the behavior of the sensitivities of the inputs. Figure 6 implies that at all times, X_1 exhibits the highest sensitivity level followed by X_4, X_2 and X_3. The sensitivities show almost no variation as d increases from 300 to 425 units. The sensitivity levels of X_1 and X_4 drop a few points upto about 475 units after which they seem to remain roughly constant. The sensitivities of X_2 and X_3 on the other hand do not seem to change much.

Hence, a change in d does not impact the order of the sensitivities of the inputs.

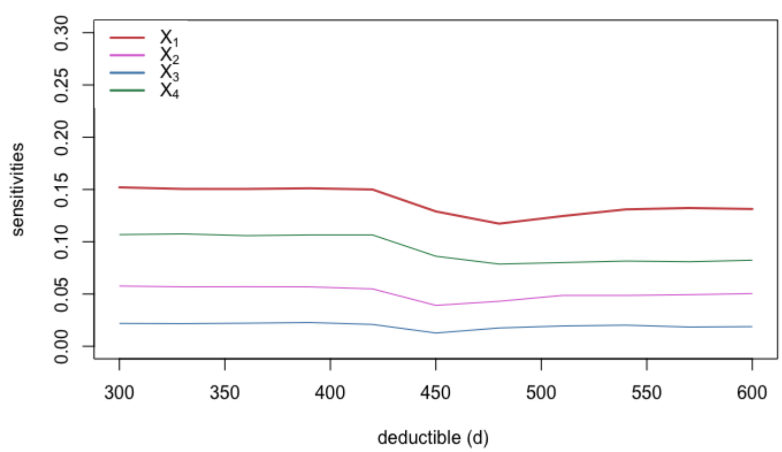


Figure 6: Sensitivity levels of X_1, X_2, X_3, X_4 when d changes from 300 to 600 units for $p = 0.95$.

Variation 3

In this variation, we have two parts. For the first part, the claims represented by X_2 is modeled using an Inverse Gamma distribution with mean 200 and standard deviation 20. As expected, X_1 and X_4 exhibit a higher sensitivity than X_2 and X_3. We obtain the same order for the sensitivities of the inputs as the benchmark model.

It is in our interest to examine the behavioral change of the sensitivities when the standard deviation is now varied from 10 to 100 units and this is shown in figure 7. At the lowest standard deviation considered (10 units), the order of the sensitivities are as follows: X_1, X_4, X_2 and X_3. As the standard deviation of X_2 increases, the sensitivity of X_1 decreases whereas, X_2 and X_4 increase. The sensitivity level of X_3 remains roughly at the same level.

There is a steep increase in the change of the sensitivity level of X_2 as the standard deviation increases and this is expected. X_2 has more than a 60% sensitivity level and X_4 has a 40% approximately when standard deviation is 100 for part 3(b), due to the increased volatility of the claim size.

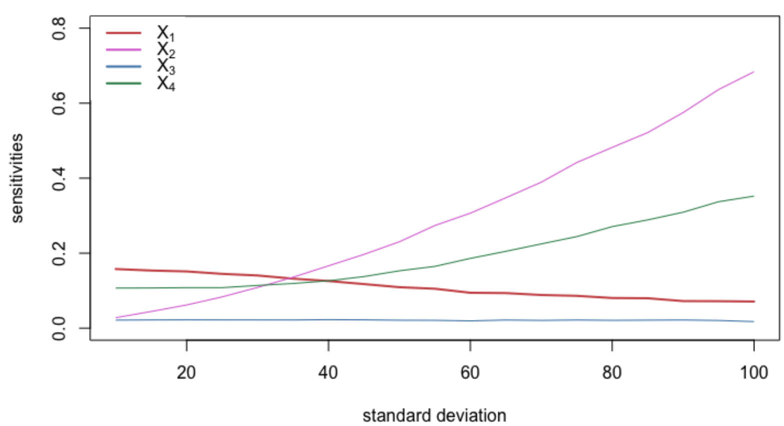


Figure 7: Sensitivity levels of X_1, X_2, X_3, X_4 when the standard deviation of X_2 changes from 10 to 100 units for $p = 0.95$.

4.3 Comments

The benchmark model indicates that the output has a higher sensitivity to X_1 followed by X_4, X_2 and X_3. We obtain consistent results with Pesenti et al. (2018) as they also report a higher sensitivity to X_1 and X_4 for the same model. Therefore, it is reasonable to state that X_1 and X_4 are the most influential factors of the model.

A few aspects in the way the sensitivities were specified to the model inputs were noted. The first being that the percentage increase in VaR and ES produced in Tables 1-4 contained inconsistent trends as the value of p was altered. For instance in Table 1, we see that the sensitivity levels for each distortion increases as p increases. However, in Tables 2-4, the sensitivity levels for each distortion increases till $p = 0.95$ and then decreases at $p = 0.99$.

Further, VaR and ES are used to rank the inputs based on comparing the sensitivity levels at each p value for the distortions. If an input A has higher percentage increases for all the corresponding values than input B, then input A exhibits a greater sensitivity than input B. We might have instances where all the percentage increases in VaR and ES for one input are not higher or lower than another input. So the ranking may not be very obvious and hence, depending on the purpose for which the sensitivity analysis is used, different values of p may be more important.

5 Conclusion

The proposed framework was effective for understanding the relationships between inputs and the output. We had used distortion risk measures to stress the probability distributions and also employed VaR and ES for examining the behavioral changes in the model. The methodology adopted in this paper leads to consistent results with other sensitivity methods such as those obtained by Pesenti et al. (2018).

In addition to this, we were able to examine the cascading effects on other inputs while one input was altered. As a result, we were able to point out those aspects of the model which needed the most attention. In such a case, it is possible to spot unexpected relationships between inputs and outputs from running the model. Other factors such as GDP, interest rates, returns from assets, claims from adjacent underwriting years can be included in the model for a more realistic application. Nevertheless, the framework would remain unaltered even if other inputs are being integrated into the model.

6 Appendix

According to Denuit et al. (2005), $\rho_g(X)$ is coherent if, and only if, g is concave. We provide a simpler proof which is more intuitive.

Let us suppose that $\rho_g(X) = E(g\prime(1-U)F_X^{-1}(U))$, U is any uniform distribution. This naturally satisfies positive homogeneity, translation invariance, monotonicity. For $\rho_g(X)$ to be coherent, it has to satisfy the property of sub-additivity. We know that concavity and sub-additivity are concomitant.

Let us choose the uniform to be U_X such that it is co-monotonic to X.

Hence,

$$\rho_g(X) = E(g\prime(1-U_X)X) \qquad \text{since } F_X^{-1}(U_X) = X$$

Now, let

$$\begin{aligned}
\rho_g(X+Y) &= E[g\prime(1-U_{X+Y})(X+Y)] \\
&= E[g\prime(1-U_{X+Y})X] + E[g\prime(1-U_{X+Y})Y] \\
&\leq E[g\prime(1-U_X)X] + E[g\prime(1-U_Y)Y] \\
&= \rho_g(X) + \rho_g(Y)
\end{aligned}$$

The inequality follows from the result that $E[A.B] \leq E[A^*.B^*]$ if A^* and B^* are co-monotonic and have the same marginal distributions as A and B, for any two random variables A and B (Kaas et al., 2002).

Hence, as X and U_X are co-monotonic, $E[g\prime(1-U_X)X] \geq E[g\prime(1-U_{X+Y})X]$. Similarly as Y and U_Y are co-monotonic, $E[g\prime(1-U_Y)Y] \geq E[g\prime(1-U_{X+Y})Y]$. Further, $g\prime(1-U_X)$ can only be increasing in U_X if g is concave.

Hence, ρ_g is sub-additive and therefore coherent.

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