A new Dividend Strategy in the Brownian Risk Model[1]

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Abstract

In risk theory, the *stability problem* addresses the safety of an insurance company. By supposing to use the probability of ruin as a stability criterion, actuarial literature focused on its minimization during the first part of the 20th century. However, minimizing the ruin probability for an infinite length of time implies that companies should let their surplus grow to infinity. De Finetti's modification (De Finetti, 1957), allowing the insurance company to distribute dividends to its shareholders, led to a wealth of publications over the following decades (see (Avanzi B. , 2009), for a review of this research), nearly all of them dealing with one essential question:

How should dividends be paid to the shareholders, if the ultimate aim is to maximize the expectation of the discounted dividends until possible ruin of the insurance company?

In continuous time, some very explicit calculations can be made when assuming that the insurance company's surplus (before distribution of dividends) is modeled by a Wiener process (Brownian motion) with a constant positive drift $\mu > 0$ and variance per unit of time σ^2. Although even in this surplus model, the *classical barrier strategy*, cf. (Gerber & Shui, 2004), turned out to be the ultimate best solution to the mathematical problem described above (see for instance (Schmidli, 2008)[Theorem 2.55]), the resulting dividend stream is far from practical acceptance. Furthermore, the associated probability of ruin is 1. This realization finally led to the idea of imposing restrictions on the nature of the dividend stream, resulting in optimization problems with further constraints.
Furthermore, it is widely accepted and also found by empirical research (see for instance (Brav, Graham , Harvey, & Michaely, 2005)) that companies also aim at a non-decreasing, stable flow of dividends because of the signals it sends to the market. Based on this quantity, (Avanzi & Wong, 2009) acknowledged its importance by specifying a dividend strategy (a so-called *mean-reverting dividend strategy*) that - in contrast to a classical barrier strategy - yields a smooth flow of dividend payments over time.

At the center of this paper, we establish new dividend strategy in the Brownian risk model, as being a combination of the classical barrier and the mean-reverting strategy. According to this strategy, dividends are paid at a constant rate $g > 0$ of the company's current (modified) surplus process, whenever the process is above a barrier level $b > 0$, whereas below b, no dividends are paid. Thus, we obtain a dividend strategy that is governed by two parameters, a barrier level $b > 0$ and a dividend payout rate $g > 0$. After deriving the corresponding expected present value of all dividends until ruin we discuss the optimal choice of parameters and consequential effects on the value function. Finally, we conclude with an embedment of the new dividend strategy in a comparison of the classical barrier and the mean-reverting, regarding the present value of dividends as well as the variability of dividend payouts.

[1] These results are taken from the diploma thesis entitled "Dividend Strategies in the Brownian Risk Model" by Marco Ehlscheid, November 2011, University of Cologne

1. Introduction

The main focus of this work will be the consideration, creation and analysis of different types of dividend strategies using the example of an insurance company. For that purpose, we assume that the insurer's surplus process $(U_t)_{t\geq 0}$ before any distribution of dividends is modeled by a Brownian motion with drift

$$U_t = x + \mu t + \sigma W_t, \quad t \geq 0, \tag{1}$$

with initial capital $x > 0$, positive drift μ, volatility $\sigma > 0$ and $(W_t)_{t\geq 0}$ being a standard Brownian motion. The filtration $\{\mathcal{F}_t\}$ is the filtration generated by the Brownian motion. Thus, justifying its name *Brownian risk model*.

The motivation of this assumption is based on the following.

Although the classical risk model seems to be quite simple, it is often difficult to obtain accurate numerical calculations. For that reason one looks for approximations. A useful method is the so-called *diffusion approximation*. The idea is to replace the random part of the collective risk model in (1) by a diffusion process.

Definition 1. *A stochastic process* $(X_t)_{t\geq 0}$ *that is the unique solution to the stochastic differential equation (SDE)*

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \quad t \geq 0, \tag{2}$$

is called a ***diffusion process*** with ***infinitesimal drift function*** $\mu(t,x)$ and ***infinitesimal variance*** $\sigma^2(t,x)$ at (t,x), given that $\sigma^2(t,x) > 0$ and $x \in E$, where $E \subset \mathbb{R}$ is the state space of $(X_t)_{t\geq 0}$.

Furthermore, let D_t, $t \geq 0$, denote the undiscounted aggregate dividends paid by time t and assume that the payment of dividends has no influence on the business. Thus, after the distribution of dividends the modified surplus of the company at time t is $X_t := U_t - D_t$.

In our studies, we will only deal with dividend strategies that are said to be *admissible*. That is, we assume that the modified surplus process $(X_t)_{t\geq 0}$ fulfils

$$\mathbb{P}_x[X_t \geq 0, \textit{ for all } t \geq 0] = 1.$$

Let $\delta > 0$ be a constant discount factor, the so-called *force of interest* for valuation. Then, δ can be interpreted as reflecting the preference of shareholders to receive dividend payments earlier rather than later during the lifetime of the surplus process. Furthermore, let D denote the present value of all dividends until ruin. Thus,

$$D = \int_{0-}^{\tau-} e^{-\delta t}\mathrm{d}D_t, \tag{3}$$

where

$$\tau := \inf\{t \geq 0; X_t < 0\} \tag{4}$$

is the *time of ruin*. As usual, we let $\inf\{\varnothing\} = \infty$. In (3) we include the point 0 into the integration area in order to take an immediate dividend at $D_0 > 0$ into the value. But we do not take a possible dividend at time τ into consideration, to prevent from ruin caused by a dividend payment. Note that ruin - as considered in this context - does not mean that the insurer is bankrupt, but rather that the capital set aside for the risk was not enough, so that the insurance company is no longer able to cover its liabilities by own means.

When analyzing each of the following dividend strategies, we will always aim at determining and afterwards maximizing the expected present value of all dividend payments until the time of ruin, if it occurs. This means that we are interested in the function

$$V(x) := \mathsf{E}_x[D] \overset{(3)}{=} \mathsf{E}_x\left[\int_{0-}^{\tau-} e^{-\delta t} \mathrm{d}D_t\right],$$

with $x > 0$ representing the insurance company's initial capital at time $t = 0$.

2. A Classical Barrier Strategy in the Brownian Risk Model

In the theory of dividend strategies, the classical *barrier strategy* with a constant barrier level $b > 0$ is definitely the most common and prevalent strategy studied and discussed for diverse surplus models over the last decades. In (Gerber & Shui, Optimal dividends: Analysis with Brownian, 2004), the authors went back to the roots and considered the continuous counterpart of de Finetti's model, assuming that the surplus of the insurance company is a Brownian motion with a positive drift. By referring to (Gerber & Shui, Optimal dividends: Analysis with Brownian, 2004), we will next give an introduction of this strategy in the underlying risk model and state its most important concepts and results.

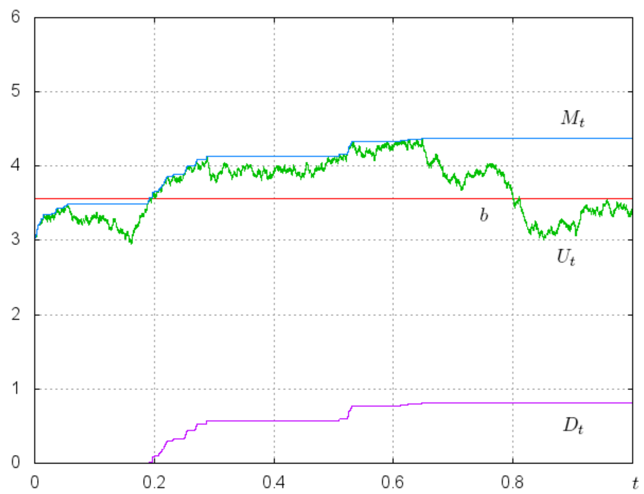

Figure 1: Sample paths of $(U_t)_{t\geq 0}$, $(M_t)_{t\geq 0}$ and its associated aggregate dividends process $(D_t)_{t\geq 0}$.

2.1. The Strategy and Basic Results

Assume that the insurance company will pay dividends to its shareholders according to a *barrier strategy* with parameter $b > 0$. So whenever the (modified) surplus process $(X_t)_{t\geq 0}$ reaches level b, the excess (or "overflow") will immediately be paid as dividends. Thus, the level b acts as a *reflective barrier*.

A formal definition can be given in terms of the running maximum

$$M_t := \max_{0\leq s\leq t} U_s.$$

Then, the aggregate dividends paid by time t are

$$D_t := (M_t - b)^+ = \begin{cases} 0, & \text{if} M_t \leq b \\ M_t - b, & \text{if} M_t > b. \end{cases} \tag{5}$$

A sample path of the surplus process and its associated accumulated dividends process is illustrated in figure 1. Note that the process $(D_t)_{t\geq 0}$ is continuous and increasing, and therefore of bounded variation. With respect to (5) denote by $V(x;b)$ the expectation of D, i.e.

$$V(x;b) = \mathsf{E}[D \mid X_0 = x] = \mathsf{E}_x[D], \quad x \geq 0.$$

Since the shareholders of the company might be interested in maximizing the present value of dividends under this strategy, we will focus on $V(x;b)$ from now on.

It can be easily shown that as a function of the intial surplus $x > 0$, $V(x;b)$ satisfies the homogeneous second-order differential equation

$$\frac{\sigma^2}{2} V''(x;b) + \mu V'(x;b) - \delta V(x;b) = 0, \ \mathit{for} \ 0 < x < b. \tag{6}$$

According to the classical barrier strategy, the function $V(x;b)$ satisfies the initial conditions

$$V(0;b) = 0, \tag{7}$$

$$V'(b;b) = 1. \tag{8}$$

Condition (7) is quite obvious, because $\tau = 0$ if $x = X_0 = 0$ by the fluctuations of the Brownian motion, so that ruin is immediate and no dividends are paid. Condition (8) follows from the fact that the process $(D_t)_{t\geq 0}$ only increases at points, where $X_t = b$, i.e. $V'(b;b) = 1$. For a rigorous proof, we refer to (Gerber, Games of economic survival with discrete- and continuousincome, 1972). Subject to the first boundary condition (7), we derive that

$$V(x;b) = c_b(e^{rx} - e^{sx}), \quad for\ 0 \le x \le b, \tag{9}$$

with coefficient c_b being independent of x, and

$$r := \frac{-\mu + \sqrt{\mu^2 + 2\delta\sigma^2}}{\sigma^2} \ and \ s := \frac{-\mu - \sqrt{\mu^2 + 2\delta\sigma^2}}{\sigma^2} \tag{10}$$

being the roots of the characteristic equation:

$$\frac{\sigma^2}{2}\xi^2 + \mu\xi - \delta = 0. \tag{11}$$

Now, using the second condition (2.9), we receive

$$c_b = \frac{1}{re^{rb} - se^{sb}},$$

so that

$$V(x;b) = \frac{e^{rx} - e^{sx}}{re^{rb} - se^{sb}}, \quad for\ 0 \le x \le b. \tag{12}$$

Since r denotes the positive root and s denotes the negative root of equation (10), both numerator and denominator of (12) are positive by the properties of the exponential function.

For $x > b$, it follows from (5) that

$$V(x;b) = x - b + V(b;b)$$

$$\overset{(12)}{=} x - b + \frac{e^{rb} - e^{sb}}{re^{rb} - se^{sb}}, \ x > b, \tag{13}$$

which can be explained as follows:

If the initial surplus x exceeds the dividend barrier b, the difference is immediately paid out as dividend. In view of the fact that this payment is executed at time $t = 0$, no discounting is necessary. Therefore, starting with an initial capital larger than the barrier level b is equivalent to starting with an initial capital of the amount b and adding the payment of $(x - b)$ at $t = 0$.

2.2. The Optimal Barrier Level

For a given initial surplus $U_0 = X_0 = x$, let b^* denote the optimal value of b, that is, the value that maximizes the expectation of D, i.e.

$$V(x;b^*) = \max_{b>0} V(x;b).$$

As a function of $b > 0$, $V(x;b)$ can be written as

$$V(x;b) = \begin{cases} x - b + \dfrac{\varphi(b)}{\varphi'(b)}, & \text{if } 0 < b < x \\ \dfrac{\varphi(x)}{\varphi'(b)}, & \text{if } b \geq x, \end{cases}$$

where $\varphi(x) := e^{rx} - e^{sx}$ and $V(x;b)$ is continuous.

Taking the derivative with respect to b, we obtain

$$\frac{\partial}{\partial b} V(x;b) = \begin{cases} -V(b;b)\dfrac{\varphi''(b)}{\varphi'(b)}, & \text{if } 0 < b < x \\ -V(x;b)\dfrac{\varphi''(b)}{\varphi'(b)}, & \text{if } b \geq x. \end{cases} \tag{14}$$

Furthermore,

$$\frac{\varphi''(0)}{\varphi'(0)} = \frac{r^2 - s^2}{r - s} = r + s \overset{(10)}{=} -\frac{2\mu}{\sigma^2} < 0.$$

Hence, $\dfrac{\partial}{\partial b} V(x;b)$ is positive for small values of b. Since $V(x;b) \to 0$ for $b \to \infty$, we gather that $V(x;b)$ attains its maximum for a finite and positive value of b.

According to equation (14), the first order condition reads

$$\varphi''(b) = 0.$$

It turns out that this equation has a unique solution

$$b = b^* = \frac{\ln(s^2) - \ln(r^2)}{r - s} = \frac{1}{r - s} \ln\left(\frac{s^2}{r^2}\right) > 0. \tag{15}$$

Consequently, (15) is the barrier that maximizes $V(x;b)$, independently of the initial surplus x.

Remark *In stochastic control theory, for example Schmidli showed in* (Schmidli, 2008), *[theorem 2.55], that under the assumption of unrestricted dividend payments - meaning that all increasing adapted càdlàg processes D are allowed - this barrier strategy with barrier level b^*, as given in (15), is optimal under all dividend strategies in the Brownian risk model as long as the barrier level b is not primarily specified.*

3. A Mean-Reverting Strategy in the Brownian Risk Model

A so-called *mean-reverting dividend strategy* within the Brownian risk model, that had already been studied by Avanzi and Wong in (Avanzi & Wong, 2009).

In comparison to the classical barrier strategy, they eliminated the dividend barrier level and specified a dividend strategy that yields a smooth flow of dividends over time based on continuous dividend payments at a rate $g > 0$ of the company's current surplus process.

Its motivation is justified by the companies' additional aim to guarantee a non-decreasing, stable flow of dividends because of the signal it sends to the market. This is also found by empirical research. For instance, in (Brav, Graham , Harvey, & Michaely, 2005)[Table 4] it turns out that this quality is one of the most important ones for decision makers.

3.1. The Strategy and Basic Results

In contrast to the classical barrier dividend strategy, dividends are now paid continuously over time at a rate $g > 0$ of the current (modified) surplus process $(X_t)_{t\geq 0}$.

A formal definition can be given by

$$dD_t = g \cdot X_t \mathrm{d}t, \quad t \geq 0. \tag{16}$$

Then, the company's modified surplus has dynamics

$$dX_t = (\mu - gX_t)dt + \sigma dW_t, \quad t \geq 0, \tag{17}$$

with $X_0 = U_0 = x$.

This means that the surplus process is a *diffusion process*, see definition 1, with drift $(\mu - gx)$ and variance σ^2 per unit of time, where x is the current surplus. In other words, the surplus process resembles an *Ornstein-Uhlenbeck process* with additional drift μ.

Remember that for an Ornstein-Uhlenbeck process, the drift is a linear function of x with negative slope, providing a mean-reversion tendency. More precisely, at time t, the drift of the modified surplus applies to

$$\begin{cases} \mu - gX_t > 0, & \text{if} X_t < \frac{\mu}{g} \\ \mu - gX_t < 0, & \text{if} X_t > \frac{\mu}{g}. \end{cases} \quad (18)$$

Thus, the surplus process $(X_t)_{t\geq 0}$ is reverting around the level $l := \frac{\mu}{g}$, while the dividend payout process $(gX_t)_{t\geq 0}$ itself reverts around an average rate

$$g \cdot l = \mu \quad (19)$$

- the original drift of the company's surplus before distribution of dividends. Consequently, the dividend strategy is mean-reverting around μ, and this, whichever parameter g or l is selected; hence justifying its particular name.

In this context, the process $(X_t)_{t\geq 0}$ operates as a buffer reservoir to yield a smoother dividend flow with target annual rate μ, irrespectively of the value of g. Notice that $(gX_t)_{t\geq 0}$ is a continuous process that - in comparison to any of the dividend strategies discussed so far - cannot lead to periods without dividends at all.

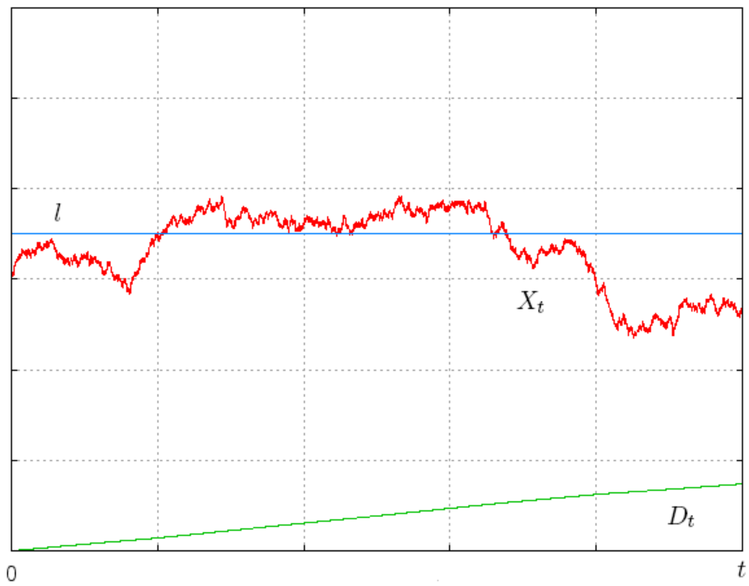

Figure 2: Sample path of $(X_t)_{t\geq 0}$ and $(D_t)_{t\geq 0}$ for $\mu = \sigma = 1$, $l = 9/2$ and $x = 3$.

Figure 2 illustrates a typical sample path of the company's surplus process $(X_t)_{t\geq 0}$ and its associated (undiscounted) aggregate dividends process $(D_t)_{t\geq 0}$.

For the sake of completeness, we confirm at this point that the diffusion process, defined in (17), is *recurrent*, meaning that for any x, $y \in \mathbb{R}$, it is

$$\mathbb{P}[T_y \mid X_0 = x] = 1,$$

where $T_y = \inf\{t \geq 0; X_t = y\}$ is the first hitting time of y. Since required calculations are quite substantial and would therefore exceed the goal of our work, we omit the proof and refer instead to (Klebaner, 2005)[Theorems 6.27-6.28].

Hence, $\mathbb{P}[\tau < \infty \mid X_0 = x] = 1$, for all $x > 0$, and we infer that ruin is certain under a mean-reverting dividend strategy.

Since it is reasonable to assume that the shareholders of the insurance company will prefer a strategy that maximizes the expected present value of all dividends until ruin, we are next interested in determining the optimal mean-reverting rate g^* - from which the optimal mean-reverting level l^* follows immediately by (19). Since we are no longer dealing with a barrier level b, denote by $V(x;l)$ the expected present value of dividends until ruin for an initial surplus $X_0 = x > 0$ and mean-reverting level l.

Then, by virtually the same proceeding as in section 2.1, $V(x;l)$ can be shown to be the solution to the following IDE

$$\frac{\sigma^2}{2}V''(x;l) + (\mu - gx)V'(x;l) - \delta V(x;l) = 0.$$

After some transformations, this IDE can be modified into a *confluent hypergeometric differential equation* (also called *Kummer differential equation*) of the form

$$th''(t) + (\frac{1}{2} - t)h'(t) - \frac{\delta}{2g}h(t) = 0, \tag{20}$$

with parameters $\frac{\delta}{2g}$ and $\frac{1}{2}$; see (Cai, Gerber, & Yang, 2006).

Since the determination of initial conditions to solve (20) in this general context in not trivial, Avanzi and Wong used an alternative way to derive an explicit representation of the expected present value of all dividends paid until the surplus process $(X_t)_{t\geq 0}$ is absorbed at zero.

Within this framework, we will state the main facts and results of their studies and refer for detailed information to their realizations in (Avanzi & Wong, 2009)[sec. 3-5].

As a first step, Avanzi and Wong assumed that the surplus process $(X_t)_{t\geq 0}$ is never absorbed, i.e. the company's business is allowed to continue even if $(X_t)_{t\geq 0}$ becomes negative. Thus, whenever the surplus is negative, the dividend payouts are negative as well. However, this can only be possible if the surplus is part of a bigger portfolio so that negative dividends can then be interpreted as transfers from other surpluses within the same portfolio, since the shareholders' liability is limited to their investment.

As a second step, using the present value of all dividends paid from $t = 0$, they determined its conditional expectation for a given reverting level l and initial capital $X_0 = x > 0$ and finally arrived at

$$\mathsf{E}[D_x \mid X_0 = x] = \mathsf{E}_x[D_x] = \frac{\mu}{\delta} - \frac{g}{g+\delta}(l-x). \tag{21}$$

Remark *The expected present value in (21) is equal to a perpetuity of rate μ plus the difference between the initial surplus x and the mean-reverting level l, corrected by a certain factor which is indeed a function of both g and δ. Here, the subtraction can be interpreted as the profit (or cost) incurred by the initial trip of the process to its reverting level l. Additionally, if $X_0 = l$, the present value of dividends is thus the same as the one of a perpetuity with payment rate μ, that could have been anticipated in view of (19).*

Now, assume that the business immediately stops when the modified surplus passes the specific level zero for the first time, which is the classical definition of ruin. Then, $V(x;l)$, can be derived with the help of (19) and the time of ruin τ. We have,

$$V(x;l) = \mathsf{E}[D_{x;0} \mid X_0 = x]$$

$$= \mathsf{E}[D_x - e^{-\delta\tau} D_0 \mid X_0 = x]$$

$$= \mathsf{E}_x[D_x] - \mathsf{E}_x[e^{-\delta\tau}] \cdot \mathsf{E}_x[D_0]$$

$$\overset{(21)}{=} \frac{\mu}{\delta}\left[1 - \mathsf{E}_x[e^{-\delta\tau}]\right] - \frac{g}{g+\delta}\left[(l-x) - l \cdot \mathsf{E}_x[e^{-\delta\tau}]\right] \tag{22}$$

$$= \frac{g}{g+\delta}\left[\left(\frac{\mu}{\delta} + x\right) - \frac{\mu}{\delta} \cdot \mathsf{E}_x[e^{-\delta\tau}]\right], \tag{23}$$

where $\mathsf{E}_x[e^{-\delta\tau}]$ is the *Laplace transform* of the probability density function (pdf) of τ, and at the same time, the expected present value of a payment of 1 at the time of ruin. For a precise derivation we refer to (Avanzi & Wong, 2009)[sec. 4].

Remark *Similar to (21), the expected present value of all dividend payments until ruin in (22) equals the expected present value of an annuity certain plus a correction due to the process trip to level l first and to level zero then.*

3.2. The Optimal Reverting Level

As before, we still pursue the objective of maximizing the expected value of all dividend payouts until ruin, i.e.

$$\max_{l>0} V(x;l),$$

for a given initial surplus $U_0 = X_0 = x > 0$. To this, let g^* denote the optimal mean-reverting rate - from which the optimal reverting level l^* immediately follows by (19).

Hence, setting the derivative of (22) with respect to g equal to zero, we obtain by a simple rearrangement that the optimal payout rate g^* is the solution to

$$\frac{\delta}{g+\delta}\left(\frac{\mu + x\delta}{\mu} - \mathsf{E}_x[e^{-\delta\tau}]\right) = g \cdot \frac{\partial}{\partial g}\mathsf{E}_x[e^{-\delta\tau}]. \tag{24}$$

Since $x > 0$ and $0 < \mathsf{E}_x[e^{-\delta\tau}] < 1$ the left hand side of (24) decreases monotonically from a (strictly) positive number to zero as g goes to infinity. Whereas the right hand side of (24) increases monotonically from zero to infinity as $g \to \infty$, since the derivative $\frac{\partial}{\partial g}\mathsf{E}_x[e^{-\delta\tau}] > 0$.

Consequently, according to the *immediate value theorem* a solution $g^* > 0$ to (24) always exists. In addition, based on the mentioned monotonicity properties and the assumption $\delta > 0$, the solution g^* is also unique.

Even though we cannot determine the optimal value of g^* explicitly, however, we are able to state the shape of the expected present value of all dividend payments until ruin under the optimal payout rate. Some elementary transformations of (24) using (23) yield

$$V(x;l^*) = \frac{g^2\mu}{\delta^2}\left(\frac{\partial}{\partial g}\mathsf{E}_x[e^{-\delta\tau}]\right)\Bigg|_{g=g^*}. \tag{25}$$

Thus, the optimal reverting level $l^* = l^*(x)$ (or equivalently $g^* = g^*(x)$) depends on the initial surplus x. The $l^*(x)$ are the ones that maximize the expected present value if the initial surplus is x, and if one cannot modify the reverting level in the future. Otherwise, if one could chance the reverting level dynamically, calculations would act as lower bounds for the optimal present value of dividends with a (dynamic) mean-reverting strategy.

Remark *Although we cannot quote the optimal reverting level $l^*(x)$ explicitly, we are still able to speculate about its behavior according to the initial surplus level x. As a matter of fact, as long as x is around $l^*(x)$ or above, its effect should be insignificant. But if x is relatively close to zero, the optimal reverting level $l^*(x)$ (the optimal payout rate $g^*(x)$) will be higher (lower) in order to counteract the risk of ruin.*

4. A new Dividend Strategy in the Brownian Risk Model

The motivation of this new dividend strategy results from the idea of teaming the optimality property of the classical barrier strategy due to the maximization of the present value of all dividend payments until ruin with a continuous flow of dividends whenever the modified surplus process $(X_t)_{t\geq 0}$ exceeds the barrier level $b > 0$. The new strategy is of barrier type and thus, it can be treated as a combination of the classical barrier strategy from section 2.1 and the mean-reverting strategy from section 3.1.

Although being aware that, without any constraints concerning the barrier level b, we will not receive a superior strategy as the classical barrier strategy with optimal barrier level b^*, given by (15), in case of unrestricted dividend payouts, we rather expect to achieve at least an improvement of the mean-reverting strategy in terms of present value of dividends until ruin. In this respect, it will be interesting to investigate in how far a contingent enhancement of the value function bears on the variability of dividend payouts.

4.1. The Strategy and Basic Results

When talking about combining the classical barrier strategy with the mean-reverting strategy, we assume that the insurance company pays dividends to its shareholders according to a dividend strategy, which is now governed by both parameters the barrier level $b > 0$ and the mean-reverting strategy's payout rate $g > 0$.

As already specified in the classical barrier strategy, no dividends are paid whenever the (modified) surplus $X_t = U_t - D_t$, $t \geq 0$, is below the barrier b. Whereas above b, dividends are distributed continuously at a positive rate g of the current modified surplus, which is similar to the mean-reverting strategy. Therefore, as long as the surplus process with initial point $X_0 = x > b$ does not reach the barrier level b, $(X_t)_{t\geq 0}$ equals the Ornstein-Uhlenbeck process (17) of section 3.1 with its mean-reversion tendency. The same also applies in the event that the surplus process passes the barrier level.

Hence, the level b once again plays the role of a break point. Furthermore, this strategy yields at least a piecewise continuous flow of dividends over time.

A formal definition of this strategy can be given by

$$dD_t = gX_t \cdot 1_{\{X_t > b\}} \mathrm{d}t, \quad for\ t \geq 0. \tag{26}$$

Then,

$$D_t = g\int_0^t X_s \cdot 1_{\{X_s > b\}} \mathrm{d}t,$$

such that expression (3) can be rewritten as

$$D = g\int_0^\tau e^{-\delta t} X_t \cdot 1_{\{X_t > b\}} \mathrm{d}t. \tag{27}$$

A typical sample path of the company's surplus process $(X_t)_{t\geq 0}$ and its associated (undiscounted) aggregate dividends process $(D_t)_{t\geq 0}$ under this strategy is illustrated in figure 3.

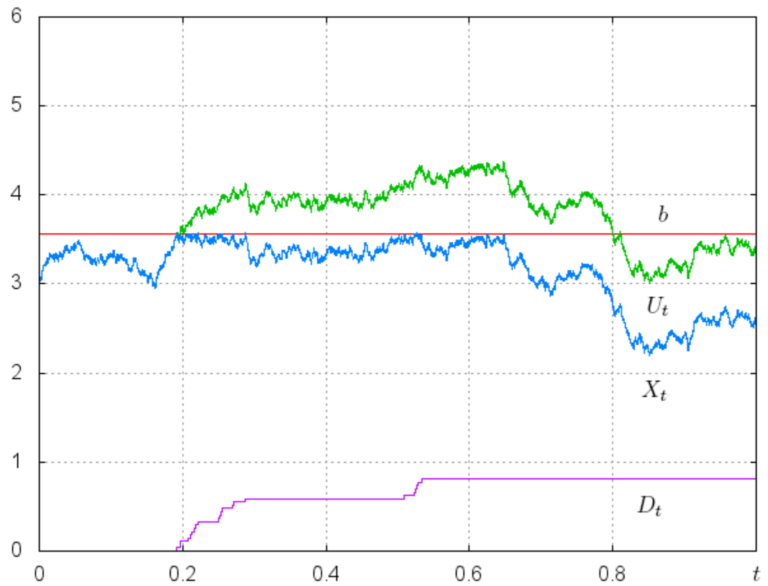

Figure 3: Sample paths of the processes $(U_t)_{t\geq 0}$, $(X_t)_{t\geq 0}$ and $(D_t)_{t\geq 0}$ for $\mu = 1$, $\sigma = 1$, $\delta = 0.05$, $b = 3.5$ and $g = 103$.

For the sake of comparability, we are again interested in the optimal choice of governed parameters that maximize the expectation of (27).

Regarding this, we first determine the company's value function, denoted by $V_g(x;b)$, i.e. the expected present value of all dividend payouts until ruin. Since we are dealing with a barrier strategy, the value function $V_g(x;b)$ will once more depend on whether starting with an initial capital x larger or lower than the barrier level b.

Based on the fact that no dividends are paid below the barrier, $V_g(x;b)$ is again given as the solution to the IDE

$$\frac{\sigma^2}{2}V_g''(x;b)+\mu V_g'(x;b)-\delta V_g(x;b)=0, \quad for\ 0<x<b, \tag{28}$$

with initial condition

$$V_g(0;b)=0, \tag{29}$$

since $\tau=0$ if $x=0$ and ruin is immediate by the fluctuations of the Brownian motion. Such as in the previous chapter, we obtain

$$V_g(x;b)=v_{b,g}(e^{rx}-e^{sx}), \quad for\ 0\le x\le b, \tag{30}$$

with coefficient $v_{b,g}$ being independent of x and r, s being the roots of the characteristic equation

$$\frac{\sigma^2}{2}\xi^2+\mu\xi-\delta=0, \text{ as given in (10).}$$

On the other hand, if starting with an initial capital $x=X_0$ larger than b, the surplus process has drift $(\mu-gX_t)$ at time t and variance σ^2 per unit of time. Thus, as long as $(X_t)_{t\ge0}$ remains above b, which obviously depends on the values of μ and g, the process is an OU-process with its mentioned mean-reversion property.

Then, the value function $V_g(x;b)$ satisfies the non-homogeneous second-order differential equation

$$\frac{\sigma^2}{2}V_g''(x;b)+(\mu-gx)V_g'(x;b)-\delta V_g(x;b)+gx=0, \quad for\ x>b. \tag{31}$$

In order to solve the integro-differential equation (31) to obtain $V_g(x;b)$, one will need the theory of so-called *(confluent) hypergeometric differential equations* and their solutions and refer for basic definitions and properties to (Slater, 1960). After some transformations one finds that a complete solution to (31) is given by

$$V_g(x;b)=\kappa_{b,g}\cdot M\left(\frac{\delta}{2g},\frac{1}{2};\frac{(gx-\mu)^2}{g\sigma^2}\right)+\rho_{b,g}\cdot U\left(\frac{\delta}{2g},\frac{1}{2};\frac{(gx-\mu)^2}{g\sigma^2}\right)+\frac{g}{g+\delta}\cdot x+\frac{\mu g}{\delta(g+\delta)}, \tag{62}$$

where $\kappa_{b,g}$, $\rho_{b,g}\in\mathbb{R}$.

Remark *In terms of integral representation, the Kummer (confluent hypergeometric) functions* of the *first kind* and the *second kind denoted by* $M(a,c;x)$ *and* $U(a,c;x)$*, respectively, read*

$$M(a,c;x) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{xt}t^{a-1}(1-t)^{c-a-1}\mathrm{d}t,\ \textit{for}\ a,c>0,$$

and

$$U(a,c;x) := \frac{1}{\Gamma(a)}\int_0^\infty e^{-xt}t^{a-1}(1-t)^{c-a-1}\mathrm{d}t,\ \textit{for}\ x,a>0,$$

with $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$ *being the generic Gamma function.*

The concept of (confluent) hypergeometric differential equations and Kummer functions had already been studied intensively in literature over the last decades. Thus, one can find for instance in (Slater, 1960), that both confluent hypergeometric function of the first and the second kind show a specific asymptotic behavior for large x, in particular

$$U(a,c;x) = x^{-a}[1+O(|x|^{-1})] \quad \text{and} \quad M(a,c;x) = \frac{\Gamma(c)}{\Gamma(a)}e^x x^{a-c}[1+O(|x|^{-1})],$$

as $x \to \infty$, so that

$$U(a,c;x) : x^{-a}, \qquad \textit{for}\ a>0; \tag{33}$$

$$M(a,c;x) : \frac{\Gamma(c)}{\Gamma(a)}e^x x^{a-c}, \qquad \textit{for}\ x>0, \tag{34}$$

as $x \to \infty$; (see (Slater, 1960)[p.60]).

Therefore, we conclude that $U(a,c;x) \to 0$ and $M(a,c;x) \to \infty$ if x tends to infinity.

By transferring this to solution given in (32), it becomes apparent that the above condition for $V_g(x;b)$, namely being bounded by a linear function for increasing x, can only hold if and only if $\kappa_{b,g} = 0$.

Consequently, (32) reduces to

$$V_g(x;b) = \rho_{b,g}\cdot U\left(\frac{\delta}{2g},\frac{1}{2};\frac{(gx-\mu)^2}{g\sigma^2}\right) + \frac{g}{g+\delta}\cdot(x+\frac{\mu}{\delta}),\ \textit{for}\ x>b. \tag{35}$$

Now, using the continuity of the functions $V_g(x;b)$ and $V_g'(x;b)$ at $x=b$ to obtain the coefficients $\nu_{b,g}$ and $\rho_{b,g}$ we finally arrive after some simple calculations at

$$V_g(x;b) = \frac{g}{g+\delta} \cdot [\frac{[(\delta b+\mu)(gb-\mu)\cdot B_b + g\sigma^2 \cdot A_b]\cdot(e^{rx}-e^{sx})}{(re^{rb}-se^{sb})g\sigma^2 \cdot A_b - (e^{rb}-e^{sb})(\mu-gb)\delta \cdot B_b}], \tag{36}$$

for $0 \le x \le b$, and

$$V_g(x;b) = \frac{g}{g+\delta}[x+\frac{\mu}{\delta} - \frac{(\delta b+\mu)A_x}{\delta A_b} + \frac{(e^{rb}-e^{sb})[(\delta b+\mu)(gb-\mu)\cdot B_b + g\sigma^2 \cdot A_b]\cdot A_x}{[(re^{rb}-se^{sb})g\sigma^2 \cdot A_b - (e^{rb}-e^{sb})(\mu-gb)\delta \cdot B_b]\cdot A_b}]$$

$$= \frac{g}{g+\delta} \cdot [x+\frac{\mu}{\delta} - \frac{A_x}{A_b}\cdot\left(b+\frac{\mu}{\delta}\right)] + \frac{A_x}{A_b}\cdot \nu_{b,g} \cdot (e^{rb}-e^{sb}), \tag{37}$$

for $x > b$, wherein

$$A_b := U\left(\frac{\delta}{2g}, \frac{1}{2}; \frac{(gb-\mu)^2}{g\sigma^2}\right), \tag{38}$$

$$B_b := U\left(\frac{\delta}{2g}+1, \frac{3}{2}; \frac{(gb-\mu)^2}{g\sigma^2}\right). \tag{39}$$

We know that the value function V_g is a solution to the system of differential equations (28) and (31). Using *"Tanaka's formula"* (see (Karatzas & Shreve, 2005)[chap. 3.6A]) one obtains that the value function fulfils the smooth-fit property, which means that V_g is smooth in C^1 through the barrier b. Now, if there would be an infinite number of solutions, then also $\beta \cdot V_g$, $\beta \in \mathbb{R}$, would be a solution. Plugging this into IDE (31) yields a contradiction.

As the new dividend strategy is a combination of the classical barrier and the mean-reverting strategy according to which the dividend payout is not fixed, but rather comply with the value of the current modified surplus process, provided that the process is above a constant barrier $b > 0$, we expect that ruin will also occur in finite time, independent of the value of the payout rate $g > 0$. A mathematical proof can be found in the related diploma thesis[2].

[2] Diploma thesis entitled "Dividend Strategies in the Brownian Risk Model" by Marco Ehlscheid, November 2011, University of Cologne

4.2. The Optimal Barrier and Optimal Pay-Out Rate

For a given dividend payout rate $g \in (0,\infty)$, let b^* once again denote the optimal value of b, that is, the value that maximizes the value function $V_g(x;b)$.

As a function of the barrier level b, $V_g(x;b)$ reads

$$V_g(x;b) = \begin{cases} \rho_{b,g} \cdot U\left(\frac{\delta}{2g}, \frac{1}{2}; \frac{(gx-\mu)^2}{g\sigma^2}\right) + \frac{g}{g+\delta} \cdot \left(x - \frac{\mu}{\delta}\right), & \text{for} 0 < b < x; \\ \nu_{b,g} \cdot (e^{rx} - e^{sx}), & \text{for} b \geq x; \end{cases} \tag{40}$$

With same coefficients $\rho_{b,g}$ and $\nu_{b,g}$ as given (36) and (37), respectively, and being independent of x. Furthermore, using simple algebra it can be shown that the optimal barrier level b^* does not even depend on the initial capital $x > 0$.

However, even if not being able to provide an explicit expression for the optimal choice of $b > 0$ for this dividend strategy, the thesis states at least the proof of the following

Corollary 3.4 *Assume that* $\delta \in (0,1)$ *and* $g, \mu, \sigma > 0$ *are arbitrary but fixed. Then, there exists an optimal barrier level* $b^* < \infty$*, such that*

$$V_g(x;b^*) = \max_{b \geq 0} V_g(x;b), \forall\, x > 0.$$

Remark *Note that* $\lim_{b\to\infty} V_g(x;b) = 0, \forall\, x > 0.$ *As* $x > 0$ *is fixed, this limit can also be seen as the result of a trade-off between the value of the barrier level* b *and the probability that the surplus process* $(X_t)_{t\geq 0}$ *reaches or even exceeds this level. More precisely, since the process drift is constant anywhere below* b*, the larger* b*, the lower the probability that the surplus process will reach the barrier level before passing the absorption level at zero, and this, whichever dividend payout rate* $g > 0$ *is selected.*

Let us now take a brief look at the optimal payout rate g^*. For a given dividend barrier level $b > 0$, let g^* (l^*) symbolize the optimal dividend payout rate (the optimal reverting level). That is, the value that maximizes the value function $V_g(x;b)$ in g, provided that such a $g > 0$ exists.

To begin with, we first investigate the value function's limit value behavior. In doing so, consider the corresponding value function V_g as given in (36) and (37) for $x > b$ and $x \in [0,b]$ respectively, but this time, as a function of g.

Since the value function represents the expected accumulated discounted dividend payments and $g > 0$ the rate at which dividends are proportionally distributed from the surplus process above the barrier, we obtain as g approaches zero, that

$$\lim_{g \to 0} V_g(x;b) = 0.$$

On the other hand, using the asymptotic behavior of the Kummer function U we get

$$\lim_{g \to \infty} A_b = 1 \quad and \quad \lim_{g \to \infty} B_b = 0,$$

as g tends to infinity, which yields

$$\lim_{g \to \infty} v_{b,g} = \frac{1}{re^{rb} - se^{sb}}.$$

Hence we maintain,

$$\lim_{g \to \infty} V_g(x;b) = \frac{e^{rx} - e^{sx}}{re^{rb} - se^{sb}}, \ for \ 0 \le x \le b,$$

which equals $V(x;b)$ in (12). On the other hand, if $x > b$, we obtain

$$\lim_{g \to \infty} V_g(x;b) = (x-b) + \frac{e^{rb} - e^{sb}}{re^{rb} - se^{sb}}, \ for \ x > b,$$

which equals (13).

Consequently, as g goes from 0 to infinity (or alternatively, when l goes from infinity to 0), the expectation of the present value of dividends tends to the value function of the classical barrier strategy from section 2.1. Thus the barrier strategy can be seen as the limit $g \to \infty$. This property might have already been expected, since the dividend payouts above the barrier level $b > 0$ according to the new strategy are restricted. once $g \in (0,\infty)$ has been specified.

The thesis proceeds with a variety of examples and (37) illustrations discussing the influence of the parameters x and b on the optimal payout rate g^* as well as in how far the volatility σ could infect the optimal dividend payout rate g^*. We explicitly point out that all gained results, given interpretations and explanations concerning the optimal choice of the payout rate g subject to the value of a predefined barrier level $b > 0$ also hold for other constellations of parameters. But, since a general proof is rather complex and requires an extensive knowledge and understanding of the theory of confluent hypergeometric functions, it is abandoned in our work.

A Note on Simultaneous Optimization

Following an intensive examination of the separate optimization of the strategies' parameters b and g, we will finally complete with commenting on the question:

What if maximizing the value function $V_g(x;b)$ with respect to both parameters b and g simultaneously?

In this case, the value function with respect to the pair (b,g) reads

$$\Theta : \mathrm{R}^+ \times \mathrm{R}^+ \to \mathrm{R}_0^+$$

$$(b,g) \mapsto V_g(x;b) = \begin{cases} (36) & \text{for } 0 \le x \le b \\ (37) & \text{for } x > b. \end{cases}$$

Now, we are interested in the optimal pair (b^*, g^*) such that

$$V_{g^*}(x;b^*) = \max_{(b,g)} V_g(x;b).$$

for a given initial capital $x > 0$.

But without going into more detail, we are already able to answer the above mentioned question by just using the optimization criterion, if no barrier level $b > 0$ is predefined. As mentioned in one of the previous chapters, without any constraints concerning the barrier level $b > 0$ or even the amount of dividend payments, the classical barrier strategy with optimal barrier level b^*, given by (15), turns out to be the optimal strategy due to the maximization of all dividend payments until ruin in the Brownian risk model. Because of this, we draw the conclusion that, independent of the company's initial surplus $x > 0$, if neither the barrier level b, nor the payout rate $g > 0$ is predefined (for instance, by the company itself or the controlling authority), it is most profitable to choose the optimal barrier level as

$$b^* = \frac{\ln(s^2) - \ln(r^2)}{r - s},$$

resulting from the optimization problem according to the classical barrier strategy from section 2.1, and at the same time let the payout rate g grow to infinity. This means that dividends should be paid at highest rate g, whenever the optimal barrier level b^* is exceeded, so that an immediate return to the barrier level b is guaranteed.

But in general, this assumption does generally not reflect the reality. Usually, at least one of the parameters is specified, for example, due to the company's policy, its shareholders, a board decision or even due to a ruling of the controlling authority. This means that maximizing the present value of all dividends until ruin with respect to (b, g) according to this dividend strategy mostly reduces to a maximization of the value function $V_g(x;b)$ with respect to only one of the parameters b or g.

Closing this section about the optimal choice of g, we maintain the following:

As long as the dividend barrier level b is not primarily specified by restrictions, for example, by the company's policy or a supervision order, it is always most favorable to choose the optimal barrier b^* (CBS) from the classical barrier strategy and distribute dividends at a maximum rate of $g > 0$, whenever the surplus process $(X_t)_{t\geq 0}$ exceeds this level. In other words, one should stick to the classical barrier dividend strategy in order to ensure a maximum dividend payout to the shareholders. The same applies to the optimal payout rate g^* in case that a predefined barrier level b fulfils $b \geq b^*$. Nevertheless, this should not be the case, once a predefined barrier level b is located below the optimal classical barrier level b^* (CBS). Then, distributing dividends at a lower rate of the current surplus process above the barrier is more profitable than an immediate dividend payment of the entire overflow, whenever the surplus process exceeds the barrier. Therefore, in such a case, the new dividend strategy is preferable.

4.3. Comparison with a Barrier and Mean Reverting Strategy

In the following we compare the expected present value of dividends in all three strategies including the cases in which one of the governed parameters b or g is predefined (section 4.3.1). Subsequently, we conclude with evaluating the variability of dividend payouts for different values of b and g (section 4.3.2). All numerical results in this section have been computed using Mathematica.

4.3.1. Present Value of Dividends

Tables 1-3 display the numerical results of the expected present value of dividends for different constellations of parameters in all three strategies, the classical barrier strategy (CBS), the mean-reverting strategy (MRS) and the new dividend strategy of barrier type with its mean-reverting tendency.

As already mentioned at the end of the previous section, it is not very realistic that both the barrier level b and the payout rate g are arbitrary from the outset. Concerning this, we added the results of our new strategy assuming that at least $g = g^*$ (MRS) is given by the optimal payout rate from the mean-reverting strategy.

Moreover, recall from the previous section that, due to the optimization problem concerning the payout rate g, the assumption that the optimal barrier level b^* from the classical barrier strategy is given, yields the same result as in case of simultaneous optimization of the pair (b, g).

In terms of expectation, the mean-reverting strategy is very close to the classical barrier strategy, especially for high x, low σ and low δ. But, and this is what all three tables highlight, our new dividend strategy always outperforms the mean-reverting strategy, even for the same payout rate g; compare for instance columns 6 and 14 in tables 1-3.

This can be explained as the result of merging an efficient and effective dividend strategy with the structure of the optimal strategy in the underlying Brownian risk model. More precisely, when adopting a dividend barrier level $b > 0$ to the mean-reverting strategy, dividends are only paid above this level, so that during the lifetime of the surplus process there will be periods of time without dividends at all. Within this period, the surplus process behaves like a Brownian motion with constant positive drift μ and variance σ^2 per unit of time. Thus, its survival probability increases compared to the survival probability of a surplus process carrying a smooth flow of dividends over time. Consequently, we obtain a larger amount of dividend payments until ruin.

Furthermore, notice that the results of the new strategy are either equal to or at most slightly below those of the classical barrier strategy. However, based on what we have figured out in the previous section, this is not surprising:

On the one hand, if none of the governed parameters b or g is specified or equivalently, the optimal classical barrier level b^* is assumed as given, then, the payout rate g tends to infinity and the corresponding value function converges to the one obtained when applying the classical barrier strategy. On the other hand, as long as the selected payout rate g is finite, the respective optimal barrier b^* is smaller than the optimal barrier of the classical barrier strategy. Hence, we always receive a minor expected present value of dividends than in case of applying the classical barrier strategy in its optimal state. However, for increasing values of g, the appropriate optimal barrier level b^*, and consequently, the value function $V_g(x; b^*)$ will more and more approximate its respective counterpart of the classical barrier strategy.

	Classical Barrier Strategy		Mean-Reverting Strategy			BS with MR-Tendency (given b)				BS with MR-Tendency given g			
x	b^*	$V(x; b^*)$	g^*	l^*	$V(x; l^*)$	b^*	g^*	l^*	$V_{g^*}(x; b^*)$	g^* MRS	l^* MRS	b^*	$V_{g^*}(x; b^*)$
$0.5b^*$	1.256	19.266	0.531	1.884	18.297	1.256	1288.09	0.001	19.266	0.531	1.884	0.994	18.675
b^*	1.256	20.000	0.560	1.787	19.269	1.256	1288.09	0.001	20.000	0.560	1.787	1.005	19.458
$1.5b^*$	1.256	20.628	0.570	1.754	19.896	1.256	1288.09	0.001	20.628	0.570	1.754	1.009	20.071
$2b^*$	1.256	21.256	0.576	1.735	20.491	1.256	1288.09	0.001	21.259	0.576	1.735	1.012	20.667

Table 1 : Expected present value of dividends for different initial surpluses x with $\mu = 1$, $\delta = 5\%$ and $\sigma = 0.5$.

	Classical Barrier Strategy		Mean-Reverting Strategy			BS with MR-Tendency (given b)				BS with MR-Tendency given g			
σ	b^*	$V(x; b^*)$	g^*	l^*	$V(x; l^*)$	b^*	g^*	l^*	$V_{g^*}(x; b^*)$	g^* MRS	l^* MRS	b^*	$V_{g^*}(x; b^*)$
0.10	0.083	19.954	7.275	0.137	19.873	0.083	3167.16	0.0003	19.954	7.275	0.137	0.070	19.899
0.50	1.256	19.266	0.531	1.884	18.297	1.256	1288.09	0.0008	19.266	0.531	1.884	0.994	18.675
1	3.563	17.908	0.209	4.788	15.951	3.563	3305.43	0.0003	17.908	0.209	4.788	2.724	16.778

Table 2 : Expected present value of dividends for different volatilities σ with $x = 0.5b^*(\sigma)$, $\mu = 1$ and $\delta = 5\%$.

	Classical Barrier Strategy		Mean-Reverting Strategy			BS with MR-Tendency (given b)				BS with MR-Tendency given g			
δ	b^*	$V(x; b^*)$	g^*	l^*	$V(x; l^*)$	b^*	g^*	l^*	$V_{g^*}(x; b^*)$	g^* MRS	l^* MRS	b^*	$V_{g^*}(x; b^*)$
0.01	1.668	99.047	0.376	2.660	97.530	1.668	184.37	0.0054	99.047	0.376	2.660	1.363	98.070
0.05	1.256	19.266	0.531	1.884	18.297	1.256	1288.09	0.0008	19.266	0.531	1.884	0.994	18.675
0.1	1.075	9.369	0.648	1.543	8.644	1.075	7600.04	0.0001	9.369	0.648	1.543	0.837	8.939

Table 3 : Expected present value of dividends for different interest rates δ with $x = 0.5b^*(\delta)$, $\mu = 1$ and $\sigma = 0.5$.

4.3.2. Variability of Dividend Payouts

For expository purposes, we focus in this section on a given set of parameters. More precisely, assume that the parameters of the surplus process have been selected as $\mu = \sigma = 1$, $\delta = 5\%$ and $x = 3$. Based on the fact that the new dividend strategy is only expedient if either a predefined barrier level b fulfils $b < \frac{1}{r-s} \ln\left(\frac{s^2}{r^2}\right) = b^*$ (CBS), or a fixed payout rate $g > 0$ is given, we will restrict ourselves for a comparison of the variability of dividend payouts to some specified values for b and g. These values as well as their corresponding optimal parameter are displayed in table 4. At this point, we notice that the properties described in this section also hold for other constellations of parameters.

	Given Parameters		**Optimal Parameters**		
Dividend Strategy	b	$g\ (l)$	b^*	$g^*\ (l^*)$	$V_g(3;b)$
CBS	-	-	3.563	-	19.433
MRS	-	-	-	0.222 (4.51)	17.802
New Strategy					
(1)	3.5	-	-	4.561 (0.45)	19.431
(2)	2.5	-	-	0.371 (2.70)	18.904
(3)	1.753	-	-	0.266 (3.76)	18.314
(4)	0.8	-	-	0.227 (4.41)	17.877
(5)	-	1.00 (1.00)	3.409	-	19.407
(6)	-	0.50 (2.00)	3.228	-	19.295
(7)	-	0.25 (4.00)	2.581	-	18.625
(8)	-	0.22 (4.51)	2.7681	-	18.364

Table 4: Optimal choice of parameters b^* and g^* for $\mu = \sigma = 1$, $\delta = 5\%$ and $x = 3$.

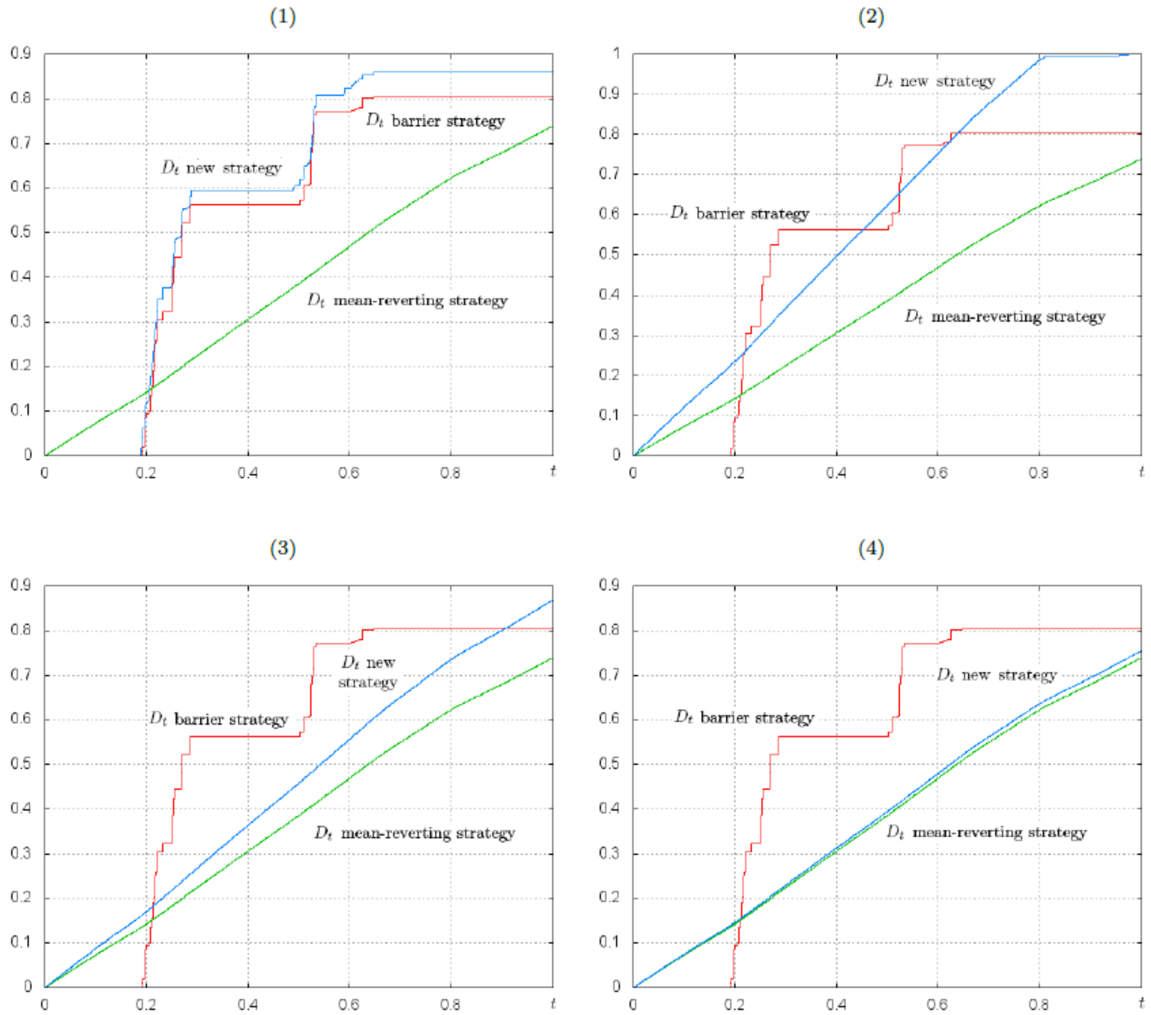

Figure 4: Comparison of the sample paths of the aggregate dividends process $(D_t)_{t\geq 0}$ with a classical barrier strategy (red), a mean-reverting strategy (green) and the new strategy (blue) for decreasing barrier levels b.

Figure 4 and figure 5 illustrate how the classical barrier strategy (red path) and the mean-reverting strategy (green path) would have performed for their optimal choice of parameter in comparison to the new dividend strategy (blue path) for the same sample path. The numbering refers to the appropriate choice of parameters in table 4. We find that the higher variability of the classical barrier payouts is apparent, as well as the periods without dividends at all; compare the red and green path in all charts of figures 4 and 5.

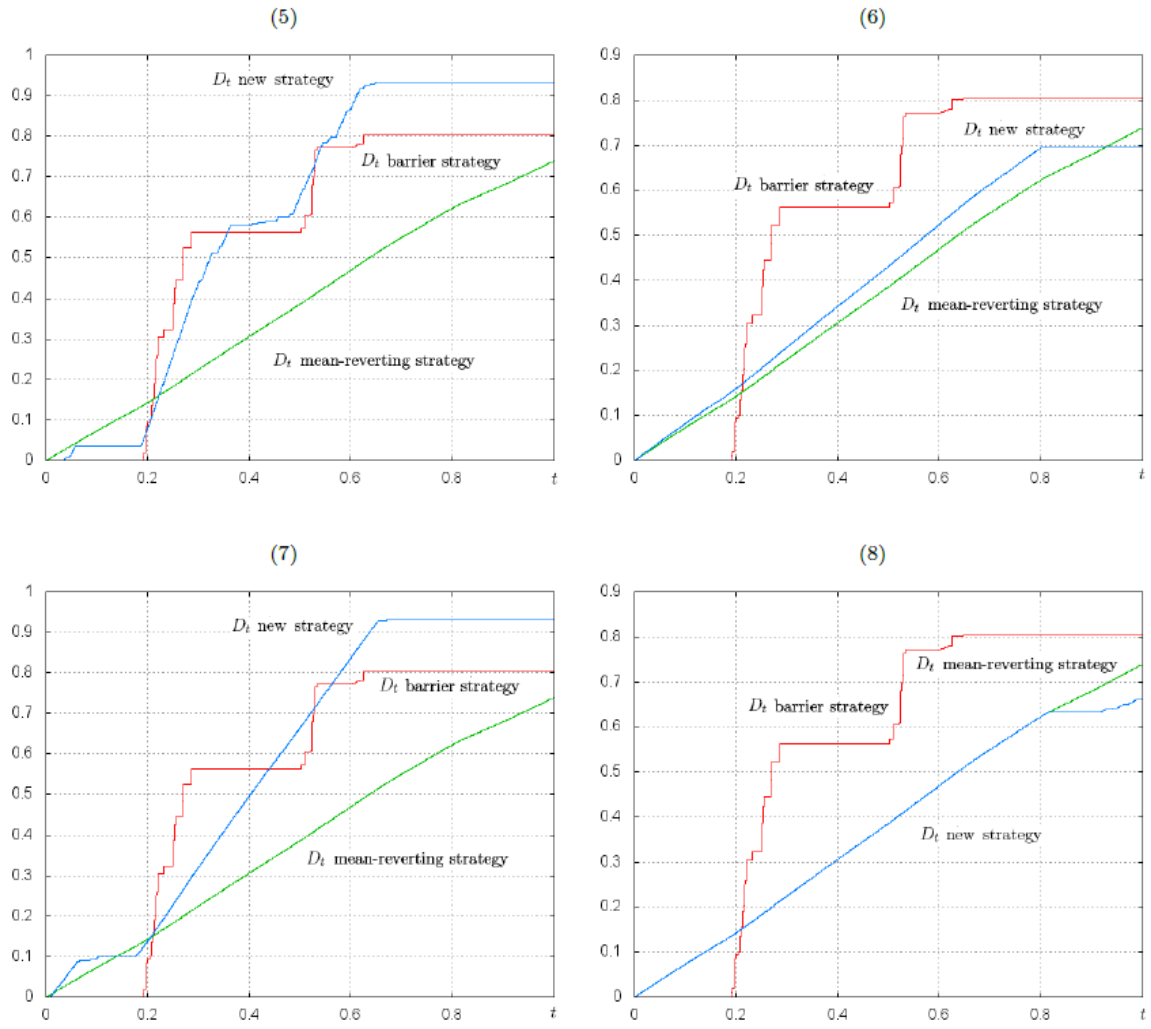

Figure 5: Comparison of the sample paths of the aggregate dividends process $(D_t)_{t\geq 0}$ with a classical barrier strategy (red), a mean-reverting strategy (green) and the new strategy (blue) for decreasing payout rate g.

When analyzing the performance of the blue path in each chart of figure 4, it is particularly striking that for decreasing predefined values of $b < b^* = 3.563$ we observe a smoothing of dividend payments that results in a considerable reduction of the variability of payouts. In order to give a suitable explanation, we recall from the new dividend strategy's structure that, based on continuous dividend payments above the barrier level b, the surplus process $(X_t)_{t\geq 0}$ resembles an OU-process with mean-reverting level $l = \mu/g$. Whereas below b, it behaves like a Brownian motion with constant drift $\mu > 0$. Consequently, the smaller the barrier level b, the higher the probability that the surplus will exceed this level, once it has fallen below. Furthermore, the optimal payout rate g^* is an increasing function of b with lower bound

g^* (MRS), as being the optimal payout rate according to the mean-reverting strategy for the same constellation of parameters. This bound results from the fact that the mean-reverting strategy can be seen as the limit of the new dividend strategy as b approaches zero. In addition, note that a small value of g^* is tantamount to a high value of the optimal reverting level l^*. Thus, a decrease in the value of b yields a slope of the optimal reverting level l^*. Hence, $l^* > b$, which results in more regular dividends because of the mean-reverting nature of the modified surplus process above the barrier level b (see, for instance, the appropriate values in parentheses in table 4).

All this eventually leads to a reduction of the periods in which no dividends are paid, and at the same time, it warrants a smoother flow of dividends over time based on continuous payments above the barrier. However, this causes a smaller present value of dividend payments in return (see table 4, last column).

Conversely, the more the barrier level b approaches the optimal classical barrier level $b^* = 3.563$, the stronger the growth in the value of g^*. Then, the optimal reverting level l^* is close to zero and the strategy resembles the classical barrier strategy from section 2.1 (see for instance chart (1) in figure 4). Even though this effect provides a larger amount of dividends until ruin, it results in a higher variability of dividend payouts and increases the number of periods without dividends at all.

In case of an increasing predefined payout rate $g \geq g^*$ (MRS), see figure 5 charts (8) to (4) and the corresponding values of parameters in table 4, we observe a similar performance of the blue sample path as in case of a predefined barrier level (figure 4). The larger the payout rate $g > 0$ is selected, the stronger the convergence of the new dividend strategy to the classical barrier strategy. This is no surprise, because, according to our new dividend strategy, the optimal barrier level b^* is also an increasing function of g with upper bound b^* (CBS). Therefore, an analogue explanation to the one above can be used to comment on the variability of dividend payouts for decreasing and increasing values of g.

5. Conclusion

In our work[3] we established a new dividend strategy for an insurance company whose surplus process, before any distribution of dividends, is modeled by a Brownian motion with constant positive drift. We aimed at investigating the strategy's performance and profit due to the optimal choice of parameters that provides a maximization of the present value of all dividends until possible ruin. As a combination of the classical barrier strategy with a constant barrier level $b > 0$ and the mean-reverting strategy of Avanzi and Wong with a mean-reverting rate $g > 0$, we finally constructed a strategy that - even though it also leads to ruin almost surely - yields a piecewise

[3] Diploma thesis entitled "Dividend Strategies in the Brownian Risk Model" by Marco Ehlscheid, November 2011, University of Cologne

smooth flow of dividends based on continuous dividend payments at a rate $g > 0$ of the current surplus process, whenever it exceeds the barrier level $b > 0$.

After having determined the corresponding value function, we focused on the optimal choice of parameters b and g. As it is often unrealistic to assume that both the barrier level and the payout rate are arbitrary from the outset, we included the cases in which one of the governed parameters had been predefined.

Having started with a given rate $g > 0$, we derived that the optimal barrier level b^* is an increasing function of g. Therefore, the higher the value of g, the higher the optimal barrier level needs to be chosen in order to counteract the risk of precocious ruin. As the classical barrier strategy with optimal barrier b^*(CBS) had already been proven as the optimal dividend strategy in the Brownian risk model, b^*(CBS) acts as an upper bound for the optimal b^* according to the new dividend strategy as g tends to infinity. Consequently, for large values of g, it is reasonable to distribute dividends according to the classical barrier strategy with optimal barrier level b^*(CBS) to ensure a maximization of the dividend cash flow, even though this causes a high variability in the dividend payouts and increases the number of periods without any dividends at all.

On the other hand, for a given barrier level $b > 0$, we figured out that it could sometimes be more profitable to apply the new dividend strategy and pay dividends at a lower rate g of the company's current surplus process to maximize the value function, instead of sticking to the payout policy of the classical barrier strategy. This would be the case, if and only if the selected barrier level b was located strictly below the corresponding optimal classical barrier level for the same constellation of parameters. Hence, the new dividend strategy with a finite optimal rate $0 < g^* < \infty$ yielded a larger present value of dividends caused by an increase of the surplus process survival probability. As expected, g^* is increasing in b with lower bound g^*(MRS), being the optimal payout rate according to Avanzi's mean reverting strategy, as b approaches zero. In the unlikely event that none of the governed parameters were specified, we concluded that the classical barrier strategy with optimal barrier b^*(CBS), as being the optimal strategy in the underlying risk model, should serve as the preferable dividend strategy.

In conclusion, we maintain that, according to this dividend strategy, a general response to the issue of the optimal choice of parameters in order to maximize the expected present value of all dividends until ruin is not trivial. In most cases, it depends on the company's business environment, which mainly affects and influences the choice of the barrier level b, the dividend payout rate g or even both. Beyond that, the preference for either a maximum dividend cash flow, implicating a high variability of the barrier payouts ,or a rather continuous flow of dividends and accepting a lower present value of dividends in return, should also be taken into account when applying this dividend strategy and specifying its governed paramters.

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