Dissertation

Asymptotic Ruin Probabilities and Optimal Investment for an Insurer

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Zusammenfassung

Die vorliegende Dissertation befaßt sich mit dem klassischen Modell der Ruintheorie, in dem der Risikoprozess durch einen Poisson–Zählprozess, unabhängige, identisch verteilte Einzelschäden und eine konstante Prämienrate (pro Zeiteinheit) gegeben ist. Allerdings wurde dieses Modell insofern erweitert, daß das Versicherungsunternehmen seine Reserven nun nicht nur in eine risikolose Anleihe mit null Zinsen, sondern auch in eine riskante Anlageform investieren kann.

Im ersten Kapitel führen wir zunächst den Risikoprozess ein (Abschnitt 1.1). Dann besprechen wir kurz den Unterschied zwischen „kleinen“ und „großen“ Schäden anhand ihrer Verteilungsfunktionen (Abschnitt 1.2) und präsentieren klassische Resultate über Ruinwahrscheinlichkeiten (für unendlichen Zeithorizont) (Abschnitt 1.3). Schließlich stellen wir die Investitionsmöglichkeit vor, die durch eine geometrische Brownsche Bewegung modelliert wird (Abschnitt 1.4). Das erste Kapitel dient hauptsächlich dazu, Notationen und Begriffe einzuführen und gibt eine erste Vorstellung von den Problemen, die in weiterer Folge untersucht werden sollen.

Im ersten Abschnitt des zweiten Kapitels finden wir zunächst eine exponentielle obere Schranke für die minimale Ruinwahrscheinlichkeit eines Versicherungsunternehmens, das ausschließlich „kleine“ Schäden (mit exponentiellen Momenten) versichert und in eine riskante Anlageform (z.B. eine Aktie oder ein Aktienindex) investiert, die durch eine geometrische Brownsche Bewegung modelliert wird. (Die minimale Ruinwahrscheinlichkeit wird durch die Wahl einer geeigneten Handelsstrategie erzielt.) Die Schranke ist kleiner als die klassische Cramér–Lundberg–Schranke, die man ohne Investitionsmöglichkeit erhält. Die Handelsstrategie, die zu unserer exponentiellen Schranke führt, kann explizit berechnet werden und besteht darin, einen konstanten Geldbetrag in die riskante Aktie zu investieren. In Abschnitt 2.2 geben wir eine exponentielle untere Schranke für die minimale Ruinwahrscheinlichkeit an, und zwar mit demselben Exponenten wir für die obere Schranke; dafür benötigen wir die Annahme eines gleichmäßigen exponentiellen Moments in der Tail–Verteilung. Mit Hilfe dieses Resultats können wir zeigen, daß die konstante Handelsstrategie, die die exponentielle obere Schranke geliefert hat, asymptotisch optimal ist für unendlich große Startreserve der Versicherung (Abschnitt 2.3). In Abschnitt 2.4 zeigen wir – ohne die Annahme eines gleichmäßigen exponentiellen Moments in der Tail–Verteilung, für beliebige Startreserve – daß das Versicherungsunternehmen entweder unendlich reich oder irgendwann ruiniert wird, wenn man einen unendlichen Zeithorizont betrachtet. In Abschnitt 2.5 vergleichen wir die konstante Handelsstrategie, die wir explizit erhalten haben, mit der konstanten Handelsstrategie, die in der wohlbekannten Diffusionsapproximation optimal ist, für exponentiell verteilte Schäden. Dann betrachten wir die Verallgemeinerung der obigen Resultate auf ein Modell, in dem es mehr als eine Aktie gibt (Abschnitt 2.6). Schließlich präsentieren wir ein Beispiel für eine Schadensverteilung mit exponentiellen Momenten, die kein gleichmäßiges exponentielles Moment in der Tail–Verteilung hat (Abschnitt 2.7).

Im dritten Kapitel wird die Verallgemeinerung der Resultate aus Kapitel 2 für ein Modell betrachtet, in dem die Investitionsmöglichkeit durch einen exponentiellen Lévy–Prozess modelliert wird. Abschnitt 3.1 stellt die nötigen Vorkenntnisse aus der Theorie der Lévy–Prozesse bereit.

Im vierten Kapitel leiten wir das asymptotische Verhalten der minimalen Ruinwahrscheinlichkeit für ein Versicherungsunternehmen her, das „große" Schäden, mit einer regulär variierenden Tail–Verteilung, versichert. In Abschnitt 4.1 fassen wir kurz die Resultate aus der Theorie der Funktionen regulärer Variation zusammen, die in weiterer Folge benötigt werden. Abschnitt 4.2 enthält das Hauptresultat des vierten Kapitels, welches besagt, daß – für Schäden mit regulär variierender Tail–Verteilung – die minimale Ruinwahrscheinlichkeit eines Versicherungsunternehmens, das in eine geometrische Brownsche Bewegung investieren kann, wieder von regulärer Variation ist, und zwar mit dem gleichen Variationsindex wie die Tail–Verteilung. In Abschnitt 4.3 präsentieren wir die Hamilton–Jacobi–Bellman Gleichung zu dem Problem der Minimierung der Ruinwahrscheinlichkeit durch die Wahl einer geeigneten Handelsstrategie. In Abschnitt 4.4 beweisen wir Hilfsresultate, die wir für den Beweis des Hauptresultats benötigen, der schließlich in Abschnitt 4.5 gegeben wird.

Abstract

This thesis deals with the classical model of ruin theory given by a Poisson claim number process with identically and independently distributed single claims and a constant premium flow, but with the additional feature that the insurer has the possibility to invest in a risky asset.

In Chapter 1 we first introduce the risk process (Section 1.1). Then we briefly discuss the distinction between light-tailed claim distributions and heavy-tailed claim distributions (Section 1.2) and classical results on (infinite time) ruin probabilities without investment possibility (Section 1.3). We finally introduce the investment possibility, modelled by geometric Brownian motion, in Section 1.4. Chapter 1 mainly serves the purpose of introducing notions and notations and gives a first idea of the problems to be considered thereafter.

In Chapter 2, Section 2.1, we give an exponential upper bound on the minimal ruin probability of an insurer, who is incurring small claims (with exponential moments) and who invests in a risky asset modelled by geometric Brownian motion. (The minimal ruin probability is obtained by investing in an appropriate way.) The bound is smaller than the classical exponential Cramér–Lundberg bound without investment. The trading strategy yielding this exponential bound can be calculated explicitly. In Section 2.2 we give an exponential lower bound for the minimal ruin probability with the same exponent as in the case of the upper bound, under the assumption of a uniform exponential moment in the tail distribution. With the help of the latter result we can show that the constant investment strategy yielding the upper exponential bound is asymptotically optimal, as the initial reserve of the insurer tends to infinity (Section 2.3). In Section 2.4 we show that – without the assumption of a uniform exponential moment in the tail distribution – for arbitrary initial reserve, the insurer either gets ruined, or she becomes infinitely rich as time goes to infinity. In Section 2.5 we compare the constant investment strategy, which we obtain explicitly, with the constant investment strategy, which is optimal in the well-known diffusion approximation, for exponentially distributed claims. Then the generalization of the above results to a setting with more than one risky asset is discussed (Section 2.6). Finally, we present an example for a claim size distribution with exponential moments, but no uniform exponential moment in the tail distribution (Section 2.7).

Chapter 3 deals with the generalization of the results of Chapter 2 to a setting, where the investment possibility is modelled by an exponential Lévy process. Section 3.1 provides the necessary tools from the theory of Lévy processes.

In Chapter 4 we derive the asymptotic behavior of the ruin probability for an insurer, who incurs claims with a regularly varying tail distribution (i.e. ‘large’ claims). In Section 4.1 we briefly recall the results from the theory of regularly varying functions, which will be needed in the sequel. Section 4.2 contains the main theorem of Chapter 4, which states that for claims with regularly varying tail distribution, the minimal ruin probability of an insurer, who may invest in a geometric Brownian motion, is also of regular variation with the same index of variation as the tail distribution. In Section 4.3 we present the Hamilton–Jacobi–Bellman equation for the problem of minimizing the ruin probability by choosing an appropriate investment strategy. In Section 4.4 we state and prove auxiliary results, which are needed for the proof of the Main Theorem, which is finally given in Section 4.5.

Contents

Preface

In 1903, F. Lundberg laid the foundation of risk theory in his thesis [23], where he introduced a risk model based on a homogeneous Poisson claim number process. Since that time, this risk model has attracted much attention by mathematicians and actuaries. In particular, we mention H. Cramér who incorporated F. Lundberg's ideas into the theory of stochastic processes ([4], [5]). The famous Cramér–Lundberg theorem for small claims (Theorem 1.3.1) states – among other things – that the infinite time ruin probability of an insurer can be bounded from above by an exponential function with an explicitly given exponent, if we assume a positive safety loading (see page 12). The classical proof of this result uses renewal theory (see [9]). In 1973, H. Gerber [13] gave an alternative proof, making use of martingale methods.

It has only been recently that a more general problem has been considered: If an insurer, additionally, has the opportunity to invest in a risky asset (modelled, e.g., by geometric Brownian motion), what is the minimal ruin probability she can obtain? In particular, can she do better than keeping the funds in the bonds? And if yes, to which extent can she do better?

S. Browne investigated this problem, but under the assumption that the risk process follows a Brownian motion (the so called 'diffusion approximation'). In this simpler setting, the investment strategy which minimizes the ruin probability consists in holding a constant amount of wealth in the risky asset, and the corresponding minimal ruin probability is given by an exponential function. C. Hipp and M. Plum [17] investigated the general problem, for a risk process of the compound Poisson form, and derived the corresponding Hamilton–Jacobi–Bellman equation for the maximal survival probability. This nonlinear second order integro–differential equation is generally very hard to solve. However, Hipp and Plum present a special example with exponentially distributed claims, where the solution can be explicitly calculated and decreases as an exponential function of the initial wealth, with a sharper exponent than the classical Lundberg exponent without investment.

On the other hand, A. Frovola, Yu. Kabanov and S. Pergemnshchikov [10] showed that for an insurer, who invests a constant fraction of wealth in the risky asset and who incurs exponentially distributed claims, the ruin probability either equals one for all initial reserves or decreases asymptotically for large wealth like a negative power function (depending on the model parameters). H. Gjessing and J. Paulsen (see

[25], [26]) and Kalashnikov and Norberg [19] obtained similar results. These results indicate that the additional investment possibility may increase the risk of getting ruined significantly.

It is shown, however, in J. Gaier, P. Grandits and W. Schachermayer [12] that the minimal ruin probability – depending on the initial wealth – for an insurer with investment possibility can be bounded from above and from below by an exponential function with a better, i.e greater, exponent than the classical Lundberg exponent without investment, for small claims, i.e. claims with exponential moments. There is an investment strategy that yields this exponential function as the corresponding ruin probability: This investment strategy consists in holding a fixed amount of wealth in the risky asset and can be explicitly calculated. It turns out that this investment strategy is asymptotically optimal for large initial wealth.

So far, we have only treated risk theory with *small* claims. What happens to the ruin probability of an insurer, who may invest in a risky asset (modelled by geometric Brownian motion) and is incurring *large* claims with heavy tails, which do not have exponential moments? Classically – without investment possibility – for subexponential tails, the ruin probability decreases like the integrated tail distribution (see, e.g., P. Embrechts and N. Veraverbeke [8]). C. Klüppelberg and U. Stadtmüller have shown in [22] that for the case of constant interest rate and no risky investment, and claims with regularly varying tails, the ruin probability is again of regular variation with the same index as the tail distribution. It was shown in J. Gaier and P. Grandits [11] that for the case of zero interest and risky investment, under the assumption of a regularly varying tail distribution of the claims, the minimal ruin probability is also of regular variation with the same index. For claims with tails of regular variation, the order of decrease of the integrated tail distribution, as a function of the claim size, is greater by one power of the claim size than the order of decrease of the tail distribution (for large claims); thus, for claims with tails of regular variation, both for nonzero interest rate and for a risky investment possibility, the ruin probability decreases much faster than in the classical setting without investment.

The articles [11] and [12] form an essential part of this thesis (Chapters 2 and 4).

Chapter 1

Introduction

1.1 The Risk Process

The *risk process* is supposed to model the time evolution of the reserves of an insurance company. In the following, we will use the classical Cramér–Lundberg model to describe the risk process R (see, e.g., the books by ASMUSSEN [1], EMBRECHTS, KLÜPPELBERG AND MIKOSCH [7], or GERBER [14]): the process R is given by a Poisson process N with intensity λ and by a positive random variable X, independent of the process N, with distribution function F in the following way

$$R(t,x) = x + ct - \sum_{i=1}^{N(t)} X_i, \tag{1.1.1}$$

where $x > 0$ is the initial reserve of the insurance company, $c \in \mathbb{R}$ is the (constant) premium rate over time and $(X_i)_{i=1}^{\infty}$ is an i.i.d. sequence of copies of X, where X_i is modelling the i-th loss of the insurance company. The process $(N(t))_{t\geq 0}$ models the number of claims that occur in the time interval $[0,t]$.[1]

It has been a central topic of classical risk theory to obtain information about the *ruin probability* of an insurance company: the event '*ruin*' is defined as the first point of time, where the reserve of the insurance company drops below the level 0. The ruin probability is therefore defined as

$$\Psi(x) = \mathbb{P}[R(t,x) < 0 \text{ for some } t \geq 0], \tag{1.1.2}$$

and the corresponding *time to ruin*, $\tau(x)$,

$$\tau(x) := \inf\{t \geq 0 : R(t,x) < 0\}. \tag{1.1.3}$$

[1] We assume the existence of an underlying complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying the process N and the random variables $(X_i)_{i=1}^{\infty}$, possibly also carrying other objects (see Section 1.4).

It is easy to convince oneself that

$$\lim_{t\to\infty} \frac{\mathbb{E}[R(t,x)]}{t} = c - \lambda\mathbb{E}[X]. \tag{1.1.4}$$

Therefore, in order to exclude the trivial case of almost sure (a.s.) ruin, it is reasonable to impose $c - \lambda\mathbb{E}[X] > 0$. This leads to the definition of the (relative) *safety loading*,

$$\rho := \frac{c}{\lambda\mathbb{E}[X]} - 1. \tag{1.1.5}$$

The condition $c - \lambda\mathbb{E}[X] > 0$ then is equivalent to a *positive safety loading* $\rho > 0$.

1.2 Claim Size Distributions

One can roughly classify the most popular classes of distributions which have been used to model the claim size X into two groups, *light-tailed distributions* and *heavy-tailed distributions*.

1.2.1 Light-tailed Claim Distributions

Light-tailed claim distributions are defined as distributions whose *tail distribution function* $\overline{F}(x) := 1 - F(x)$ satisfies $\overline{F} = O(e^{-rx})$ for some $r > 0$, where O is the standard Landau-symbol, defined in Section 4.1. For obvious reasons, such light-tailed distributions are also called *distributions with exponential moments*. Examples for distributions that are light-tailed are given in Table 1.1.

Name	Tail $\overline{F}$ or density f	Parameters
Exponential	$\overline{F}(x) = e^{-\lambda x}$	$\lambda > 0$
Gamma	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$	α, $\beta > 0$
Weibull	$\overline{F}(x) = e^{-cx^\tau}$	$c > 0$, $\tau \geq 1$
Truncated Normal	$f(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2}$	
Any distribution with bounded support		

Table 1.1: Light-tailed claim size distributions. All distributions have support $(0, \infty)$.

1.2.2 Heavy-tailed Claim Distributions

For our setting, we define *heavy-tailed distributions* as distributions F that satisfy $\hat{F}(r) = \infty$ for all $r > 0$, where $\hat{F}(r) := \int_0^\infty e^{rx} dF(x)$ is the *moment generating function of X at r*. We shall denote the set of all heavy-tailed distributions by $\mathcal{H}$: it can be further divided into several classes, which will be defined below. In order to do so, we still need some definitions.

Definition 1.2.1. *Let ℓ be a positive, Lebesgue measurable function, defined on some neighbourhood $[a, \infty)$ of infinity, and satisfying*

$$\lim_{x\to\infty} \frac{\ell(\lambda x)}{\ell(x)} = 1 \quad \forall \lambda > 0, \tag{1.2.1}$$

then ℓ is said to be slowly varying (in Karamata's sense)*.*

A Lebesgue measurable function $f > 0$ satisfying

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\rho \quad \forall \lambda > 0, \tag{1.2.2}$$

for some $\rho \in \mathbb{R}$, is called regularly varying *of* index ρ*. We write $f \in \mathcal{R}_\rho$.*

Thus, $\mathcal{R}_0$ is the class of slowly varying functions.

Definition 1.2.2. *Let f and g be Lebesgue-measurable. Then the* convolution *of f and g, denoted by $f * g$, is the function*

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x - z)g(z)\, dz. \tag{1.2.3}$$

We denote by f^{n*} the n-fold convolution *of f with itself*

$$f^{n*}(x) := \underbrace{(f * f * \ldots * f)}_{n \text{ times}}(x). \tag{1.2.4}$$

Now we are able to define the following classes of heavy-tailed distributions, where df is a shortcut for distribution function: the class (without name)

$$\mathcal{L} := \{F \text{ df on } (0, \infty) : \lim_{x\to\infty} \overline{F}(x - y)/\overline{F}(x) = 1 \ \forall y > 0\}; \tag{1.2.5}$$

the *class of subexponential distributions*,

$$\mathcal{S} := \{F \text{ df on } (0, \infty) : \lim_{x\to\infty} \overline{F^{n*}}(x)/\overline{F}(x) = n \text{ for all } n \geq 2\}; \tag{1.2.6}$$

the *class of distributions with regularly varying tails*,

$$\mathcal{R} := \{F \text{ df on } (0, \infty) : \overline{F} \in \mathcal{R}_{-\rho} \text{ for some } \rho \geq 0\}; \tag{1.2.7}$$

and the *class of dominatedly varying distributions*,

$$\mathcal{D} := \{F \text{ df on } (0,\infty) : \limsup_{x\to\infty} \overline{F}(x/2)/\overline{F}(x) < \infty\}. \tag{1.2.8}$$

The following relations hold:

$$\text{(i)} \quad \mathcal{R} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H} \text{ and } \mathcal{R} \subset \mathcal{D}, \tag{1.2.9}$$

$$\text{(ii)} \quad \mathcal{L} \cap \mathcal{D} \subset \mathcal{S}, \tag{1.2.10}$$

$$\text{(iii)} \quad \mathcal{D} \not\subseteq \mathcal{S} \text{ and } \mathcal{S} \not\subseteq \mathcal{D}. \tag{1.2.11}$$

In Table 1.2 we list claim size distributions with heavy tails.

Name	Tail $\overline{F}$ or density f	Parameters	class
Lognormal	$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-(\ln x-\mu)^2/(2\sigma^2)}$	$\mu \in \mathbb{R}$, $\sigma > 0$	$\mathcal{S}$
Pareto	$\overline{F}(x) = \left(\frac{\kappa}{\kappa+x}\right)^\alpha$	κ, $\alpha > 0$	$\mathcal{R}$
Burr	$\overline{F}(x) = \left(\frac{\kappa}{\kappa+x^\tau}\right)^\alpha$	κ, α, $\tau > 0$	$\mathcal{R}$
Bektander-type-I	$\overline{F}(x) = (1 + 2(\beta/\alpha)\ln x)$ $e^{-\beta(\ln x)^2-(\alpha+1)\ln x}$	α, $\beta > 0$	$\mathcal{S}$
Bektander-type-II	$\overline{F}(x) = e^{-\alpha/\beta} x^{-(1-\beta)} e^{-\alpha x^\beta/\beta}$	$\alpha > 0$, $0 < \beta < 1$	$\mathcal{S}$
Weibull	$\overline{F}(x) = e^{-cx^\tau}$	$c > 0, 0 < \tau < 1$	$\mathcal{S}$
Loggamma	$f(x) = \frac{\alpha^\beta}{\Gamma(\beta)}(\ln x)^{\beta-1} x^{-\alpha-1}$	α, $\beta > 0$	$\mathcal{R}$
Truncated α-stable	$\overline{F}(x) = \mathbb{P}[\lvert X\rvert > x]$, where X is an α-stable r.v.[2]	$1 < \alpha < 2$	$\mathcal{R}$

Table 1.2: Heavy-tailed claim size distributions. All distributions have support $(0,\infty)$, except for the Bektander cases and the Loggamma with support $(1,\infty)$.

[2] See Embrechts, Klüppelberg and Mikosch [7], Definition 2.2.1.

1.3 Classical Results on Ruin Probabilities without Investment Possibility

In the next two subsections we will present the classical estimates of the ruin probability for an insurer without investment possibility.

1.3.1 Light-tailed Claim Distributions

In order to do so, for light-tailed claim distribution functions, we define the function $h : \mathbb{R}_+ \to \mathbb{R}_+$

$$h(r) = \mathbb{E}[e^{rX}] - 1 = \hat{F}(r) - 1 = \int_0^\infty e^{rz} dF(z) - 1, \quad r \geq 0. \tag{1.3.1}$$

We will assume that there exists $r_\infty \in (0, \infty]$ such that $h(r) < \infty$ for $r < r_\infty$ and such that $h(r) \to \infty$, for $r \uparrow r_\infty$. The function h has the following properties: $h(0) = 0$, h is increasing, convex, and continuous on $[0, r_\infty)$ (cf. Grandell [15]).

The following Cramér–Lundberg estimates of the ruin probability, when the claim size has exponential moments, are fundamental in risk theory.

Theorem 1.3.1 (Cramér–Lundberg theorem for small claims). *Consider the Cramér–Lundberg model, described in Section 1.1, and assume that the claim size distribition has exponential moments (see page 12). Assume further a positive safety loading $\rho > 0$. Then there exists a number $0 < \nu < r_\infty$ such that*

$$\lambda h(\nu) = c\nu, \tag{1.3.2}$$

and the following relations hold.

(i) For all $x \geq 0$,

$$\Psi(x) \leq e^{-\nu x}. \tag{1.3.3}$$

(ii) If, moreoever,

$$\int_0^\infty x e^{\nu x} \overline{F}(x) dx < \infty, \tag{1.3.4}$$

then

$$\lim_{x \to \infty} e^{\nu x} \Psi(x) = C < \infty, \tag{1.3.5}$$

where

$$C := \frac{\rho}{\nu} \frac{\mathbb{E}[X]}{\int_0^\infty x e^{\nu x} \overline{F}(x) dx}. \tag{1.3.6}$$

(iii) In case of an exponential distribution function $F(x) = 1 - e^{-x/\theta}$,

$$\Psi(x) = \frac{1}{1+\rho} e^{-\frac{\rho}{\theta(1+\rho)}x}. \tag{1.3.7}$$

Remarks.

1. Inequality (1.3.3) is the famous *Lundberg inequality*, the exponent ν is called *Lundberg* or *adjustment coefficient*, or *Lundberg exponent* (Gerber [14], Asmussen [1] or Grandell [15]).
2. The fundamental, so called *Cramér–Lundberg condition* (1.3.2), can also be written as

$$\lambda \int_0^\infty e^{\nu z} \overline{F}(z) dz = c \tag{1.3.8}$$

3. Actually, we need not assume that $h(r) \to \infty$ for $r \uparrow r_\infty$. If h were to jump to infinity at r_∞, we still get an exponential bound on the ruin probability $\Psi(x)$: If there exists $\nu < r_\infty$ such that $\lambda h(\nu) = c\nu$, then the bound is $e^{-\nu x}$, otherwise it is simply $e^{-r_\infty x}$.

In Section 2.1 (resp. Section 3.2), we prove an exact analogue of Theorem 1.3.1 *(i)* for an insurer, who additionally has the possibility to invest in a risky asset modelled by geometric Brownian motion (resp. an exponential Lévy process).

1.3.2 Heavy-tailed Claim Distributions

Definition 1.3.2. *For a distribution function $F : \mathbb{R}_+ \to [0,1]$, we define the* integrated tail distribution *as*

$$F_I(x) := \frac{1}{\mu} \int_x^\infty \overline{F}(y)\, dy, \tag{1.3.9}$$

where $\mu = \int_0^\infty y\, dF(y)$, and $\overline{F}$ is the tail distribution. We denote the tail of the integrated tail distribution *by $\overline{F}_I(x) = 1 - F_I(x)$.*

The most general result about the asymptotic behavior of the ruin probability for claims with heavy tails holds for claims which have subexponential integrated tails and goes as follows.

Theorem 1.3.3 (Cramér–Lundberg theorem for large claims). *Consider the model for the risk process as described in Section 1.1 with a positive safety loading $\rho > 0$. Then the following assertions are equivalent*

(i) $F_I \in \mathcal{S}$,

(ii) $1 - \Psi(x) \in \mathcal{S}$,

(iii) $\lim_{x\to\infty} \Psi(x)/\overline{F}_I(x) = 1/\rho$.

The proof of $(i) \Rightarrow (iii)$ relies on the following relation, which holds in the Cramér–Lundberg model under the assumption of a positive safety loading

$$\Psi(x) = \frac{\rho}{1+\rho} \sum_{n=0}^{\infty} (1+\rho)^{-n} \overline{F_I^{n*}}(x). \tag{1.3.10}$$

(For a proof of equation (1.3.10) we refer to Feller [9], Volume 2, Chapter XI, *Renewal Theory.*) Since one can interchange limits and sums (see Theorem 1.3.6. in Embrechts, Klüppelberg and Mikosch [7]), it is clear from formula (1.3.10) that for claims with subexponential integrated tails we obtain

$$\begin{aligned} \lim_{x\to\infty} \frac{\Psi(x)}{\overline{F}_I(x)} &= \lim_{x\to\infty} \frac{\rho}{1+\rho} \sum_{n=0}^{\infty} (1+\rho)^{-n} \frac{\overline{F_I^{n*}}(x)}{\overline{F}_I(x)} \\ &= \frac{\rho}{1+\rho} \sum_{n=0}^{\infty} (1+\rho)^{-n} n \\ &= \frac{1}{\rho}. \end{aligned} \tag{1.3.11}$$

In the following, we will state several sufficient conditions for the distribution function F such that the integrated tail distribution function $F_I \in \mathcal{S}$; unfortunately, $F \in \mathcal{S}$ does not imply $F_I \in \mathcal{S}$ and vice versa.

Theorem 1.3.4. *(i) If $F \in \mathcal{D}$, then $F_I \in \mathcal{S}$.*

(ii) If $F \in \mathcal{R}$, then $F_I \in \mathcal{S}$.

The proof of (i) is an immediate consequence of Proposition 1.4.4 in Embrechts, Mikosch and Klüppelberg [7], whereas for (ii), it follows immediately from Karamata's Theorem (see Section 4.1) that $F \in \mathcal{R}$ implies $F_I \in \mathcal{R}$, and hence $F_I \in \mathcal{S}$ by (1.2.9).

Combining Theorems 1.3.3 and 1.3.4 yields the following result for claims with distributions with regularly varying tails.

Theorem 1.3.5. *Consider the Cramér–Lundberg model, described in Section 1.1, and assume a positive safety loading $\rho > 0$. For claim size distributions with regularly varying tail, the probability of ruin, $\Psi(x)$, for large initial capital x is essentially determined by the tail of the integrated tail distribution $\overline{F}_I$ for large values of x, in the sense that*

$$\Psi(x) \sim \frac{1}{\rho}\overline{F}_I(x), \quad x \to \infty. \tag{1.3.12}$$

(For the definition of the symbol '$\sim$', see Definition 4.1.2.)

We will prove an analogue of this theorem in Chapter 4, where we additionally allow the insurance company to invest in a risky asset, modelled by geometric Brownian motion.

1.4 Introducing an Investment Opportunity: Geometric Brownian Motion

The classical model for the risk process, described in Section 1.1, does not account for interest on the reserve: in modern terms this may be expressed by saying that the insurance company may only invest in a bond with zero interest rate. We will stick to the assumption of zero interest in the following, apart from the remark at the end of Section 2.1, where we treat a straightforward generalization of the main result of Section 2.1 to a setting with zero *real interest force.*

Now we deviate from the classical setting of Section 1.1 and assume, in addition, that the company may also invest in a stock or market index, described by geometric Brownian motion (GBM)

$$dS(t) = S(t)(a\,dt + b\,dW(t)), \tag{1.4.1}$$

where $a, b \in \mathbb{R}$ are fixed constants and W is a standard Brownian motion independent of the process R.

We will denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by the processes R and S and use $\mathbb{E}_t[\,.\,]$ as a shorthand notation for the conditional expectation $\mathbb{E}[\,.\,|\mathcal{F}_t]$.

If at time t the insurer has wealth $Y(t)$, and invests an amount $K(t)$ of money in the stock and the remaining reserve $Y(t) - K(t)$ in the bond (which in the present model yields no interest), her wealth process Y can be written as

$$\begin{aligned} Y(t, x, K) &= x + ct - \sum_{i=1}^{N(t)} X_i + (\frac{K}{S} \cdot S)(t) \\ &= R(t, x) + (K \cdot W_{a,b})(t), \end{aligned} \tag{1.4.2}$$

where $W_{a,b}(t)$ denotes the generalized Wiener process $W_{a,b}(t) = at + bW(t)$ with drift a and standard deviation b, and $(K \cdot W_{a,b})$ denotes the stochastic integral of the process K with respect to the process $W_{a,b}$ (see, e.g., PROTTER [27]).

We are interested in the infinite time ruin probability of the insurance company, defined by

$$\Psi(x, K) = \mathbb{P}[Y(t, x, K) < 0, \text{ for some } t \geq 0], \tag{1.4.3}$$

depending on the initial wealth x and the investment strategy K of the insurer. We further define the time of ruin (depending on K)

$$\tau(x, K) := \inf\{t \geq 0 : Y(t, x, K) < 0\}. \tag{1.4.4}$$

The set $\mathcal{K}$ of *admissible strategies* K is defined as

$$\begin{aligned} \mathcal{K} := \{&K = (K(t))_{t\geq 0} : \ K \text{ is predictable and adapted to } \mathbb{F} \\ &\text{and } \mathbb{P}[\int_0^t K(s)^2 ds < \infty] = 1 \text{ for all } t \in [0, \infty)\}. \end{aligned} \tag{1.4.5}$$

Note that $K \in \mathcal{K}$ is a necessary and sufficient condition for the stochastic integral $(K \cdot W_{a,b})$ with respect to (w.r.t.) the generalized Wiener process appearing in (1.4.2) to exist (see Karatzas and Shreve [21]).

Furthermore we define the *minimal ruin probability*

$$\Psi^*(x) = \inf_{K \in \mathcal{K}} \Psi(x, K), \quad x \geq 0. \tag{1.4.6}$$

If this infimum is attained for an admissible strategy $K^* \in \mathcal{K}$, we will call this strategy an *optimal strategy* with respect to the initial reserve x.

In the following chapters it will be our main goal to analyze the function Ψ^* and its asymptotic properties as the initial wealth x goes to infinity.

Remark. In Chapter 3, we will consider a generalization of the setting introduced in this section, namely when the investment possibility can be described by an exponential Lévy process (see Definition 3.1.5).

Chapter 2

Ruin Probabilities for Small Claims and Asymptotically Optimal Investment in a Geometric Brownian Motion

2.1 Exponential Upper Bound

The classical Cramér–Lundberg model without investment possibility (see Sections 1.1 and 1.3.1) is, of course, a special case of the model described in Section 1.4, namely by letting $a = b = 0$. Remember that, under the assumption $c > \lambda\mathbb{E}[X]$, in this setting the ruin probability – which then is independent of the investment strategy K – can be bounded from above by $e^{-\nu x}$, where ν is the positive solution of the equation

$$\lambda h(r) = cr, \tag{2.1.1}$$

see Theorem 1.3.1.

The main result of this section is an analogue of this result for the model with nontrivial investment possibility, obtained by combining the assumptions of Sections 1.1 and 1.4 (with $b \neq 0$), and is summarized in the following theorem. It will be a consequence of Theorem 2.1.3.

Theorem 2.1.1 (Main theorem - small claims, geometric Brownian motion (GBM)). *For the model described in Sections 1.1 and 1.4, assume that $b \neq 0$. Then the minimal ruin probability $\Psi^*(x)$ of an insurer, investing in a stock market, can be bounded from above by*

$$\Psi^*(x) \leq e^{-\hat{r}x}, \tag{2.1.2}$$

where $0 < \hat{r} < r_\infty$ *is the positive solution of the equation (compare Figure 2.1)*

$$\lambda h(r) = cr + \frac{a^2}{2b^2}. \tag{2.1.3}$$

If $\mathbb{E}[X] < c/\lambda$*, i.e., if the Lundberg coefficient* $\nu > 0$ *exists, and if* $a \neq 0$*, then* $\hat{r} > \nu$*, such that one obtains a sharper bound for* $\Psi^*(x)$*. Dropping the assumption* $\mathbb{E}[X] < c/\lambda$*, for* $a \neq 0$*, we still obtain* $\hat{r} > 0$*, i.e. an exponential decay of the minimal ruin probability.*

For later use we introduce the following process, for fixed numbers $x, r \in \mathbb{R}_+$ and a fixed admissible strategy $K \in \mathcal{K}$,

$$M(t, x, K, r) := e^{-rY(t,x,K)}. \tag{2.1.4}$$

This process is already familiar from Gerber's approach to risk theory via martingale inequalities (Gerber [13]).

Lemma 2.1.2 (The process $M(t, x, \hat{K}, \hat{r})$ is a martingale.). *Let* $x > 0$*, and* $a \neq 0, b \neq 0$*. There exists a unique* $0 < \hat{r} < r_\infty$ *satisfying the equation*

$$\lambda h(\hat{r}) = \frac{a^2}{2b^2} + c\hat{r}. \tag{2.1.5}$$

For this $\hat{r}$ *and the constant process* $\hat{K}(t) \equiv a/\hat{r}b^2$*, the process* $M(t, x, \hat{K}, \hat{r})$ *is a martingale w.r.t. the filtration* $\mathbb{F}$*.*

Proof. The existence and uniqueness of $\hat{r}$ are easy consequences of the properties of h (cf. Figure 2.1 and page 15).

If we define $f : \mathbb{R} \times [0, r_\infty) \to \mathbb{R}$ by

$$f(K, r) := \lambda h(r) - (Ka + c)r + \frac{1}{2}K^2 b^2 r^2, \tag{2.1.6}$$

then it can be easily checked that $f(\hat{K}, \hat{r}) = 0$. Now, in order to show that the process $M(t, x, \hat{K}, \hat{r})$ is a martingale w.r.t. $\mathbb{F}$, we proceed as follows (see, e.g., the book by Asmussen [1]): for arbitrary $t \geq 0$,

$$\begin{aligned}
\mathbb{E}[M(t, 0, \hat{K}, \hat{r})] &= \mathbb{E}[e^{-\hat{r}(ct - \sum_{i=1}^{N(t)} X_i + \hat{K}W_{a,b}(t))}] \\
&= e^{-\hat{r}(c+\hat{K}a)t}\mathbb{E}[e^{\hat{r}\sum_{i=1}^{N(t)} X_i}]\mathbb{E}[e^{-\hat{r}\hat{K}bW(t)}] \\
&= e^{-\hat{r}(c+\hat{K}a)t} e^{h(\hat{r})\lambda t} e^{(\hat{r}^2\hat{K}^2b^2/2)t} \\
&= e^{f(\hat{K},\hat{r})t} \\
&= 1.
\end{aligned} \tag{2.1.7}$$

Since $Y(t,x,\hat{K})$ has stationary independent increments, we obtain, for $0 \le t \le T$,

$$\begin{aligned}
\mathbb{E}_t[M(T,x,\hat{K},\hat{r})] &= \mathbb{E}_t[e^{-\hat{r}Y(T,x,\hat{K})}] \\
&= e^{-\hat{r}Y(t,x,\hat{K})}\mathbb{E}_t[e^{-\hat{r}(Y(T,x,\hat{K})-Y(t,x,\hat{K}))}] \\
&= e^{-\hat{r}Y(t,x,\hat{K})}\mathbb{E}[e^{-\hat{r}(Y(T-t,x,\hat{K})-Y(0,x,\hat{K}))}] \\
&= e^{-\hat{r}Y(t,x,\hat{K})}\mathbb{E}[e^{-\hat{r}Y(T-t,0,\hat{K})}] \\
&= e^{-\hat{r}Y(t,x,\hat{K})} \\
&= M(t,x,\hat{K},\hat{r}), \qquad (2.1.8)
\end{aligned}$$

and therefore $M(t,x,\hat{K},\hat{r})$ is a martingale w.r.t. the filtration $\mathbb{F}$. □

Remark. The above argument also shows that for each $r \in [0,\hat{r})$, there exist two constant processes $K_{1,2}(r) \in \mathcal{K}$ such that the process $M(t,x,K_{1,2}(r),r)$ is a martingale. The values $K_{1,2}(r)$ are given in the following way

$$K_{1,2}(r) = \frac{a}{b^2 r} \pm \sqrt{\Delta(r)}, \qquad (2.1.9)$$

where

$$\Delta(r) := \frac{2}{b^2 r^2}\left(\frac{a^2}{2b^2} + cr - \lambda h(r)\right) \ge 0, \quad \text{for } r \le \hat{r}, \qquad (2.1.10)$$

and $K_{1,2}(r)$ satisfy $f(K_{1,2}(r),r) = 0$.

Note that for $r = \hat{r}$, we obtain $\Delta(\hat{r}) = 0$, and therefore $K_1(\hat{r}) = K_2(\hat{r}) = \hat{K}$.

From now on we shall always consider the processes M and Y, stopped at the time of ruin, so we define

$$\tilde{M}(t,x,K,r) := M(t \wedge \tau(x,K),x,K,r) \qquad (2.1.11)$$

and

$$\tilde{Y}(t,x,K) := Y(t \wedge \tau(x,K),x,K), \qquad (2.1.12)$$

where we use the standard notation $t \wedge \tau(x,K) := \min(t,\tau(x,K))$.

Theorem 2.1.3 (Exponential upper bound, for investment modelled by GBM). *Let $a \neq 0, b \neq 0$. For $\hat{r}$, $0 < \hat{r} < r_\infty$, defined by equation* (2.1.5)*, and for the constant investment strategy $\hat{K}(t) \equiv a/\hat{r}b^2$, the ruin probability can be bounded from above by (for all $x \in \mathbb{R}_+$)*

$$\Psi(x,\hat{K}) \le e^{-\hat{r}x}. \qquad (2.1.13)$$

Proof. From Lemma 2.1.2 we know that $M(t,x,\hat{K},\hat{r})$ is a martingale w.r.t. the filtration $\mathbb{F}$. Therefore, also the stopped process $\tilde{M}(t,x,\hat{K},\hat{r})$ is a martingale w.r.t. $\mathbb{F}$ (see Theorem

(II.77.5) in ROGERS AND WILLIAMS VOL.1 [29]; note that M is non-negative). Using this, we obtain similarly as in GERBER [13], for $t \geq 0$,

$$\begin{aligned} e^{-\hat{r}x} &= \tilde{M}(0,x,\hat{K},\hat{r}) \\ &= \mathbb{E}[\tilde{M}(t,x,\hat{K},\hat{r})] \\ &= \mathbb{E}[M(t\wedge\tau(x,\hat{K}),x,\hat{K},\hat{r})] \\ &= \mathbb{E}[M(\tau(x,\hat{K}),x,\hat{K},\hat{r})\chi_{\{\tau(x,\hat{K})<t\}}] \\ &\quad +\mathbb{E}[M(t,x,\hat{K},\hat{r})\chi_{\{t\leq\tau(x,\hat{K})\}}] \\ &\geq \mathbb{E}[M(\tau(x,\hat{K}),x,\hat{K},\hat{r})\chi_{\{\tau(x,\hat{K})<t\}}], \end{aligned} \tag{2.1.14}$$

where χ_A is the indicator function of the set A, and where we used the fact that the process M is nonnegative.

Monotone Convergence yields that

$$\lim_{t\to\infty}\mathbb{E}[M(\tau(x,\hat{K}),x,\hat{K},\hat{r})\chi_{\{\tau(x,\hat{K})<t\}}] = \mathbb{E}[M(\tau(x,\hat{K}),x,\hat{K},\hat{r})\chi_{\{\tau(x,\hat{K})<\infty\}}]. \tag{2.1.15}$$

Hence

$$e^{-\hat{r}x} \geq \mathbb{E}[M(\tau(x,\hat{K}),x,\hat{K},\hat{r})|\tau(x,\hat{K})<\infty]\,\mathbb{P}[\tau(x,\hat{K})<\infty]. \tag{2.1.16}$$

We finally arrive at

$$\begin{aligned} \Psi(x,\hat{K}) &= \mathbb{P}[\tau(x,\hat{K})<\infty] \\ &\leq \frac{e^{-\hat{r}x}}{\mathbb{E}[M(\tau(x,\hat{K}),x,\hat{K},\hat{r})|\tau(x,\hat{K})<\infty]}. \end{aligned} \tag{2.1.17}$$

Since the random variable $M(\tau(x,\hat{K}),x,\hat{K},\hat{r})$ is a.s. greater than or equal to 1 on the set $\{\tau(x,\hat{K})<\infty\}$, the result follows. □

Proof of 2.1.1. The Main Theorem now is an immediate consequence of Theorem 2.1.3, observing that $\hat{r} > \nu$ (assuming that $b \neq 0$ and $a \neq 0$). We discuss the latter fact in the following.

As we have mentioned before, the classical Lundberg exponent ν is the positive solution to

$$h(r) = \frac{c}{\lambda}r. \tag{2.1.18}$$

If now, in addition, the insurance company has the opportunity to invest in the market, the corresponding exponent $\hat{r}$ is the positive solution of

$$h(r) = \frac{c}{\lambda}r + \frac{a^2}{2\lambda b^2}. \tag{2.1.19}$$

The right hand side of (2.1.19) is just the right hand side of (2.1.18), but shifted by the positive constant $a^2/2\lambda b^2$. From the properties of h it is obvious that $\hat{r} > \nu$, if $a \neq 0$, and that $\hat{r} = \nu$, for $a = 0$ (see also Figure 2.1).

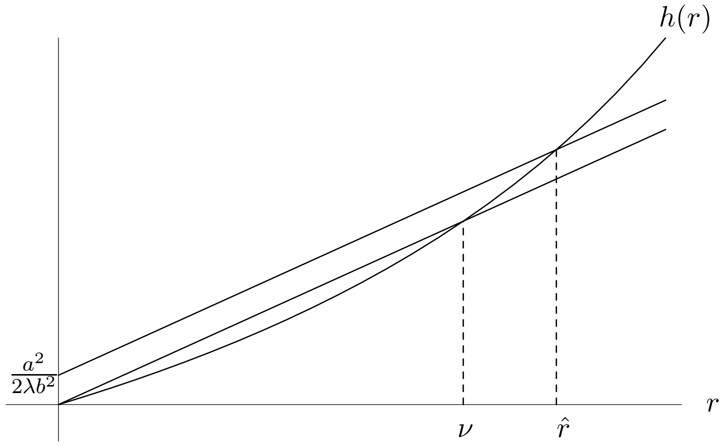


Figure 2.1: $h(r)$, $\frac{c}{\lambda}r$ and $\frac{c}{\lambda}r + \frac{a^2}{2\lambda b^2}$ for exponentially distributed claims and parameter values $\theta = 10$, $c = 15$, $\lambda = 1$, $a = 0.06$ and $b = 0.15$. In this case we obtain $\nu = 1/30 = 0.0\dot{3}$ and $\hat{r} = 0.041$.

What about the assumption $c > \lambda\mathbb{E}[X]$? In the classical setting without investment, this condition is equivalent to $h'(0) = \mathbb{E}[X] < c/\lambda$, and guarantees that h and the line with slope c/λ through 0 have a strictly positive intersection. In the present model with investment the picture changes (see Figure 2.2): It is easily seen that for $a \neq 0$, equation (2.1.19) always possesses a strictly positive solution $\hat{r}$. (See also the remark after the example.)

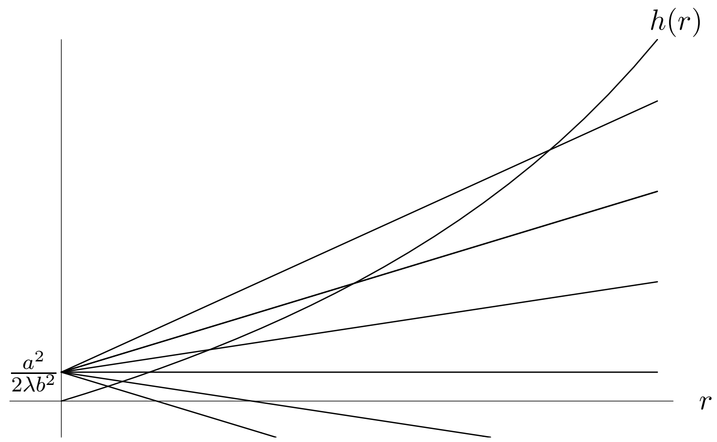


Figure 2.2: $h(r)$ and $\frac{c}{\lambda}r + \frac{a^2}{2\lambda b^2}$ for exponentially distributed claims, parameter values $\theta = 10$, $\lambda = 1$, $a = 0.06$, $b = 0.15$ and different values of c.

Thus we have completed the proof of the Main Theorem and now pass on to an illustrative example.

Example. Consider the situation for the classical Erlang model when claim sizes are exponentially distributed with parameter θ, i.e. $dF(x) = (e^{-x/\theta}/\theta)dx$. In this case $h(r) = \theta r/(1-\theta r)$, $r \in [0, 1/\theta)$. A plot of this function is shown in Figure 2.1 for $\theta = 10$. Equation (2.1.18) has two solutions, namely 0 and $\nu = \rho/(\rho+1)\theta$, where the relative safety loading ρ equals $c/\lambda\theta - 1$ (see page 12). Note that ν is only positive if $c > \lambda\theta$. An elementary calculation reveals that on the other hand the coefficient $\hat{r}$ equals

$$\nu + \left(\sqrt{(\frac{\nu + a^2/2b^2c}{2})^2 + \frac{a^2}{2b^2c}(\frac{1}{\theta} - \nu)} - \frac{\nu + a^2/2b^2c}{2}\right). \tag{2.1.20}$$

Remarks. 1. At first sight it seems very amazing that one obtains an exponential bound on the ruin probability Ψ for *arbitrary* values of the parameters c, λ and $\mathbb{E}[X]$. The premium rate c might even be negative!

This stunning fact can be explained as follows: remember that the process $\hat{K}$ is given by $\hat{K}(t) \equiv a/\hat{r}b^2$, $t \geq 0$. For 'unfavourable' parameters of the risk process, $\hat{r}$ is small and therefore $\hat{K}$ is large. This leads to an arbitrarily large drift of the wealth process from the investment. This way, the very large constant investment $\hat{K}$ leads eventually to an exponential decay of the ruin probability.

This result also gives some theoretical justification for the technique of '*cash flow underwriting*' which – at least from time to time – enjoys some popularity among re-insurers: according to this technique the re-insurer sometimes accepts contracts which will probably result in a technical loss, hoping that the financial gains obtained from a 'good' (i.e. a risky) investment of the premiums will outweigh this loss.

2. If we drop the assumption that the bond yields zero interest rate, it turns out that the case of zero *real interest force* i, when the interest force on the bond is equal to the inflation force (cf. Delbaen and Haezendonck [6]), can be treated with essentially the same methods as the ones described in the previous section. The stochastic differential equation for the wealth process $Y^{(i)}$ with interest $i > 0$ is given by

$$\begin{aligned} & dY^{(i)}(t) \\ = \; & (ce^{it} + (i(Y^{(i)}(t-) - K(t)) + aK(t)))dt + bK(t)dW(t) - e^{it}X_{N(t)}dN(t). \end{aligned} \tag{2.1.21}$$

If we introduce the *present value process* $\overline{Y}^{(i)}(t) := e^{-it}Y^{(i)}(t)$, we obtain

$$d\overline{Y}^{(i)}(t) = e^{-it}\left((ce^{it} + (a-i)K(t))dt + bK(t)dW(t) - e^{it}X_{N(t)}dN(t)\right). \tag{2.1.22}$$

Defining the process $\overline{M}^{(i)}(t) := e^{-r\overline{Y}^{(i)}(t)}$ for $r \in \mathbb{R}_+$, it follows the same way as with zero interest rate that $\overline{M}^{(i)}(t \wedge \tau, x, \hat{K}^{(i)}, \hat{r}^{(i)})$ is a martingale, where $\hat{r}^{(i)}$ is the solution to

$$\lambda h(r) = cr + \frac{(a-i)^2}{2b^2}, \tag{2.1.23}$$

and the process $\hat{K}^{(i)} \in \mathcal{K}$ is given by

$$\hat{K}^{(i)}(t) = \frac{a-i}{\hat{r}^{(i)}b^2}e^{it}. \tag{2.1.24}$$

Then by the same line of argument as in the case of zero interest it can be shown that the ruin probability $\Psi(x, \hat{K}^{(i)})$ for the strategy $\hat{K}^{(i)}$ can be bounded from above by

$$\Psi(x, \hat{K}^{(i)}) \leq e^{-\hat{r}^{(i)}x}. \tag{2.1.25}$$

2.2 Exponential Lower Bound for Uniform Exponential Tail Moment

In this section we want to work in the direction of an asymptotic optimality resp. asymptotic uniqueness result for the constant investment strategy $\hat{K}$ and the exponent $\hat{r}$, which will finally be given in Section 2.3. We will need the following assumption on the tail distribution of the claim size:

Definition 2.2.1. *Let* $0 < r < r_\infty$ *be given. We say that* X *has* a uniform exponential moment in the tail distribution for r, *if the following condition holds true*

$$\sup_{y \geq 0} \mathbb{E}[e^{-r(y-X)} | X > y] < \infty. \tag{2.2.1}$$

Remark. From now on we shall assume that the random variable X, that models the claim size, has a uniform exponential moment in the tail distribution for $\hat{r}$. Partly we do so for the ease of exposition, partly because we need the assumption: First to go from a local submartingale to a true submartingale in the proof of Theorem 2.2.2, and second in order to obtain a positive constant C in Theorem 2.2.4. In Section 2.4, we present several of the results, that are proved in this section, without the assumption of a uniform exponential moment in the tail distribution. In Section 2.7, we present an example for a claim size distribution with exponential moments for some $r_\infty > 0$, but without uniform exponential moment of the tail distribution for any $0 < r < r_\infty$.

Under Assumption (2.2.1) (for $\hat{r}$), we can prove the following theorem.

Theorem 2.2.2 ($M(t, x, K, \hat{r})$ is a UI submartingale for all $K \in \mathcal{K}$, GBM). *Assume that* X *has a uniform exponential moment in the tail distribution for* $\hat{r}$. *Then for each* $K \in \mathcal{K}$, *the process* $(\tilde{M}(t, x, K, \hat{r}))_{t \geq 0}$ *is a uniformly integrable submartingale.*

Proof. Application of Itô's Lemma to the process M yields, for arbitrary $K \in \mathcal{K}$ and $r \in \mathbb{R}_+$,

$$\begin{aligned} \frac{dM(t,x,K,r)}{M(t-,x,K,r)} &= \left(-(c + K(t)a)r + \frac{1}{2}r^2b^2K(t)^2\right)dt \\ &\quad - rbK(t)dW(t) + \left(e^{rX_{N(t)}} - 1\right)dN(t). \end{aligned} \tag{2.2.2}$$

This can be rewritten as

$$\begin{aligned}\frac{dM(t,x,K,r)}{M(t-,x,K,r)} &= \left(-(c+K(t)a)r+\frac{1}{2}r^2b^2K(t)^2+\lambda h(r)\right)dt\\ &\quad -rbK(t)dW(t)\\ &\quad +(e^{rX_{N(t)}}-1)dN(t)-\lambda\mathbb{E}[e^{rX_{N(t)}}-1]dt\\ &= f(K(t),r)dt-rbK(t)dW(t)\\ &\quad +(e^{rX_{N(t)}}-1)dN(t)-\lambda\mathbb{E}[e^{rX_{N(t)}}-1]dt. \end{aligned} \tag{2.2.3}$$

Therefore the stopped process $\tilde{M}(t,x,K,\hat{r})$ can be expressed in terms of stochastic integrals as

$$\begin{aligned} &\tilde{M}(t,x,K,\hat{r})-e^{-\hat{r}x}\\ = &\int_0^{t\wedge\tau} M(s-,x,K,\hat{r})f(K(s),\hat{r})ds- \ rb\int_0^{t\wedge\tau} M(s-,x,K,\hat{r})K(s)dW(s)\\ &+\int_0^{t\wedge\tau} M(s-,x,K,\hat{r})(e^{\hat{r}X_{N(s)}}-1)dN(s)\\ &-\mathbb{E}[e^{\hat{r}X}-1]\int_0^{t\wedge\tau} M(s-,x,K,\hat{r})\lambda ds. \end{aligned} \tag{2.2.4}$$

Since by assumption, the process $K\in\mathcal{K}$ is integrable with respect to the Brownian motion and since $0\le M(s-,x,\hat{K},\hat{r})\le 1$, for $0\le s\le\tau$, the stochastic integral w.r.t. the Brownian motion in (2.2.4) gives a local martingale (see Theorem IV.29 in Protter [27]). Furthermore, it is shown at the end of this section that the difference of the two processes

$$\int_0^{t\wedge\tau} \tilde{M}(s-,x,\hat{K},\hat{r})(e^{rX_{N(s)}}-1)dN(s) \tag{2.2.5}$$

and

$$\lambda\mathbb{E}[e^{\hat{r}X}-1]\int_0^{t\wedge\tau} \tilde{M}(s-,x,\hat{K},\hat{r})ds \tag{2.2.6}$$

is a martingale (Lemma 2.2.5).

Finally, with the help of the defining equation (2.1.5) for $\hat{r}$, it is easy to show that for all $K\in\mathbb{R}$,

$$\begin{aligned} f(K,\hat{r}) &= \frac{1}{2}\hat{r}^2b^2(K-\hat{K})^2.\\ &\ge 0. \end{aligned} \tag{2.2.7}$$

Hence, for all $0\le t\le T$,

$$\int_{t\wedge\tau}^{T\wedge\tau} \tilde{M}(s-,x,K,\hat{r})f(K(s),\hat{r})ds\ge 0. \tag{2.2.8}$$

Putting the pieces together, it is an easy consequence that $\tilde{M}(t, x, K, \hat{r})$ is a local submartingale.

To proceed from this to the conclusion that $\tilde{M}(t, x, K, \hat{r})$ indeed is a true submartingale, and even uniformly integrable, we use Assumption (2.2.1). Using the standard notation $\tilde{M}^* := \sup_{t\geq 0} |\tilde{M}(t)|$, it follows that

$$\begin{aligned} \mathbb{E}[\tilde{M}^*] &\leq \mathbb{E}[\tilde{M}(\tau, x, K, \hat{r}) | \tau < \infty] \\ &\leq \mathbb{E}[\tilde{M}(\tau, x, K, \hat{r}) | \tau < \infty, Y(\tau-) > 0], \end{aligned} \tag{2.2.9}$$

since $M(\tau, x, K, \hat{r})$ is a.s. equal to 1 on $\{\tau < \infty, Y(\tau-) = 0\}$, where ruin occurs a.s. through the Brownian motion, and $M(\tau, x, K, \hat{r}) \geq 1$ a.s. on $\{\tau < \infty, Y(\tau-) > 0\}$, where ruin occurs through a jump.

Now we proceed similarly as in Asmussen [1], p. 77. Let $H(dt, dy)$ denote the joint probability distribution of τ and $Y(\tau-)$ conditional on the event that ruin occurs, and that it occurs through a jump. Then, given $\tau = t$ and $Y(\tau-) = y > 0$, a claim has distribution function $dF(z)/\int_y^\infty dF(u)$ (for $z > y$). Therefore

$$\begin{aligned} \mathbb{E}[\tilde{M}^*] &\leq \mathbb{E}[\tilde{M}(\tau(x, K), x, K, \hat{r}) | \tau < \infty, Y(\tau-) > 0] \\ &= \int_0^\infty \int_0^\infty H(dt, dy) \int_y^\infty e^{-\hat{r}(y-z)} \frac{dF(z)}{\int_y^\infty dF(u)} \\ &\leq \left(\sup_{y\geq 0} \int_y^\infty e^{-\hat{r}(y-z)} \frac{dF(z)}{\int_y^\infty dF(u)} \right) \int_0^\infty \int_0^\infty H(dt, dy) \\ &= \sup_{y\geq 0} \int_y^\infty e^{-\hat{r}(y-z)} \frac{dF(z)}{\int_y^\infty dF(u)} \\ &< \infty \end{aligned} \tag{2.2.10}$$

by assumption (2.2.1).

A standard argument, using Dominated Convergence, together with (2.2.10) implies that $\tilde{M}$ indeed is a uniformly integrable submartingale (see, e.g., Protter [27], Theorem I.47). □

The following lemma will be useful in the sequel (see also the more general Proposition 2.4.2 in Section 2.4).

Lemma 2.2.3 (Ruin or infinite wealth, uniform exponential tail moment, GBM). *If X has a uniform exponential moment in the tail distribution for $\hat{r}$, then for arbitrary $K \in \mathcal{K}$ and $x \in \mathbb{R}_+$, the stopped wealth process $(\tilde{Y}(t, x, K))_{t\geq 0}$ converges almost surely on $\{\tau(x, K) = \infty\}$ to ∞ for $t \to \infty$. In other words, either ruin occurs, or the insurer becomes infinitely rich.*

Proof. From Theorem 3.3.1 we know that $\tilde{M}(t, x, K, \hat{r})$ is a uniformly integrable submartingale. Applying Doob's Supermartingale Convergence Theorem (Rogers and

Williams Vol. 1 [29], Theorem (II.69.1)) to $-\tilde{M}$, it follows that $\lim_{t\to\infty} \tilde{M}(t,x,K,\hat{r})$ exists a.s. Therefore, also the stopped wealth process $\tilde{Y}(t,x,K)$ converges a.s for $t \to \infty$.

There must exist $d > 0$ such that $\mathbb{P}[X > d] > 0$. If we define the events $E_n := \{X_n > d\}$, then $\mathbb{P}[E_n^c] < 1$, and the events $\{E_j\}_{j=1}^{\infty}$ are mutually independent. Therefore,

$$\mathbb{P}[\bigcup_{k=1}^{\infty} \bigcap_{n\geq k} E_n^c] = \lim_{k\to\infty} \mathbb{P}[\bigcap_{n\geq k} E_n^c] = \lim_{k\to\infty} \prod_{n\geq k} \mathbb{P}[E_n^c] = 0. \tag{2.2.11}$$

Hence, $\mathbb{P}[\bigcap_{k=1}^{\infty} \bigcup_{n\geq k} E_n] = 1$. In other words, with probability 1, a jump of size greater than d occurs infinitely often.

On the other hand, the stochastic integral $K \cdot W_{a,b}$ is a.s. continuous, and therefore the jumps of the compound Poisson process underlying the liabilities, greater than d, which will occur infinitely often a.s., cannot be compensated for by the a.s. continuous stochastic integral $K \cdot W_{a,b}$. As a result, the wealth process, stopped at time of ruin, cannot converge to a nonzero finite value with positive probability. □

With the help of the two preceding lemmas we obtain the following result.

Theorem 2.2.4 (Reverse inequality, uniform exponential tail moment, GBM). *Assume that X has a uniform exponential moment in the tail distribution for $\hat{r}$. Then the ruin probability satisfies, for every admissible process $K \in \mathcal{K}$,*

$$\Psi(x,K) \geq C\, e^{-\hat{r}x}, \tag{2.2.12}$$

where

$$\begin{aligned} C &= \inf_{y\geq 0} \frac{\int_y^{\infty} dF(u)}{\int_y^{\infty} e^{-\hat{r}(y-z)} dF(z)} \\ &= \frac{1}{\sup_{y\geq 0} \mathbb{E}[e^{-\hat{r}(y-X)} | X > y]} > 0. \end{aligned} \tag{2.2.13}$$

Proof. As $\tilde{M}(t,x,K,\hat{r})$ is a uniformly integrable submartingale, it follows from Doob's Optional Sampling Theorem (see Rogers and Williams Vol. 1 [29], Theorem (II.77.5)) that (using τ as a shorthand notation for $\tau(x,K)$)

$$\begin{aligned} \tilde{M}(0,x,K,\hat{r}) &= e^{-\hat{r}x} \\ &\leq \mathbb{E}[\tilde{M}(\tau,x,K,\hat{r})]. \end{aligned} \tag{2.2.14}$$

Now we proceed similarly as in the proof of Theorem 2.1.3, but use Lemma 2.2.3.

$$\begin{aligned} &\mathbb{E}[\tilde{M}(\tau,x,K,\hat{r})] \\ =\ &\mathbb{E}[\tilde{M}(\tau,x,k,\hat{r})|\tau<\infty]\mathbb{P}[\tau<\infty] \\ &+ \mathbb{E}[\lim_{t\to\infty} \tilde{M}(t,x,K,\hat{r})|\tau=\infty]\mathbb{P}[\tau=\infty] \\ =\ &\mathbb{E}[\tilde{M}(\tau,x,K,\hat{r})|\tau<\infty]\mathbb{P}[\tau<\infty]. \end{aligned} \tag{2.2.15}$$

Plugging this into equation (2.2.14), and using (2.2.9) and (2.2.10), we obtain

$$\Psi(x,K) \geq e^{-\hat{r}x}\frac{1}{\mathbb{E}[\tilde{M}(\tau,x,K,\hat{r})|\tau<\infty]} \geq Ce^{-\hat{r}x}. \tag{2.2.16}$$

This completes the proof. □

Remark. In the classical Erlang model, i.e. for claims with an exponential distribution (with parameter θ), one obtains the value $C = 1/(h(\hat{r})+1) = 1-\theta\hat{r}$.

Now, we present the proof for the following statement, which we used in the proof of Theorem 2.2.2, in the beginning of this section, in order to show that the process $\tilde{M}(t,x,K,\hat{r})$ is a local submartingale for all admissible trading strategies $K \in \mathcal{K}$.

Lemma 2.2.5. *Let $0 \leq r < r_\infty$ and $K \in \mathcal{K}$. The difference of the processes*

$$\lambda\mathbb{E}[e^{rX}-1]\int_0^{t\wedge\tau} M(s-,x,K,r)ds \tag{2.2.17}$$

and

$$\int_0^{t\wedge\tau} M(s-,x,K,r)(e^{rX_{N(s)}}-1)dN(s) \tag{2.2.18}$$

is a martingale w.r.t. the filtration $\mathbb{F}$.

Proof. Note that $N = (N(t))_{t\geq 0}$ is a finite variation process. Therefore the stochastic integral w.r.t. N in (2.2.18) makes sense (a.s.) as a pathwise Lebesgue-Stieltjes integral (see, e.g., Protter [27]). Let $\{T_n\}_{n=1}^\infty$ denote the arrival times of N. Then

$$\begin{aligned}&\int_0^{t\wedge\tau} M(s-,x,K,r)(e^{rX_{N(s)}}-1)dN(s)\\ =\ &\sum_{n=1}^{\infty} M(T_n-,x,K,r)(e^{rX_n}-1)\chi_{\{t\wedge\tau\geq T_n\}}.\end{aligned} \tag{2.2.19}$$

Taking expectations we obtain for $0 \leq t \leq T$

$$\begin{aligned}
& \mathbb{E}_{t\wedge\tau}[\int_{t\wedge\tau}^{T\wedge\tau} M(s-,x,K,r)(e^{rX_{N(s)}}-1)dN(s)] \\
= \; & \mathbb{E}_{t\wedge\tau}[\sum_{n=1}^{\infty} M(T_n-,x,K,r)(e^{rX_n}-1)\chi_{\{T\wedge\tau\geq T_n>t\wedge\tau\}}] \\
= \; & \mathbb{E}_{t\wedge\tau}[\sum_{n=1}^{\infty} \mathbb{E}_{T_n-}[M(T_n-,x,K,r)(e^{rX_n}-1)\chi_{\{T\wedge\tau\geq T_n>t\wedge\tau\}}]] \\
= \; & \mathbb{E}_{t\wedge\tau}[\sum_{n=1}^{\infty} \mathbb{E}_{T_n-}[e^{\hat{r}X_n}-1]M(T_n-,x,K,r)\chi_{\{T\wedge\tau\geq T_n>t\wedge\tau\}}] \\
= \; & \mathbb{E}_{t\wedge\tau}[\sum_{n=1}^{\infty} \mathbb{E}[e^{\hat{r}X}-1]M(T_n-,x,K,r)\chi_{\{T\wedge\tau\geq T_n>t\wedge\tau\}}] \\
= \; & \mathbb{E}[e^{rX}-1]\mathbb{E}_{t\wedge\tau}[\int_{t\wedge\tau}^{T\wedge\tau} M(s-,x,K,r)dN(s)] \\
= \; & \mathbb{E}[e^{rX}-1]\mathbb{E}_{t\wedge\tau}[\int_{t\wedge\tau}^{T\wedge\tau} M(s-,x,K,r)\lambda\, ds], \qquad (2.2.20)
\end{aligned}$$

where from the fourth to the fifth line we have used that X_n and $\mathcal{F}_{T_n-}$ are independent and from the sixth to the seventh line we have used that $N(t) - \lambda t$ is a martingale (see, e.g., PROTTER [27], p. 39). Thus the difference of (2.2.17) and (2.2.18) is a martingale w.r.t. the stopped filtration $(\mathcal{F}_{t\wedge\tau})_{t\geq 0}$. Then a standard argument (PROTTER [27], p. 11) shows that the difference between (2.2.17) and (2.2.18) also is a martingale w.r.t. the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. □

We now pass over to the asymptotic uniqueness of the constant investment strategy $\hat{K}$.

2.3 Asymptotic Optimality for Uniform Exponential Tail Moment

HIPP AND PLUM showed in [17] that, for the case of locally bounded density of the jump size, the problem of minimizing the ruin probability over all admissible trading strategies possesses a solution that is Markovian. That is to say that at time t, the trading strategy depends on $\mathcal{F}_t$ only through the current level of wealth $Y(t-,x,K)$. Therefore from now on we shall restrict our attention to such strategies. We will write $k : \mathbb{R}_+ \to \mathbb{R}$ for the function that describes the dependency on wealth of a certain strategy $K \in \mathcal{K}$. Then the corresponding investment at time t equals $K(t) = k(Y(t-,x,K))$. We will show that, if the optimal strategy – as a function of wealth – converges to a constant as wealth tends to infinity, then the limiting constant must be $\hat{K} = a/b^2\hat{r}$ (Corollary 2.3.2). We will even show the stronger result that a Markovian strategy, which is asymptotically bounded away from this constant strategy, leads to an exponentially worse (i.e. larger) ruin probability than the one obtained by using the constant strategy $\hat{K}$.

Theorem 2.3.1 (Asymptotic optimality of $\hat{K}$, uniform exponential tail moment, GBM). *Let X have a uniform exponential moment in the tail distribution for $\hat{r}$. Suppose further that $K \in \mathcal{K}$ is a Markovian strategy and let $k : \mathbb{R}_+ \to \mathbb{R}$ be its defining function. If there exist $\alpha > 0$ and $x_\alpha \geq 0$ such that*

$$|k(x) - \hat{K}| \geq \alpha, \quad \text{for } x \geq x_\alpha, \tag{2.3.1}$$

then there are $r_\alpha < \hat{r}$ and $A_\alpha > 0$ such that

$$\Psi(x, K) \geq A_\alpha e^{-r_\alpha x}. \tag{2.3.2}$$

Proof. We split up the proof into several steps.
Step 1. For α and x_α as in the Theorem, we define the stopping time

$$\tau_\alpha := \inf\{t \geq 0 : Y(t, x, K) \leq x_\alpha\}, \tag{2.3.3}$$

which is only nontrivial for $x > x_\alpha$.
Step 2. We show that, for $x > x_\alpha$, there exists $r_\alpha < \hat{r}$ such that $\tilde{M}(t \wedge \tau_\alpha, x, K, r_\alpha)$ is a uniformly integrable submartingale: we know that $f(\hat{K}, \hat{r}) = 0$, that $f(k, \hat{r}) = \hat{r}^2 b^2 (k - \hat{K})^2/2 > 0$ for $k \neq \hat{K}$, and that $\lim_{k \to \infty} f(k, r) = \infty$, for all $r \in (0, \hat{r})$. Using these facts and the continuity of f it is straightforward to show that, for α as before, there exists some $0 < r_\alpha < \hat{r}$ such that we have $f(k, r_\alpha) \geq 0$, for $|k - \hat{K}| > \alpha$. Now one proceeds in the same way as in Section 2.2 to prove that $\tilde{M}(t \wedge \tau_\alpha, x, K, r_\alpha)$ is a uniformly integrable submartingale, using that $\tau_\alpha \leq \tau$ a.s., for $x > x_\alpha$, and Lemma 2.2.5. Another consequence of $\tau_\alpha \leq \tau$ a.s. is that $\tilde{M}(t \wedge \tau_\alpha, x, K, r_\alpha) = M(t \wedge \tau_\alpha, x, K, r_\alpha)$ a.s.

Step 3. Using that the process $M(t \wedge \tau_\alpha, x, K, r_\alpha)$ is a uniformly integrable submartingale and Lemma 2.2.3, we obtain

$$\begin{aligned} e^{-r_\alpha x} &\leq \mathbb{E}[M(\tau_\alpha, x, K, r_\alpha)] \\ &= \mathbb{E}[\lim_{t \to \infty} M(t, x, K, r_\alpha) | \tau_\alpha = \infty] \mathbb{P}[\tau_\alpha = \infty] \\ &\quad + \mathbb{E}[M(\tau_\alpha, x, K, r_\alpha) | \tau_\alpha < \infty] \mathbb{P}[\tau_\alpha < \infty] \\ &\leq 0 \cdot \mathbb{P}[\tau_\alpha = \infty] + \frac{1}{C_\alpha} e^{-r_\alpha x_\alpha} \mathbb{P}[\tau_\alpha < \infty], \end{aligned} \tag{2.3.4}$$

where the constant C_α is defined by

$$\frac{1}{C_\alpha} := \sup_{y \geq 0} \frac{\int_y^\infty e^{-r_\alpha (y-z)} dF(z)}{\int_y^\infty dF(u)} = \sup_{y \geq 0} \mathbb{E}[e^{-r_\alpha (y-X)} | X > y]. \tag{2.3.5}$$

Hence

$$\mathbb{P}[\tau_\alpha < \infty] \geq C_\alpha e^{-r_\alpha (x - x_\alpha)}. \tag{2.3.6}$$

Since $r_\alpha < \hat{r}$, the constant C_α satisfies $C_\alpha > C$, and therefore $C_\alpha > 0$ by assumption.

Step 4. The ruin probability can be estimated by

$$\begin{aligned}\mathbb{P}[\tau(x,K) < \infty] &= \mathbb{P}[\tau(x,K) < \infty | \tau_\alpha < \infty]\mathbb{P}[\tau_\alpha < \infty] \\ &\geq \mathbb{P}[\tau(x_\alpha,K) < \infty]\mathbb{P}[\tau_\alpha < \infty] \\ &\geq \Psi^*(x_\alpha) C_\alpha e^{-r_\alpha(x-x_\alpha)}, \end{aligned} \tag{2.3.7}$$

where for the second inequality we have used that our setting is Markovian. Note that we only obtain the inequality $\mathbb{P}[\tau(x_\alpha, K) < \infty] \leq \mathbb{P}[\tau(x,K) < \infty|\tau_\alpha < \infty]$, since one can also fall below x_α after a jump and therefore arrive at a level strictly smaller than x_α.

Step 5. We use that $\Psi^*(x_\alpha) \geq Ce^{-\hat{r}x_\alpha}$ (Theorem 2.2.4) to show that $\Psi^*(x_\alpha) > 0$, and to finally obtain

$$\Psi(x,K) \geq D_\alpha e^{-r_\alpha x}, \tag{2.3.8}$$

for a constant $D_\alpha > 0$ and for all $x > x_\alpha$.

Step 6. It is obvious that for $x \leq x_\alpha$ we can bound $\Psi(x,K)$ from below by some constant $B_\alpha > 0$.

Step 7. Finally taking A_α as the minimum of B_α and D_α, we obtain the desired result. □

Corollary 2.3.2 (Asymptotic limit of $\hat{K}$ for infinite wealth, uniform exponential tail moment, GBM). *Assume that X has a uniform exponential moment in the tail distribution for $\hat{r}$. Let $k^* : \mathbb{R}_+ \to \mathbb{R}$ be the defining function of the optimal investment strategy K^*. If this function possesses a limit for $x \to \infty$, then this limit is given by*

$$\lim_{x\to\infty} k^*(x) = \hat{K}. \tag{2.3.9}$$

Proof. Assume that $\lim_{x\to\infty} k^*(x) \neq \hat{K}$. Then there exist $\alpha, x_\alpha > 0$ such that

$$|k^*(x) - \hat{K}| > \alpha \text{ for } x \geq x_\alpha. \tag{2.3.10}$$

Therefore, using Theorem 2.3.1 one obtains that

$$\Psi^*(x) \geq A_\alpha e^{-r_\alpha x}, \tag{2.3.11}$$

for some $r_\alpha < \hat{r}$, which together with the Main Theorem yields the apparent contradiction to the optimality of K^*

$$\lim_{x\to\infty} \frac{\Psi^*(x)}{e^{-\hat{r}x}} = \infty. \tag{2.3.12}$$

□

2.4 Results without Uniform Exponential Moment in the Tail Distribution

In this section we shall examine, to which extent the results of Sections 2.2 and 2.3 can be generalized, when the assumption of a uniform exponential moment in the tail distribution (see (2.2.1)) is dropped. In particular, we will show that the statement of Lemma 2.2.3 also holds true without this assumption, i.e. for every admissible trading strategy $K \in \mathcal{K}$, the insurer a.s. either gets infinitely rich or ruined (see Proposition 2.4.2).

Proposition 2.4.1.

(i) Let $x > 0$, and let $\hat{r}$ be defined as in (2.1.5). For $z \in \mathbb{R}_+$, we define the stopping time

$$\tau_z := \inf\{t \le \tau(x,K) : \tilde{Y}(t,x,K) \ge z\},$$

which is only nontrivial, if $x < z$. For every $z \in \mathbb{R}_+$ and every admissible trading strategy $K \in \mathcal{K}$, the stopped process $\tilde{M}^{\tau_z}(t,x,K,\hat{r}) = \tilde{M}(t \wedge \tau_z, x, K, \hat{r})$ is a uniformly integrable submartingale.

Furthermore $\mathbb{P}[\{\tau_z \wedge \tau(x,K) < \infty\}] = 1$, for all $z \in \mathbb{R}_+$, i.e. with probability 1, either the insurer gets ruined or she reaches the level z.

(ii) For all $K \in \mathcal{K}$, the process $\tilde{M}(t,x,K,\hat{r})$ satisfies the submartingale inequality (for $0 \le s \le t$)

$$\tilde{M}(s,x,K,\hat{r}) \le \mathbb{E}_s[\tilde{M}(t,x,K,\hat{r})],$$

however, we also allow for the possibility that the above expressions may equal ∞.

Proof.
(i) We have already shown in the proof of Theorem 3.3.1 that the process $\tilde{M}(t,x,K,\hat{r})$ is a local submartingale, for all $K \in \mathcal{K}$. Therefore, the stopped process $\tilde{M}^{\tau_z}(t,x,K,\hat{r})$ is also a local submartingale, for all $K \in \mathcal{K}$. Observe that we have a uniform estimate for the exponential tail moments, for the stopped process $\tilde{M}^{\tau_z}(t,x,K,\hat{r})$, namely

$$\sup_{0 \le y \le z} \mathbb{E}[e^{-r(y-X)} | X > y] < \infty, \quad r \in [0, r_\infty). \tag{2.4.1}$$

Hence (cf. (2.2.10))

$$\mathbb{E}[\sup_{0 \le t < \infty} |\tilde{M}^{\tau_z}(t,x,K,\hat{r})|] < \infty, \tag{2.4.2}$$

and therefore $\tilde{M}^{\tau_z}(t,x,K,\hat{r})$ is a uniformly integrable submartingale (Protter [27], p. 35). In exactly the same way as in the proof of Lemma 2.2.3, we apply Doob's Supermartingale Convergence Theorem to show that $\lim_{t\to\infty} M^{\tau_z}$ exists a.s. Then, we deduce that, for $t \to \infty$, the insurer a.s. either gets ruined or reaches the level z from the fact that, with probability 1, infinitely many jumps of size greater than d occur, which cannot be compensated for by the a.s. continuous stochastic integral w.r.t. the Brownian motion or the a.s. continuous drift term.

(ii) We know from (i) that, for $n \in \mathbb{N}$, $\tau_n := \inf\{t \geq 0 : \tilde{Y}(t, x, K) \geq n\}$, and $0 \leq s \leq t$,

$$\tilde{M}(s \wedge \tau_n, x, K, \hat{r}) \leq \mathbb{E}_s[\tilde{M}(t \wedge \tau_n, x, K, \hat{r})]. \tag{2.4.3}$$

The l.h.s. of (2.4.3) converges a.s. to $\tilde{M}(s, x, K, \hat{r})$. The r.h.s. of (2.4.3) can be rewritten as

$$\begin{aligned} & \mathbb{E}_s[\tilde{M}(t \wedge \tau_n, x, K, \hat{r})] \\ = \; & \mathbb{E}_s[\tilde{M}(t \wedge \tau_n, x, K, \hat{r})\chi_{\{t\wedge\tau_n<\tau(x,K)\}}] \\ & + \mathbb{E}_s[\tilde{M}(\tau(x,K), x, K, \hat{r})\chi_{\{\tau(x,K)\leq t\wedge\tau_n\}}]. \end{aligned} \tag{2.4.4}$$

Letting $n \to \infty$, we can apply the conditional version of the reverse Fatou Lemma to the first term in (2.4.4) and (conditional) Monotone Convergence to the second term to obtain

$$\lim_{n\to\infty} \mathbb{E}_s[\tilde{M}(t \wedge \tau_n, x, K, \hat{r})\chi_{\{t\wedge\tau_n<\tau(x,K)\}}] \leq \mathbb{E}_s[\tilde{M}(t, x, K, \hat{r})\chi_{\{t<\tau(x,K)\}}], \tag{2.4.5}$$

and

$$\lim_{n\to\infty} \mathbb{E}_s[\tilde{M}(\tau(x,K), x, K, \hat{r})\chi_{\{\tau(x,K)\leq t\wedge\tau_n\}}] = \mathbb{E}_s[\tilde{M}(\tau(x,K), x, K, \hat{r})\chi_{\{\tau(x,K)\leq t\}}]. \tag{2.4.6}$$

To sum it up, we obtain

$$\tilde{M}(s, x, K, \hat{r}) \leq \mathbb{E}_s[\tilde{M}(t, x, K, \hat{r})], \quad \text{a.s.} \tag{2.4.7}$$

□

Proposition 2.4.2 (Ruin or infinite wealth, general version, GBM). *Let $x > 0$ and $K \in \mathcal{K}$ be given. On the set $\{\tau(x,K) = \infty\}$, the process $Y(t, x, K)$ converges a.s. to ∞ for $t \to \infty$: either the insurer gets ruined or she becomes infinitely rich.*

Proof. Assume to the contrary that $\lim_{t\to\infty} \tilde{Y}(t, x, K)$ is not a.s. equal to ∞ on the set $\{\tau(x,K) = \infty\}$ for some process $K \in \mathcal{K}$ and some initial reserve $x \in \mathbb{R}_+$. Let us work towards a contradiction.

We know from Proposition 2.4.1 (i) that, for all admissible trading strategies $K \in \mathcal{K}$ and all $n \in \mathbb{N}$,

$$\lim_{t\to\infty} \tilde{Y}^{\tau_n}(t, x, K) = n, \qquad \text{a.s. on } \{\tau(x,K) = \infty\}, \tag{2.4.8}$$

where $\tilde{Y}^{\tau_n}$ denotes the process $\tilde{Y}$, stopped at time $\tau_n := \inf\{t \geq 0 : \tilde{Y}(t,x,K) \geq n\}$. Therefore for x and K as above, there have to exist numbers $d > 0$, $\delta > 0$, and a subsequence $(n_k)_{k=1}^{\infty}$ of the natural numbers such that

$$\mathbb{P}[\bigcap_{k=1}^{\infty}\{\exists t : \tau_{n_k} \leq t < \tau_{n_{k+1}}, Y(t,x,K) \leq d\} \cap \{\tau(x,K) = \infty\}] > \delta. \quad (2.4.9)$$

This means that on the set $\{\tau(x,K) = \infty\}$, where ruin a.s. never occurs, the insurer has to reach each level $n \in \mathbb{N}$ – a consequence of Proposition 2.4.1 (i) – but on the other hand she has to fall below the level d in each of the stochastic intervals $[\![\tau_{n_k}, \tau_{n_{k+1}}[\![$ with positive probability.

The idea of the subsequent argument is the following: if the insurer falls below the level d too often, she will get ruined with too high probability. In order to make this argument rigorous, we define the following stopping times

$$\sigma_k := \inf\{t : \tau_{n_k} \leq t < \tau_{n_{k+1}}, \tilde{Y}(t,x,K) \leq d\} \wedge \tau(x,K) \wedge \tau_{n_{k+1}}, \quad k \in \mathbb{N}. \quad (2.4.10)$$

Note that, for all $k \in \mathbb{N}$, the stopping times σ_k are finite a.s.

Next, we define another sequence of stopping times

$$\rho_k := \inf\{t : t > \sigma_k, \tilde{Y}(t,x,K) \geq 2d\}, \quad k \in \mathbb{N}. \quad (2.4.11)$$

As a consequence of Proposition 2.4.1 (i), for all $k \in \mathbb{N}$, the stopping times $\rho_k \wedge \tau(x,K)$ are finite a.s. Furthermore, there exists $k_1 \in \mathbb{N}$ such that, for $k \geq k_1$, $\rho_k \wedge \tau(x,K) \leq \tau_{n_{k+1}}$ a.s.

We know from Proposition 2.4.1 (i) that, for each $k \in \mathbb{N}$, the stopped process $\tilde{M}^{\tau_{n_{k+1}}}(t,x,K,\hat{r})$ is a uniformly integrable submartingale, so we can apply Doob's Optional Sampling Theorem (Rogers and Williams Vol. 1 [29], Theorem (II.77.5)) to the process $\tilde{M}^{\tau_{n_{k+1}}}(t,x,K,\hat{r})$ and the two stopping times σ_k and $\rho_k \wedge \tau(x,K)$, $\sigma_k \leq \rho_k \wedge \tau(x,K) \leq \tau_{n_{k+1}}$, to obtain

$$M(\sigma_k) \leq \mathbb{E}_{\sigma_k}[M(\rho_k \wedge \tau(x,K), x, K, r)], \quad k \geq k_1. \quad (2.4.12)$$

Now, we define the following events

$$A_j := \{\sigma_j < \tau; \sigma_j < \tau_{n_{j+1}}\}, \quad j \in \mathbb{N}, \quad (2.4.13)$$

and

$$A^k := \bigcap_{j=1}^{k} A_j, \quad k \in \mathbb{N}. \quad (2.4.14)$$

For all $k \in \mathbb{N}$, the event A^k lies in $\mathcal{F}_{\sigma_k}$.

We multiply inequality (2.4.12), for each $k \geq k_1$, with the indicator function χ_{A^k} and take expectations to obtain

$$\mathbb{E}[M(\sigma_k, x, K, r)\chi_{A^k}] \leq \mathbb{E}[\mathbb{E}_{\sigma_k}[M(\rho_k \wedge \tau(x,K), x, K, r)]\chi_{A^k}]. \qquad (2.4.15)$$

The left hand side of (2.4.15) can be bounded by

$$e^{-rd}\mathbb{P}[A^k] \leq \mathbb{E}[M(\sigma_k, x, K, r)\chi_{A^k}], \quad k \geq k_1, \qquad (2.4.16)$$

since $M(\sigma_k, x, K, r) \geq e^{-rd}$ on the set A^k.

Our aim is to show that the probability, conditional on the event A^k, to get ruined before reaching $2d$ is strictly greater than zero, independent of k. In order to get this estimate we proceed as follows with the right hand side of (2.4.15). By definition of the conditional expectation

$$\mathbb{E}[\mathbb{E}_{\sigma_k}[M(\rho_k \wedge \tau(x,K), x, K, r)]\chi_{A^k}] = \mathbb{E}[M(\rho_k \wedge \tau(x,K), x, K, r)\chi_{A^k}]. \qquad (2.4.17)$$

Now we argue in a similar fashion as in the proof of Theorem 2.2.4

$$\begin{aligned} & \mathbb{E}[M(\rho_k \wedge \tau(x,K), x, K, r)\chi_{A^k}] \\ = \; & \mathbb{E}[M(\tau(x,K), x, K, r)\chi_{A^k}\chi_{\{\tau(x,K)<\rho_k\}}] \\ & +\mathbb{E}[M(\rho_k, x, K, r)\chi_{A^k}\chi_{\{\tau(x,K)\geq\rho_k\}}] \\ \leq \; & \mathbb{E}[M(\tau(x,K), x, K, r)\chi_{A^k}\chi_{\{\tau(x,K)<\rho_k\}}] + e^{-2rd}\mathbb{P}[A^k], \quad k \geq k_1, \qquad (2.4.18) \end{aligned}$$

using that the random variable $M(\rho_k, x, K, r)$ equals $\exp(-2rd)$, for $k \geq k_1$, on the set A^k. Then

$$\begin{aligned} & \mathbb{E}[M(\tau(x,K), x, K, r)\chi_{A^k}\chi_{\{\tau(x,K)<\rho_k\}}] \\ = \; & \mathbb{E}[M(\tau(x,K), x, K, r)|A^k \cap \{\tau(x,K) < \rho_k\}]\,\mathbb{P}[A^k \cap \{\tau(x,K) < \rho_k\}]. \qquad (2.4.19) \end{aligned}$$

Finally, we need the following inequality

$$\mathbb{E}[M(\tau(x,K), x, K, r)|A^k \cap \{\tau(x,K) < \rho_k\}] \leq \sup_{0\leq y\leq 2d} \mathbb{E}[e^{-r(y-X)}|y > X], \qquad (2.4.20)$$

which holds true, because the insurer's wealth is below the level $2d$ on the set $A^k \cap \{\tau(x,K) < \rho_k\}$. Putting (2.4.15), (2.4.16), (2.4.18) and (2.4.20) together, we obtain

$$\mathbb{P}[\tau(x,K) < \rho_k|A^k] \geq \frac{e^{-rd} - e^{-2rd}}{\sup_{0\leq y\leq 2d}\mathbb{E}[e^{-r(y-X)}|y > X]} \geq \beta, \; k \geq k_1, \qquad (2.4.21)$$

for some constant $\beta > 0$, which just depends on d and not on k.

Now, the proof of Proposition 2.4.2 is almost finished. In order to see that (2.4.9) cannot hold true for $\delta > 0$, just use

$$\begin{aligned} & \mathbb{P}[\bigcap_{k_1\leq k\leq n} \{\exists t : \tau_{n_k} \leq t < \tau_{n_{k+1}}, Y(t,x,K) \leq d\} \cap \{\tau(x,K) = \infty\}] \\ \leq \; & \mathbb{P}[\bigcap_{k_1\leq k\leq n} \{\exists t : \tau_{n_k} \leq t < \tau_{n_{k+1}}, Y(t,x,K) \leq d\}] \\ = \; & \mathbb{P}[A^n] \\ = \; & \mathbb{P}[A_n|A^{n-1}]\,\mathbb{P}[A^{n-1}]. \qquad (2.4.22) \end{aligned}$$

Since the event $\{\tau(x,K) < \rho_{n-1}\}$ excludes the event A_n, the following holds

$$\begin{aligned}\mathbb{P}[A_n|A^{n-1}]\,\mathbb{P}[A^{n-1}] &\leq (1-\mathbb{P}[\tau(x,K) < \rho_{n-1}|A^{n-1}])\,\mathbb{P}[A^{n-1}] \\ &\leq (1-\beta)\,\mathbb{P}[A^{n-1}]. \end{aligned} \tag{2.4.23}$$

The bottom line is that $\lim_{n\to\infty}\mathbb{P}[A^n] = 0$ and therefore

$$\begin{aligned} &\lim_{n\to\infty}\mathbb{P}[\bigcap_{k_1\leq k\leq n}\{\exists t : \tau_{n_k} \leq t < \tau_{n_{k+1}}, Y(t,x,K) \leq d\} \cap \{\tau(x,K) = \infty\}] \\ = &\ \mathbb{P}[\bigcap_{k_1\leq k}\{\exists t : \tau_{n_k} \leq t < \tau_{n_{k+1}}, Y(t,x,K) \leq d\} \cap \{\tau(x,K) = \infty\}] \\ = &\ 0, \end{aligned} \tag{2.4.24}$$

which is an apparent contradiction to (2.4.9). This completes the proof of Proposition 2.4.2. □

2.5 Diffusion Approximation: Comparison of Results

So far, we have been looking at the classical risk process

$$R(t,x) = x + ct - \sum_{i=1}^{N(t)} X_i, \tag{2.5.1}$$

which is a compound Poisson process with drift c. Since diffusions are much easier to deal with, much work has been done in the direction of diffusion approximations of the risk process.

Based on the following result, taken from HARRISON [16], it seems reasonable to look at a Brownian motion with drift as risk process.

Theorem 2.5.1 (Approximation of Brownian motion by compound Poisson processes, Harrison, 1977). *Let $(R_n)_{n=1}^{\infty}$ be a sequence of compound Poisson processes with drift rates c_n, jump rates λ_n and jump distributions dF_n, starting from $R_n(0) = 0$, and let $\overline{R}$ be a Brownian motion with drift μ and standard deviation σ*

$$\overline{R}(t) = \mu\, t + \sigma\, W(t), \quad \overline{R}(0) = 0, \tag{2.5.2}$$

where W is a standard Brownian motion.

Then the one-dimensional distributions of R_n converge to those of $\overline{R}$ if and only if

$$c_n - \lambda_n \int_{\mathbb{R}} x\, dF_n(x) = \mu, \quad \text{for all } n \in \mathbb{N}, \tag{2.5.3}$$

$$\lambda_n \int_{\mathbb{R}} x^2 dF_n(x) = \sigma^2, \quad \text{for all } n \in \mathbb{N}, \tag{2.5.4}$$

$$\lambda_n \int_{\{|x|>\epsilon\}} x^2 dF_n(x) \to \infty, \quad \text{as } n\to\infty \text{ for all } \epsilon > 0. \tag{2.5.5}$$

Proof. See HARRISON [16], Theorem 4.1.

□

BROWNE [3] studied the problem of minimizing the ruin probability of an insurer, whose risk process is described by a Brownian motion with drift

$$\overline{R}(t,x) = x + \mu t + \sigma W^1(t), \tag{2.5.6}$$

who may also invest in a geometric Brownian motion

$$dS(t) = S(t)(a\,dt + b\,dW(t)), \tag{2.5.7}$$

where W and W^1 are independent Brownian motions. The optimal investment strategy for this problem can be calculated explicitly and turns out to be a constant, independent of the current level of wealth

$$\overline{K}^* = \frac{a}{b^2}\frac{\sigma^2}{\mu}\frac{1}{1+\sqrt{1+(\frac{\sigma a}{\mu b})^2}}. \tag{2.5.8}$$

The minimal ruin problem as a function of the initial wealth turns out to be an exponential function

$$\overline{\Psi}^*(x) = e^{-\frac{\mu+\sqrt{\mu^2+\sigma^2 a^2/b^2}}{\sigma^2}x}. \tag{2.5.9}$$

Now let us compare $\overline{K}^*$ to our previously obtained, asymptotically optimal strategy $\hat{K}$

$$\hat{K} = \frac{a}{b^2\hat{r}}, \tag{2.5.10}$$

for exponentially distributed claims, $dF(x) = 1/\theta e^{-x/\theta}$.

Conditions (2.5.3) to (2.5.5) specialize to

$$\begin{aligned} c - \lambda\theta &= \mu, \\ 2\lambda\theta^2 &= \sigma^2, \\ \lambda\theta^3 &\sim 0. \end{aligned} \tag{2.5.11}$$

The exponent $\hat{r}$ has to satisfy the following equation

$$\lambda h(\hat{r}) = \lambda\frac{\theta\hat{r}}{1-\theta\hat{r}} = c\hat{r} + \frac{a^2}{2b^2}. \tag{2.5.12}$$

Expanding $h = \theta r/(1-\theta r)$ in a Taylor series, and using equations (2.5.11), we obtain

$$\mu\hat{r} - \frac{\sigma^2}{2}\hat{r}^2 + \frac{a^2}{2b^2} \sim 0. \tag{2.5.13}$$

The positive solution to this quadratic equation is

$$\hat{r} = \frac{\mu}{\sigma^2} + \sqrt{(\frac{\mu}{\sigma^2})^2 + \frac{a^2}{b^2\sigma^2}}. \tag{2.5.14}$$

Plugging this into $\hat{K} = a/(\hat{r}b^2)$, we arrive at $\overline{K}^*$, the optimal solution for the diffusion problem obtained by BROWNE in [3], defined by equation (2.5.8).

2.6 Generalization: More than one Risky Asset

Let us assume exactly the same setting for the risk process R as the one described in Section 1.1. Assume, additionally, that the insurer has the opportunity to invest in d ($d > 1$) risky assets $S_1, S_2, \ldots, S_d$, all modelled by geometric Brownian motion

$$\begin{aligned} dS_1(t) &= S_1(t)(a_1 dt + b_1 dW_1(t)) \\ dS_2(t) &= S_2(t)(a_2 dt + b_2 dW_2(t)) \\ &\cdots \\ dS_d(t) &= S_d(t)(a_d dt + b_d dW_d(t)), \end{aligned} \tag{2.6.1}$$

where $a_i \in \mathbb{R}, b_i > 0$, for $i = 1, \ldots, d$, and W_i, $i = 1, \ldots, d$, are d standard Brownian motions with covariances

$$\mathbb{E}[W_i(t)W_j(t)] = \rho_{ij} \cdot t, \quad i, j = 1, \ldots, d. \tag{2.6.2}$$

We still assume that all the W_i and the risk process R are independent.

We define

$$b := \begin{pmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & b_d \end{pmatrix} \tag{2.6.3}$$

$$a := \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \tag{2.6.4}$$

and the matrix

$$\rho := (\rho_{ij})_{i,j=1}^d. \tag{2.6.5}$$

We assume that ρ is non-degenerate. Further we will denote by $W_{a,b}(t)$ the vector with components

$$(W_{a,b}(t))_i := a_i + b_i W_i(t), \quad i = 1, \ldots, d. \tag{2.6.6}$$

Then, if at time t, the insurer invests an amount $K_i(t)$ of money in the i-th risky asset S_i, her wealth process $Y(t,x,K)$ can be written as follows (see also equation (1.4.2))

$$Y(t,x,K) = R(t,x) + (K \cdot W_{a,b})(t), \tag{2.6.7}$$

where $K(t) = (K_i(t))_{i=1}^d$, and where the dot '$\cdot$' denotes the sum over d stochastic integrals $\sum_{i=1}^d (K_i \cdot (W_{a,b})_i)(t)$.

Now we will proceed in exactly the same way as in Sections 2.1 and 2.2. Since the right choices of the asymptotically optimal investment process $\hat{K}$ and the exponent $\hat{r}$ are not obvious, we quickly derive their appropriate values. To this purpose we introduce (as before) the process

$$M(t,x,K,r) := e^{-rY(t,x,K)}. \tag{2.6.8}$$

We are looking for $\hat{K}$ and $\hat{r}$ such that the stopped process $\tilde{M}(t,x,\hat{K},\hat{r})$ becomes a martingale. Itô's formula applied to M yields

$$\begin{aligned}\frac{dM(t,x,K,r)}{M(t-,x,K,r)} &= \left(-(c+K(t)a)r + \frac{1}{2}r^2(bK(t))^t\rho(bK(t)) + \lambda h(r)\right)dt \\ &\quad -r(bK(t))dW(t) \\ &\quad +(e^{rX_{N(t)}}-1)dN(t) - \lambda\mathbb{E}[e^{rX_{N(t)}}-1]dt,\end{aligned}$$

where $K(t)a$ stands for the inner product of the vectors $K(t)$ and a, and $bK(t)$ for $K(t)$ multiplied by the matrix b; $dW(t)$ is the vector $(dW_1(t),\ldots,dW_d(t))$ and $(bK(t))dW(t)$ denotes an inner product.

As in Section 2.1, we define the function $f : \mathbb{R}^d \times [0,r_\infty) \to \mathbb{R}$

$$f(K,r) = -(c+Ka)r + \frac{1}{2}r^2(bK)^t\rho(bK) + \lambda h(r). \tag{2.6.9}$$

Now we proceed as follows: we differentiate $f(K,r)$ with respect to K_j, for $j = 1,\ldots,d$, and set the resulting equations equal to zero. Thus we arrive at the following expression for $\hat{K}$

$$\hat{K} = \frac{1}{r}(b\rho b)^{-1}a. \tag{2.6.10}$$

(Note that b and ρ are invertible by assumption.)

Plugging this expression for $\hat{K}$ back into $f(K,r)$ and solving $f(\hat{K},r) = 0$ for r, we obtain that $\hat{r}$ has to satisfy

$$\lambda h(\hat{r}) = c\hat{r} + \frac{1}{2}(b^{-1}a)\rho^{-1}(b^{-1}a). \tag{2.6.11}$$

Since ρ is positive semidefinite and non-degenerate, ρ^{-1} is also positive semidefinite, and therefore the expression on the right hand side of (2.6.11) satisfies

$$\begin{aligned}\frac{1}{2}(b^{-1}a)\rho^{-1}(b^{-1}a) &\geq 0, \quad a \in \mathbb{R}^d \\ &= 0 \Leftrightarrow a = 0.\end{aligned} \tag{2.6.12}$$

For $\hat{K} = (1/\hat{r}) \cdot (b\rho b)^{-1}a$ and $\hat{r}$ defined as above, every result from Sections 2.1, 2.2 and 2.3 holds true. As a consequence also the results from Section 2.4 hold true; remember that in this section we only used results for the wealth process Y and the process M from the previous sections and did not fall back on the process underlying the investment possibility, apart from the a.s.-continuity of the investment part of Y, which also holds true in the d-dimensional setting.

2.7 Example for a Non-uniform Exponential Moment in the Tail Distribution

In this section we will briefly present an example of a claim size distribution that has exponential moments for some $r_\infty > 0$, but where for all r, $0 < r \le r_\infty$ the following holds

$$\sup_{y>0} \mathbb{E}[e^{-r(y-X)}|X > y] = \infty. \tag{2.7.1}$$

The idea of the example consists of having a distribution function that is decaying exponentially fast, but has peaks very far to the right (corresponding to very large claims), whose distance relatively to each other, when proceeding to the right, becomes infinitely large.

For our example we will assume that the distribution F has nonzero mass only at countably many points $\{z_1, z_2, \ldots\}$. Therefore we can write

$$dF(z) = \sum_{n=1}^{\infty} \delta(z - z_n) p_n dz, \tag{2.7.2}$$

for a sequence $(p_n)_{n=1}^{\infty}$ satisfying $0 < p_n < 1$ and $\sum_{n=1}^{\infty} p_n = 1$.

We will show that, for $r_\infty > \ln(1+\sqrt{5})/2$, the sequences $(z_n)_{n=1}^{\infty}$ and $(p_n)_{n=1}^{\infty}$ given by

$$z_n = n^2, \quad n = 1, \ldots \infty \tag{2.7.3}$$

and

$$\begin{aligned} p_n &= e^{-r_\infty z_n}, \quad n = 2, \ldots, \infty \\ p_1 &= 1 - \sum_{n=2}^{\infty} p_n \end{aligned} \tag{2.7.4}$$

provide an example, for which $\sup_{y>0} \mathbb{E}[e^{-r(y-X)}|X > y] = \infty$ holds.

First we have to show that $(p_n)_{n=1}^{\infty}$ is a probability vector. This follows from

$$\begin{aligned} \sum_{n=2}^{\infty} p_n &< \sum_{n=2}^{\infty} e^{-r_\infty n} \\ &= \frac{e^{-r_\infty}}{e^{r_\infty} - 1} < 1, \end{aligned} \tag{2.7.5}$$

$$\tag{2.7.6}$$

for $r_\infty > \ln(1+\sqrt{5})/2$.

The condition of exponential moments is equivalent to

$$\sum_{n=1}^{\infty} e^{rz_n} p_n < \infty, \quad \text{for all } r < r_\infty,$$
$$\text{and } \lim_{r \to r_\infty} \sum_{n=1}^{\infty} e^{rz_n} p_n = \infty, \tag{2.7.7}$$

which is satisfied, since

$$\begin{aligned} \sum_{n=1}^{\infty} e^{rz_n} p_n &= p_1 e^r + \sum_{n=2}^{\infty} e^{(r-r_\infty)n^2} \\ &= p_1 e^r + \Theta(e^{(r-r_\infty)}) - e^{(r-r_\infty)}, \end{aligned} \tag{2.7.8}$$

where $\Theta : E \to \mathbb{C}$ is the theta series and E is the open unit sphere in the complex plane. It is shown, e.g., in Remmert [28], Section 11.4, page 216, that $\lim_{z\in\mathbb{R}, z\to 1} \Theta(z) = \infty$.

Now we define

$$\begin{aligned} \Delta_1 &:= z_1, \\ \Delta_n &:= z_n - z_{n-1}, \quad \text{for } n = 2, 3, \ldots. \end{aligned}$$

Then we obtain, for $r \in \mathbb{R}_+$, $m \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E}[e^{-r(z_m - X)} | X > z_m] &= \frac{\sum_{n=m+1}^{\infty} e^{r\sum_{j=m+1}^{n} \Delta_j} p_n}{\sum_{n=m+1}^{\infty} p_n} \\ &\geq \frac{e^{r\Delta_{m+1}} p_{m+1}}{\sum_{n=m+1}^{\infty} p_n}. \end{aligned} \tag{2.7.9}$$

Now we use the following very easy, but nevertheless very useful lemma.

Lemma 2.7.1. *Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function, i.e. $f(n+1) > f(n)$. Then for $a > 0$,*

$$\sum_{n=m+1}^{\infty} e^{-af(n)} \leq C e^{-af(m+1)}, \tag{2.7.10}$$

where $C := \frac{1}{1-e^{-a}}$.

Proof. First note the following easy identity

$$\sum_{n=m+1}^{\infty} e^{-af(n)} \leq \sum_{n=f(m+1)}^{\infty} e^{-an}. \tag{2.7.11}$$

The right hand side is just a geometric series, thus we obtain

$$\sum_{n=f(m+1)}^{\infty} e^{-an} = e^{-af(m+1)} \sum_{n=0}^{\infty} e^{-an} = e^{-af(m+1)} \frac{1}{1-e^{-a}}. \tag{2.7.12}$$

□

Applying the previous lemma to $f(n) = n^2$ and $a = r_\infty$ we arrive at (for $m \geq 2$)

$$\begin{aligned} \mathbb{E}[e^{-r(z_m - X)}|X > z_m] &\geq \frac{e^{r\Delta_{m+1}} p_{m+1}}{C p_{m+1}} \\ &= \frac{1}{C} e^{r\Delta_{m+1}}, \end{aligned} \tag{2.7.13}$$

where $C = 1/(1 - e^{-r_\infty})$.

Since $\lim_{m\to\infty} \Delta_m = \infty$ we conclude that

$$\lim_{m\to\infty} \mathbb{E}[e^{-r(z_m - X)}|X > z_m] = \infty, \tag{2.7.14}$$

for all $r \leq r_\infty$.

Chapter 3

Investment Modelled by an Exponential Lévy Process

In this chapter, we want to discuss generalizations of the results obtained in Chapter 2 (for an investment possibility described by geometric Brownian motion), to an investment possibility modelled by a general exponential Lévy process.

3.1 Preliminaries

For the definition of exponential Lévy processes we rely on the PhD thesis of Jan Kallsen [20].

Definition 3.1.1. *A càdlàg (right continuous with limits to the left), adapted process X with $X_0 = 0$ a.s. is called a* Lévy process, *if the distribution of $X_t - X_s$ depends only on $t - s$, and if $X_t - X_s$ is independent of $\mathcal{F}_s$, for all $s, t \in \mathbb{R}_+$ with $s \leq t$.*

Remark. We could replace the condition *càdlàg* by *continuity in probability*. This means that $\lim_{t \to s} X_t = X_s$, where the limit is taken in probability. Theorem 30 in Protter [27] states that under the assumption of continuity in probability there exists a unique modification of X which is càdlàg and which is also a Lévy process.

The distribution of a Lévy process is completely characterized by its *characteristic triplet*. In order to define this triplet we will need Theorem 3.1.3. But before tackling this issue, we have to make the following definitions.

Definition 3.1.2. *1. A* random measure *on $\mathbb{R}_+ \times \mathbb{R}^d$ is a family $\mu = \{\mu(\omega; dt, dx) : \omega \in \Omega\}$ of non-negative measure on $(\mathbb{R}_+ \times \mathbb{R}^d, \mathcal{B}_+ \otimes \mathcal{B}^d)$ satisfying $\mu(\omega; 0 \times \mathbb{R}^d) = 0$ for all $\omega \in \Omega$. (See* Jacod and Shiryaev *[18]. $\mathcal{B}_+$ and $\mathcal{B}^d$ denote the Borel sigma algebras of $\mathbb{R}_+$ resp. $\mathbb{R}^d$.)*

2. *For any* $\mathbb{R}^d$*-valued càdlàg adapted process* X*, the* random measure of jumps μ^X *is defined by*

$$\mu^X(\omega;(0,t],dx) = \sum_{0<s\le t} \chi_{\{\Delta X_s \in dx, \Delta X_s \neq 0\}}(\omega). \tag{3.1.1}$$

3. *A predictable random measure* $\mu^{X,c}$ *is called* compensator *of* μ^X*, if* $\mathbb{E}[W * \mu^{X,c}] = \mathbb{E}[W * \mu^X]$ *for any predictable mapping* $W : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$*, where (for* μ *a càdlag, adapted random measure) the* integral process $W * \mu$ *is defined pathwise by*

$$W * \mu(\omega) := \begin{cases} \int_{[0,\infty)\times\mathbb{R}^d} W(\omega,s,x)\mu(\omega;ds,dx) & \text{if } \int_{[0,\infty)\times\mathbb{R}^d} |W(\omega,s,x)|\mu(\omega;ds,dx) < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

Remark. Actually, for a Lévy process, the random measure of jumps of a set $\Lambda \in \mathcal{B}^d$, $\mu^X(\omega;(0,t],\Lambda)$, is a Poisson process. See PROTTER [27], p. 26; there, this process is called N_t^Λ.

With Definition 3.1.2 at hand, we are able to introduce the characteristic triplet of a Lévy process.

Theorem 3.1.3 (Characterization of an integrable Lévy process by its characteristic triplet). *Let* X *be an integrable Lévy process in the sense that* $\mathbb{E}[|X_t|] < \infty$*, for all* $t \in \mathbb{R}_+$*.*

1. *There is a unique triplet* (a,b,ν)*, consisting of a vector* $a \in \mathbb{R}^d$*, a symmetric, non-negative definite matrix* $b \in \mathbb{R}^{d\times d}$*, and a measure* ν *on* $\mathbb{R}^d$ *satisfying* $\int(|x^2| \wedge |x|)\nu(dx) < \infty$ *and* $\nu(\{0\}) = 0$ *such that, for any* $t \in \mathbb{R}_+$*,*

   (a)

$$A(t) = at, \tag{3.1.2}$$

   where $A(t)$ *is the predictable part of finite variation in the canonical decomposition of the special martingale* X*. (For the definition of a special semimartingale and its canonical decomposition we refer to* PROTTER *[27]. That integrable Lévy processes are special semimartingales is the content of Lemma 2.2 (2.) in* KALLSEN *[20].) In particular,* $A(t)$ *is cádlád, adapted and starts at 0 a.s.*

   (b)

$$\langle X_i^c, X_j^c\rangle_t = (b^t b)_{ij}\, t, \qquad i,j = 1,\dots d. \tag{3.1.3}$$

   (c)

$$\mu^{X,c}((0,t],\Lambda) = \nu(\Lambda)t \text{ for any } \Lambda \in \mathcal{B}^d, \tag{3.1.4}$$

   where $\mu^{X,c}$ *denotes the compensator of the random measure of jumps* μ^X *of* X*. Actually, the measure* ν *can be described as* $\nu(\Lambda) = \mathbb{E}[\mu^X((0,1],\Lambda)]$*.*

2. *The triplet (a, b, ν) uniquely determines the distribution of X.*

3. *We have*

$$\mathbb{E}[e^{iu\cdot X_t}] = \exp\left(t\left(iu\cdot a - \frac{1}{2}(bu)^t bu + \int_{\mathbb{R}^d}(e^{iu\cdot x} - 1 - iu\cdot x)\nu(dx)\right)\right), \quad (3.1.5)$$

for all $t \geq 0$, and any $u \in \mathbb{R}^d$. This formula is known as Lévy–Khintchine formula *for the Fourier transform of a Lévy process.*

For the proofs of all the assertions of Theorem 3.1.3, we refer to Kallsen [20] or Jacod and Shiryaev [18].

In the following, we will restrict our attention to $d = 1$ dimension.

If X is a Lévy process with characteristic triplet (a, b, ν), $a \in \mathbb{R}$, $b \in \mathbb{R}_+$, $\int(|x^2| \wedge |x|)\nu(dx) < \infty$, then X can be written as

$$X(t) = at + bW(t) + \int_{\mathbb{R}} x(\mu^X(.;(0,t],dx) - t\nu(dx)), \quad (3.1.6)$$

where W is a standard Brownian motion ('Lévy representation theorem', see, e.g., Theorem I.42 in Protter [27], or Jacod and Shiryaev [18]).

The representation of the characteristic function, ϕ,

$$\phi(u) := \mathbb{E}[e^{iu\cdot X_t}], \quad (3.1.7)$$

given by equation (3.1.5) is not unique in the sense that we can replace the integral term

$$-i\int_{\mathbb{R}} u\cdot x\,\nu(dx) \quad (3.1.8)$$

(under the appropriate integrability conditions) by various other integrals, changing the 'drift' a simultaneously (see Shiryaev [30], page 196 f.). For precise statements, we need the following definition.

Definition 3.1.4. *A* truncation function *is a bounded function $h = h(x)$, $x \in \mathbb{R}$, with compact support, which satisfies the equality $h(x) = x$ in a neighbourhood of the origin.*

Besides (3.1.5), the characteristic function $\phi(u)$ has the following representations (valid for an arbitrary truncation function $h = h(x)$)

$$\phi(u) = \exp\left(t\left(iu\cdot a(h) - \frac{1}{2}(bu)^t bu + \int_{\mathbb{R}}(e^{iu\cdot x} - 1 - iu\cdot h(x))\nu(dx)\right)\right), \quad (3.1.9)$$

for some $a(h) \in \mathbb{R}$, depending on the truncation function h.

We point out that the integrals on the right hand sides of (3.1.5) and (3.1.9) are well defined in view of $\int(|x^2| \wedge |x|)\nu(dx) < \infty$, because the function

$$e^{iux} - 1 - iu\cdot h(x) \quad (3.1.10)$$

is bounded and it is $O(|x^2|)$ as $|x| \to 0$.

In particular, if ν satisfies the integrability condition

$$\int_{\mathbb{R}} (|x| \wedge 1)\nu(dx) < \infty, \tag{3.1.11}$$

then besides equation (3.1.5), the characteristic function $\phi(u)$ has the representation (for $h(x) = 0$)

$$\phi(u) = \exp\left(t\left(iu \cdot a_0 - \frac{1}{2}(bu)^t bu + \int_{\mathbb{R}} (e^{iu\cdot x} - 1)\nu(dx)\right)\right), \tag{3.1.12}$$

which we will use in the following. The constant

$$a_0 := a - \int_{\mathbb{R}} x\,\nu(dx) \tag{3.1.13}$$

in this representation is called the *drift* of the process X.

Now we have all the necessary tools at hand to define exponential Lévy processes.

Definition 3.1.5. *Let X be a Lévy process with characteristic triplet (a, b, ν), $a \in \mathbb{R}, b \in \mathbb{R}_+$, $\operatorname{supp} \nu \subset (-1, \infty)$. The latter assumption on ν shall guarantee that the process $e^{X(t)}$ always stays positive.*

We further assume $\int_{-1}^{\infty} e^x \nu(dx) < \infty$.

Then we will call the process S, defined by

$$S(t) = e^{X(t)}, \tag{3.1.14}$$

an exponential Lévy process.

The following holds for exponential Lévy processes.

Lemma 3.1.6 (Stochastic integral representation for an exponential Lévy process). *Let $S = e^X$ be an exponential Lévy process, where X has characteristic triplet $(a' - b'^2/2, b', \nu')$, $a' \in \mathbb{R}$, $b' \in \mathbb{R}_+$ and ν' a measure on $\mathbb{R}$ with support $(-1, \infty)$. Then S satisfies the stochastic differential equation*

$$dS(t) = S(t-)d\tilde{X}(t), \tag{3.1.15}$$

where $(\tilde{X}(t))_{t\geq 0}$ is a Lévy process with characteristic triplet (a, b, ν)

$$\begin{aligned} a &= a' + \int_{-1}^{\infty} (e^x - 1 - x)\nu'(dx), \\ b &= b', \\ \nu(dx) &= (e^x - 1)\nu'(dx). \end{aligned} \tag{3.1.16}$$

The drift a_0 of $\tilde{X}$ is given by

$$a_0 = a_0' + \int_{-1}^{\infty} (e^x - 1)(1 - x)\nu'(dx). \tag{3.1.17}$$

The proof for this Lemma can be found in KALLSEN [20], Section 4.6, p. 171.

Now we take the setting described in Section 1.1 and Subsection 1.3.1, and let the insurance company additionally invest in a stock described by an exponential Lévy process,

$$S(t) = e^{X(t)}, \tag{3.1.18}$$

where X is a Lévy process with characteristic triplet $(a' - b'^2/2, b', \nu')$, satisfying the assumptions of Definition 3.1.5. As before, we denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by the risk process R and the risky asset S.

From now on we proceed in exactly the same way as in Section 1.4. If at time t, the insurer has wealth $Y(t)$, and invests an amount $K(t)$ of money in the stock and the remaining reserve $Y(t) - K(t)$ in the bond (which in the present model also yields no interest), her wealth process Y can be written as

$$\begin{aligned} Y(t,x,K) &= x + ct - \sum_{i=1}^{N(t)} X_i + (\frac{K}{S} \cdot S)(t) \\ &= R(t,x) + (K \cdot \tilde{X})(t), \end{aligned} \tag{3.1.19}$$

where $\tilde{X}$ is the Lévy process obtained from Lemma 3.1.6 such that $dS(t) = S(t-)d\tilde{X}(t)$, i.e. $\tilde{X}$ has the characteristic triplet (a, b, ν) defined by equations (3.1.16).

As before in the setting with geometric Brownian motion, we are interested in the ruin probability

$$\Psi(x,K) = \mathbb{P}[Y(t,x,K) < 0, \text{ for some } t \geq 0], \tag{3.1.20}$$

depending on the initial wealth x and the investment strategy K of the insurer.

The set $\mathcal{K}$ of *admissible strategies* K is now defined as

$$\begin{aligned} \mathcal{K} := & \{K = (K(t))_{t\geq 0} : K \text{ is } \mathbb{F}-\text{predictable} \\ & \text{and locally bounded}\}. \end{aligned} \tag{3.1.21}$$

The condition $K \in \mathcal{K}$ is sufficient for the stochastic integral $(K \cdot \tilde{X})$ w.r.t. the Lévy process appearing in (3.1.19) to exist (see Theorem IV.15 in PROTTER [27]; there one can also find the definition of local boundedness).

3.2 Exponential Upper Bound

Theorem 3.2.1 (Exponential upper bound, for investment modelled by an exponential Lévy process). *Let $\Psi^*(x)$ be the minimal ruin probability for an insurer, whose wealth process $Y(t,x,K)$ follows the dynamics described in equation* (3.1.19)*, depending on her investment strategy K and starting from an initial capital $x \in \mathbb{R}_+$. Assume $b \neq 0$. Then Ψ^* can be bounded from above by*

$$\Psi^*(x) \leq \Psi(x,\hat{K}) \leq e^{-\hat{r}x}, \tag{3.2.1}$$

where $\hat{K}(t) \equiv \hat{k}/\hat{r}$, $\hat{k}, \hat{r} \in \mathbb{R}$, is a constant investment strategy. The constant $\hat{k}$ is defined by the following equation, which always has a solution $\hat{k} \in \mathbb{R}$,

$$a_0 - b^2 k = -\int_{-1}^{\infty} x e^{-kx} d\nu(x). \tag{3.2.2}$$

The exponent $\hat{r}$ in (3.2.1) *(which also enters the definition of the strategy $\hat{K}(t) \equiv \hat{k}/\hat{r}$) is defined as the solution to*

$$\lambda h(r) = cr + \frac{b^2}{2}\hat{k}^2 + \int_{-1}^{\infty} \left(1 - e^{-\hat{k}x} - \hat{k}x e^{-\hat{k}x}\right) d\nu(x). \tag{3.2.3}$$

Now we can distinguish three cases:

(i) If $\hat{k} \neq 0$ and $\lambda\mathbb{E}[X] \geq c$, i.e., if the classical Cramér–Lundberg exponent does not exist, then we still obtain a strictly positive coefficient $\hat{r} \in (0, r_\infty)$ as solution of (3.2.3) *and thus an exponential bound on the minimal ruin probability.*

(ii) If $\hat{k} \neq 0$ and $\lambda\mathbb{E}[X] < c$, then $\hat{r}$ satisfies $0 < \nu < \hat{r}$, i.e., we obtain a sharper bound than the classical Cramér–Lundberg inequality.

(iii) If $\hat{k} = 0$ and $\lambda\mathbb{E}[X] < c$, then the solution $\hat{r}$ of (3.2.3) *equals ν, the classical Lundberg exponent without investment.*

Remark. The solution $\hat{k}$ to equation (3.2.2) is equal to zero if and only if

$$a_0 = -\int_{-1}^{\infty} x \, d\nu(x) \Leftrightarrow a = 0. \tag{3.2.4}$$

If we look at geometric Brownian motion as a special case of an exponential Lévy process, namely where $\nu(dx)$ vanishes, then we see that condition (3.2.4) is equivalent to $a = a_0 = 0$, which is of course the corresponding condition, that we obtained in Chapter 2.

As before in Section 2.1, we will consider the following process, for fixed numbers $x, r \in \mathbb{R}_+$ and for a fixed admissible strategy $K \in \mathcal{K}$,

$$M(t, x, K, r) := e^{-rY(t,x,K)}. \tag{3.2.5}$$

Then we can show an analogous result to Lemma 2.1.2.

Lemma 3.2.2. *Let $x > 0$. There exist a unique $\hat{k} \in \mathbb{R}$ satisfying the equation*

$$-a_0 + b^2 k = \int_{-1}^{\infty} x e^{-kx} d\nu(x), \tag{3.2.6}$$

If $\hat{k} \neq 0$, or $\hat{k} = 0$ and $\lambda\mathbb{E}[X] < c$, then there exists a unique $0 < \hat{r} < r_\infty$ satisfying

$$\lambda h(r) = cr + \frac{b^2}{2}\hat{k}^2 + \int_{-1}^{\infty} \left(1 - e^{-\hat{k}x} - \hat{k}x e^{-\hat{k}x}\right) d\nu(x). \tag{3.2.7}$$

For this $\hat{r}$ and the constant process $\hat{K}(t) \equiv \hat{k}/\hat{r}$, the process $M(t, x, \hat{K}, \hat{r})$ is a martingale w.r.t. the filtration $\mathbb{F}$.

The proof of this Theorem relies on the Lévy–Khintchine formula (see equation (3.1.5)).

Proof. The existence and uniqueness of $\hat{k}$ follow because

$$\int_{-1}^{\infty} xe^{-kx} d\nu(x) \tag{3.2.8}$$

is a monotonously decreasing, continuous function in $k \in \mathbb{R}$, and hence intersects the line

$$-a_0 + b^2 k \tag{3.2.9}$$

in a unique point $\hat{k} \in \mathbb{R}$. The sign of $\hat{k}$ can be positive or negative, depending on whether

$$a_0 > -\int_{-1}^{\infty} x\,\nu(dx) \Rightarrow \hat{k} > 0, \tag{3.2.10}$$

or

$$a_0 \leq -\int_{-1}^{\infty} x\,\nu(dx) \Rightarrow \hat{k} \leq 0. \tag{3.2.11}$$

Now we define $g : \mathbb{R} \times [0, r_\infty) \to \mathbb{R}$

$$g(k, r) := \lambda h(r) - ka_0 - cr + \frac{1}{2}k^2 b^2 + \int_{-1}^{\infty} (e^{-kx} - 1)\,\nu(dx). \tag{3.2.12}$$

If we plug $\hat{k}$ into g and set $g(\hat{k}, r) = 0$, we obtain the following equation in r

$$\lambda h(r) - cr = \hat{k}a_0 - \frac{1}{2}\hat{k}^2 b^2 - \int_{-1}^{\infty} (e^{-\hat{k}x} - 1)\,\nu(dx). \tag{3.2.13}$$

Now we use the defining equation for $\hat{k}$, namely equation (3.2.6), to obtain

$$\begin{aligned} \lambda h(r) - cr &= \hat{k}(\hat{k}b^2 - \int_{-1}^{\infty} xe^{-\hat{k}x}\nu(dx)) - \frac{1}{2}\hat{k}^2 b^2 - \int_{-1}^{\infty} (e^{-\hat{k}x} - 1)\,\nu(dx) \\ &= \frac{1}{2}\hat{k}^2 b^2 - \int_{-1}^{\infty} (\hat{k}xe^{-\hat{k}x} + e^{-\hat{k}x} - 1)\,\nu(dx). \end{aligned} \tag{3.2.14}$$

The integrand on the right hand side of (3.2.14)

$$\hat{k}xe^{-\hat{k}x} + e^{-\hat{k}x} - 1 \tag{3.2.15}$$

is less then 0, for all $\hat{k} \in \mathbb{R}$, $x \in \mathbb{R}_+$, and equals zero a.s. iff $\hat{k} = 0$, i.e., if and only if $\int_{-1}^{\infty} x\,\nu(dx) = -a_0$. Therefore if $\hat{k} \neq 0$, equation (3.2.14) always has a solution $\hat{r} \in (0, r_\infty)$. If ν, the Lundberg exponent, exists, then this solution $\hat{r}$ has to be greater than ν by exactly the same arguments as for the setting described in Chapter 2, where the insurer had the possibility to invest in a risky asset modelled by geometric Brownian motion. Moreover, if $\hat{k} = 0$, equation (3.2.14) reduces to

$$\lambda h(r) = cr. \tag{3.2.16}$$

This equation has the solution ν, if $\lambda\mathbb{E}[X] < c$.

In order to show that the process $M(t, x, \hat{K}, \hat{r})$ is a martingale w.r.t. $\mathbb{F}$, we proceed in exactly the same way as in equations (2.1.7) to (2.1.8) in the proof of Lemma 2.1.2, but use the Lévy–Khintchine formula (3.1.5): for arbitrary $t \geq 0$,

$$\begin{aligned}\mathbb{E}[M(t,0,\hat{K},\hat{r})] &= \mathbb{E}[e^{-\hat{r}(ct-\sum_{i=1}^{N(t)} X_i+\hat{K}\tilde{X}(t))}] \\ &= e^{-\hat{r}ct}\mathbb{E}[e^{\hat{r}\sum_{i=1}^{N(t)} X_i}]\mathbb{E}[e^{-\hat{k}\tilde{X}(t)}] \\ &= \exp(-\hat{r}ct)\exp(h(\hat{r})\lambda t)\exp\left\{\left(-a_0\hat{k}+\hat{k}^2b^2/2+\int_{-1}^{\infty}(e^{-\hat{k}x}-1)\,\nu(dx)\right)t\right\} \\ &= e^{g(\hat{k},\hat{r})t} \\ &= 1. \end{aligned} \tag{3.2.17}$$

Since $Y(t, x, \hat{K})$ has stationary independent increments, we obtain, for $0 \leq t \leq T$,

$$\begin{aligned}\mathbb{E}_t[M(T,x,\hat{K},\hat{r})] &= \mathbb{E}_t[e^{-\hat{r}Y(T,x,\hat{K})}] \\ &= e^{-\hat{r}Y(t,x,\hat{K})}\mathbb{E}_t[e^{-\hat{r}(Y(T,x,\hat{K})-Y(t,x,\hat{K}))}] \\ &= e^{-\hat{r}Y(t,x,\hat{K})}\mathbb{E}[e^{-\hat{r}(Y(T-t,x,\hat{K})-Y(0,x,\hat{K}))}] \\ &= e^{-\hat{r}Y(t,x,\hat{K})}\mathbb{E}[e^{-\hat{r}Y(T-t,0,\hat{K})}] \\ &= e^{-\hat{r}Y(t,x,\hat{K})} \\ &= M(t,x,\hat{K},\hat{r}), \end{aligned} \tag{3.2.18}$$

and therefore $M(t, x, \hat{K}, \hat{r})$ is a martingale w.r.t. the filtration $\mathbb{F}$. □

Now we can proceed in exactly the same way as in Chapter 2, Section 2.1, to prove Theorem 3.2.1 (see page 22 ff.). Defining $\tilde{M}(t, x, K, r)$ and $\tilde{Y}(t, x, K, r)$ as the processes $M(t, x, K, r)$, respectively $Y(t, x, K, r)$, stopped at the time of ruin $\tau(x, K)$, it is an immediate consequence of Lemma 3.2.2 that $\tilde{M}(t, x, \hat{K}, \hat{r})$ is a martingale. From this, it is easy to deduce (Theorem 2.1.3) that the ruin probability, using the strategy $\hat{K}$ and starting from initial capital x, can be bounded from above by

$$\Psi(x, \hat{K}) \leq e^{-\hat{r}x}. \tag{3.2.19}$$

3.3 Exponential Lower Bound and Asymptotic Optimality

As in the previous section we will rely very strongly on the results and methods of Chapter 2, in particular, Sections 2.2 and 2.3.

What we need to show first is, that under the assumption of a uniform exponential moment in the tail distribution for $\hat{r}$, the process $\tilde{M}(t, x, K, \hat{r})$ is a uniformly integrable submartingale for all $K \in \mathcal{K}$. (This is the exact analogue of Theorem 2.2.2.) Then we can proceed exactly as in Section 2.3 to show that under the assumption of a uniform exponential moment in the tail distribution for $\hat{r}$, the wealth process $Y(t,x,K),\ K \in \mathcal{K}$, converges a.s. on $\{\tau(x,K)=\infty\}$ to ∞. After having shown these two things, we can prove – again exactly like in Section 2.3 – that the ruin probability $\Psi(x,K)$ can be bounded from below by $Ce^{-\hat{r}x}$, where the constant C is defined by equation (3.3.14).

Theorem 3.3.1. *Assume that the claim size X has a uniform exponential moment in the tail distribution (see Definition 2.2.1). Then for each $K \in \mathcal{K}$, the process $\tilde{M}(t,x,K,\hat{r})$ is a uniformly integrable submartingale.*

Proof. We use Itô's formula to obtain

$$\begin{aligned}\tilde{M}(t,x,K,\hat{r}) &= e^{-\hat{r}x} - \tilde{M}_1(t,x,K,\hat{r}) + \tilde{M}_2(t,x,K,\hat{r}) + \tilde{M}_3(t,x,K,\hat{r}) \\ &\quad + \int_0^t \tilde{M}(s-,x,K,\hat{r})\Bigg(-(c+K(t)a_0)\hat{r} + \frac{b^2}{2}\hat{r}^2K(t)^2 + \\ &\quad + \lambda h(r) + \int_{-1}^{\infty}(e^{-\hat{r}K(s)x}-1)d\nu(x)\Bigg)\,ds, \end{aligned} \tag{3.3.1}$$

where the processes $\tilde{M}_1, \tilde{M}_2$ and $\tilde{M}_3$ are local martingales. The process $\tilde{M}_1$ is defined by

$$\tilde{M}_1(t,x,K,\hat{r}) := b\int_0^t K(s)\tilde{M}(s-,x,K,\hat{r})\,dW(s), \tag{3.3.2}$$

and is a local martingale by Theorem IV.29 in Protter [27]. The process $\tilde{M}_2$, defined by

$$\begin{aligned}\tilde{M}_2(t,x,K,\hat{r}) &:= \int_0^t (e^{X_{N(s)}}-1)\tilde{M}(s-,x,K,\hat{r})dN(s) \\ &\quad -\lambda\int_0^t \mathbb{E}[e^{\hat{r}X_{N(s)}}-1]M(s-,x,K,\hat{r})\,ds,\end{aligned} \tag{3.3.3}$$

is a martingale because of Lemma 2.2.5. Finally, the process

$$\begin{aligned}\tilde{M}_3(t,x,K,\hat{r}) &:= \int_0^t\int_{-1}^{\infty}\tilde{M}(s-,x,K,\hat{r})(e^{-\hat{r}K(s)\Delta X_s}-1)\mu^X((0,s],dx)ds \\ &\quad -\int_0^t\int_{-1}^{\infty}\tilde{M}(s-,x,K,\hat{r})(e^{-\hat{r}K(s)x}-1)\nu(dx)ds\end{aligned} \tag{3.3.4}$$

is a martingale because of similar arguments as in the proof of Lemma 2.2.5.

Now we want to show that the integrand in

$$\begin{aligned}&\int_0^t \tilde{M}(s-,x,K,\hat{r})\Bigg(-(c+K(s)a_0)\hat{r} + \frac{b^2}{2}\hat{r}^2K(s)^2 + \lambda h(r) + \\ &+ \int_{-1}^{\infty}(e^{-\hat{r}K(s)x}-1)d\nu(x)\Bigg)\,ds \\ = &\int_0^t \tilde{M}(s-,x,K,\hat{r})g(K(s)\hat{r},\hat{r})\,ds\end{aligned} \tag{3.3.5}$$

is a.s. greater than or equal to zero. For this purpose we look at the function g at points $(k, \hat{r})$, for general $k \in \mathbb{R}$. Using the defining equation for $\hat{r}$, (3.2.3), we obtain

$$\begin{aligned} & g(k,\hat{r}) \\ = \; & \lambda h(\hat{r}) - c\hat{r} - ka_0 + \frac{1}{2}k^2b^2 + \int_{-1}^{\infty}(e^{-kx}-1)\nu(dx) \\ = \; & \frac{1}{2}b^2(k-\hat{k})^2 + b^2k\hat{k} - ka_0 + \int_{-1}^{\infty}(e^{-kx} - \hat{k}xe^{-\hat{k}x} - e^{-\hat{k}x})\nu(dx). \end{aligned} \tag{3.3.6}$$

Now we use the defining equation for $\hat{k}$,

$$b^2k\hat{k} = a_0k + \int_{-1}^{\infty} kxe^{-\hat{k}x}\nu(dx), \tag{3.3.7}$$

to obtain

$$g(k,\hat{r}) = \frac{1}{2}b^2(k-\hat{k})^2 - \int_{-1}^{\infty} e^{-kx}\left((\hat{k}-k)xe^{-(\hat{k}-k)x} + e^{-(\hat{k}-k)x} - 1\right)\nu(dx). \tag{3.3.8}$$

Since

$$ye^{-y} + e^{-y} - 1 \le 0, \quad \text{for all } y \in \mathbb{R}, \tag{3.3.9}$$

it follows that

$$g(k,\hat{r}) \ge 0, \quad \text{for all } k \in \mathbb{R}. \tag{3.3.10}$$

Using this and the representation (3.3.1) for $\tilde{M}(t,x,K,\hat{r})$, it is easily seen that $\tilde{M}(t,x,K,\hat{r})$ is a local submartingale for all $K \in \mathcal{K}$.

To proceed from this to the conclusion that $\tilde{M}(t,x,K,\hat{r})$ is indeed a uniformly integrable submartingale, we use the assumption of a uniform exponential moment in the tail distribution. Since the line of argument is exactly the same as for the setting with an investment possibility modelled by geometric Brownian motion, we do not repeat the argument here, but refer to Section 2.3, page 28 ff.

□

Now the only thing we still need to show, in order to adopt all the other results from Section 2.3, is the analogue of Lemma 2.2.3.

Lemma 3.3.2 (Ruin or infinite wealth, exponential Lévy process). *If X has a uniform exponential moment in the tail distribution for $\hat{r}$, then for arbitrary $K \in \mathcal{K}$ and $x \in \mathbb{R}_+$, the stopped wealth process $(\tilde{Y}(t,x,K))_{t\ge 0}$ converges a.s. on $\{\tau(x,K) = \infty\}$ to ∞ for $t \to \infty$. In other words, either ruin occurs, or the insurer becomes infinitely rich.*

Proof. We know from Theorem 3.3.1 that $\tilde{M}(t,x,K,\hat{r})$ is a uniformly integrable submartingale. Therefore we can apply Doob's Supermartingale Convergence Theorem (ROGERS AND WILLIAMS VOL. 1 [29], Theorem (II.69.1)) to $-\tilde{M}$, to obtain that $\lim_{t\to\infty}\tilde{M}(t,x,K,R)$, and therefore also $\lim_{t\to\infty}\tilde{Y}(t,x,K,\hat{r})$ exists a.s.

Now we proceed in the same fashion as in the proof of Lemma 2.2.3. There must exist $d>0$ such that $\mathbb{P}[X>d]>0$. If we define the events $E_n := \{X_n > d\}$, then $\mathbb{P}[E_n^c]<1$, and the events $\{E_j\}_{j=1}^{\infty}$ are mutually independent. Therefore,

$$\mathbb{P}[\bigcup_{k=1}^{\infty}\bigcap_{n\geq k}E_n^c] = \lim_{k\to\infty}\mathbb{P}[\bigcap_{n\geq k}E_n^c] = \lim_{k\to\infty}\prod_{n\geq k}\mathbb{P}[E_n^c] = 0. \tag{3.3.11}$$

Hence, $\mathbb{P}[\bigcup_{k=1}^{\infty}\bigcap_{n\geq k}E_n]=1$. In other words, with probability 1, a jump of size greater than d occurs infinitely often.

Remember that the jump part of a stochastic integral is purely determined by the jumps of the integrator (PROTTER [27], Theorem IV.8)

$$\Delta(K\cdot\tilde{X})(t) = K(t)\Delta\tilde{X}(t). \tag{3.3.12}$$

We know from the above that the risk process $R(t,x)$ jumps a.s. infinitely often by an amount greater than d. Since the risk process and the Lévy process $\tilde{X}$ are independent, and since we assumed that the Lévy measure ν is finite, $\int\nu(dx)<\infty$, it follows that R and $\tilde{X}$ will jump at the same point of time by an amount greater than d with probability zero. Therefore the jumps of R, greater than d, which occur a.s. infinitely often, cannot be compensated for by the stochastic integral $(K\cdot\tilde{X})$ a.s. Hence, the wealth process, stopped at the time of ruin, cannot converge to a nonzero finite value with positive probability.

□

From now on we can go on exactly like in Section 2.3. Therefore we only the state the result, but omit the proof.

Theorem 3.3.3 (Reverse inequality, exponential Lévy process). *Assume that X has a uniform exponential moment in the tail distribution for $\hat{r}$. Then the ruin probability satisfies, for every admissible process $K\in\mathcal{K}$,*

$$\Psi(x,K)\geq Ce^{-\hat{r}x}, \tag{3.3.13}$$

where

$$C := \frac{1}{\sup_{y\geq 0}\mathbb{E}[e^{-\hat{r}(y-X)}|X>y]}. \tag{3.3.14}$$

3.4 Results without Uniform Exponential Moment in the Tail Distribution

The results from Section 2.4 can be adopted one-to-one for the setting with investment possibility modelled by an exponential Lévy process.

Chapter 4

Ruin Probabilities in the Presence of Heavy Tails and Optimal Investment

4.1 Functions of Regular Variation: Properties

In the following section we will briefly recall some results about functions of regular variation, that will be needed in the sequel, mainly without proofs. The main reference for the theory of regular variation is the book by Bingham, Goldie and Teugels [2]. For the sake of completeness we repeat Definition 1.2.1.

Definition 4.1.1 (Slow and regular variation). *Let ℓ be a positive, Lebesgue measurable function, defined on some neighborhood $[a, \infty)$ of infinity, and satisfying*

$$\lim_{x\to\infty} \frac{\ell(\lambda x)}{\ell(x)} = 1 \quad \forall \lambda > 0, \tag{4.1.1}$$

then ℓ is said to be slowly varying (in Karamata's sense).

A Lebesgue measurable function $f > 0$ satisfying

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^{\rho} \quad \forall \lambda > 0, \tag{4.1.2}$$

for some $\rho \in \mathbb{R}$, is called regularly varying *of* index ρ. *We write $f \in \mathcal{R}_\rho$.*

Definition 4.1.2 (Landau symbols). *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions, and let g be positive in a neighborhood of infinity. We write*

$$f = O(g) \quad :\Leftrightarrow \quad \limsup_{x\to\infty} \frac{|f(x)|}{g(x)} < \infty, \tag{4.1.3}$$

$$f = o(g) \quad :\Leftrightarrow \quad \lim_{x\to\infty} \frac{|f(x)|}{g(x)} = 0, \tag{4.1.4}$$

and

$$f \sim g \quad :\Leftrightarrow \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1. \tag{4.1.5}$$

Theorem 4.1.3 (Representation theorem). *The function ℓ is slowly varying if and only if it may be written in the form*

$$\ell(x) = c(x) \exp \int_1^x \frac{\epsilon(y)}{y} dy, \quad \text{for } x \geq 0, \tag{4.1.6}$$

where the function c is measurable, $c(x) \to c \in (0, \infty)$, and $\epsilon(x) \to 0$ as $x \to \infty$.

Proof. See Feller [9], Chapter VIII, p. 282, or Bingham, Goldie and Teugels [2], Theorem 1.3.1. □

The next proposition is an immediate consequence of the last theorem.

Proposition 4.1.4. *Let ℓ be a function of slow variation and $\alpha > 0$. Then*

$$\lim_{x\to\infty} x^{\alpha}\ell(x) \to \infty, \quad \lim_{x\to\infty} x^{-\alpha}\ell(x) = 0. \tag{4.1.7}$$

It is easy to convince oneself that, if $f \in \mathcal{R}_\rho$, then $f \sim x^\rho \ell(x)$ for some $\ell \in \mathcal{R}_0$. Therefore, generally for $f \in \mathcal{R}_\rho$ and $\alpha > 0$,

$$x^{-\rho+\alpha} f(x) \to \infty \text{ and } x^{-\rho-\alpha} f(x) \to 0 \text{ for } x \to \infty. \tag{4.1.8}$$

The next theorem states that the convergence in the definition of regular variation (4.1.2) actually takes place uniformly on each compact subset of $(0, \infty)$. We cite from Bingham, Goldie and Teugels [2], Theorem 1.5.2.

Theorem 4.1.5 (Uniform convergence theorem for regularly varying functions). *If f varies regularly with index ρ, then (in the case $\rho > 0$, assuming that f is bounded on each interval $(0, x]$)*

$$\frac{f(\lambda x)}{f(x)} \to \lambda^\rho \ (x \to \infty) \text{ uniformly in } \lambda \tag{4.1.9}$$

(a) on each $[a, b]$ ($0 < a \leq b \leq \infty$), if $\rho = 0$.

(b) on each $(0, b]$ ($0 < b < \infty$), if $\rho > 0$.

(c) on each $[a, \infty)$ ($0 < a < \infty$), if $\rho < 0$.

Next, we present one of the key results in the theory of functions of regular variation.

Theorem 4.1.6 (Karamata's theorem). *Let ℓ be a function of slow variation and locally bounded in $[x_0, \infty)$ for some $x_0 \geq 0$. Then* *(i) for* $\alpha > -1$,

$$\int_{x_0}^{x} y^{\alpha}\ell(y)dy \sim \frac{x^{\alpha+1}}{\alpha+1}\ell(x), \quad x \to \infty, \tag{4.1.10}$$

(ii) for $\alpha < -1$,

$$\int_{x_0}^{x} y^{\alpha}\ell(y)dy \sim -\frac{x^{\alpha+1}}{\alpha+1}\ell(x), \quad x \to \infty. \tag{4.1.11}$$

Proof. See BINGHAM, GOLDIE AND TEUGELS [2], Proposition 1.5.8. □

Karamata's Theorem essentially says that integrals of regularly varying functions are again regularly varying.

The next theorem treats the converse of Karamata's Theorem, namely under which conditions the derivative of a regularly varying function is again of regular variation.

Theorem 4.1.7 (Monotone density theorem). *Let* $F(x) = \int_0^x f(y)\,dy$ *(or* $\int_x^\infty f(y)\,dy$*), where* f *is ultimately monotone (i.e.* f *is monotone on* (z,∞) *for some* $z > 0$*). If*

$$F(x) \sim cx^{\alpha}\ell(x), \quad x \to \infty, \tag{4.1.12}$$

for some $c \geq 0$, $\alpha \in \mathbb{R}$ *and* $\ell \in \mathcal{R}_0$*, then*

$$f(x) \sim c\alpha x^{\alpha-1}\ell(x), \quad x \to \infty. \tag{4.1.13}$$

For $c = 0$*, the above relations are interpreted as* $F(x) = o(x^{\alpha}\ell(x))$ *and* $f(x) = o(x^{\alpha-1}\ell(x))$*.*

Proof. see BINGHAM, GOLDIE AND TEUGELS [2], Theorem 1.7.2.

Finally we state a small, almost trivial lemma, which nevertheless lies at the heart of the proof of the main theorem.

Lemma 4.1.8. *If* $f \sim g$*, and* $f \in \mathcal{R}_\rho$*, then also* $g \in \mathcal{R}_\rho$*.*

Proof.

$$1 = \frac{\lim_{x\to\infty} \frac{g(\lambda x)}{f(\lambda x)}}{\lim_{x\to\infty} \frac{g(x)}{f(x)}} = \lim_{x\to\infty} \frac{\frac{g(\lambda x)}{g(x)}}{\frac{f(\lambda x)}{f(x)}} = \frac{\lim_{x\to\infty} \frac{g(\lambda x)}{g(x)}}{\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)}} = \lambda^{-\rho} \lim_{x\to\infty} \frac{g(\lambda x)}{g(x)}, \quad \lambda > 0. \tag{4.1.14}$$

□

4.2 Main Theorem

In the following we will always consider the model described in Chapter 1.

Theorem 4.2.1. *Consider the risk process defined in Section 1.1, and the investment opportunity, introduced in Section 1.4. Assume a positive safety loading,* $c - \lambda\mu > 0$*. Let* $\overline{F}$*, the tail distribution of the claim size, be of regular variation with index* $\rho < -1$*. Then also the minimal ruin probability is of regular variation with index* ρ*, i.e.* $\Psi^*(x) \in \mathcal{R}_\rho$*.*

We will prove Theorem 4.2.1 in Section 4.5, after providing the preliminaries in the next two sections.

4.3 Hamilton–Jacobi–Bellman Equation, Existence and Verification

First, we present the Hamilton–Jacobi–Bellman (HJB) equation for the problem at hand. Let $\delta^* : \mathbb{R}_+ \to [0,1]$ denote the *optimal survival probability*, defined by

$$\delta^*(x) := 1 - \Psi^*(x). \tag{4.3.1}$$

Then the HJB-equation for δ^* is, for $x > 0$,

$$\sup_{\{K:\mathbb{R}_+\to\mathbb{R}\}} \left(\lambda\mathbb{E}[\delta(x-X) - \delta(x)] + (c + K(x)a)\delta'(x) + \frac{1}{2}b^2K(x)^2\delta''(x) \right) = 0, \tag{4.3.2}$$

see Hipp and Plum [17].

The supremum over $K(x)$ exists whenever $\delta''(x) < 0$, and in this case

$$K^*(x) = -\frac{a}{b^2}\frac{\delta'(x)}{\delta''(x)}. \tag{4.3.3}$$

Substituting K^*, equation (4.3.2) turns into

$$\lambda \int_0^\infty [\delta(x-z) - \delta(x)]dF(z) + c\delta'(x) = \frac{a^2}{2b^2}\frac{(\delta'(x))^2}{\delta''(x)}. \tag{4.3.4}$$

Let us turn to the boundary conditions that the optimal survival probability will have to satisfy. The first one is obviously

$$\lim_{x\to\infty} \delta^*(x) = 1. \tag{4.3.5}$$

The second one comes from the fact, that with zero wealth $x = 0$, it is surely optimal not to trade at all, since otherwise the Brownian motion will cause immediate ruin a.s.

Carrying through the infinitesimal argument based on the dynamic programming principle and Itô's formula, which also leads to the HJB-equation, one arrives at

$$\delta^{*'}(0) = \delta^*(0)\frac{\lambda}{c}. \tag{4.3.6}$$

Hipp and Plum [17] show that a solution of the HJB-equation (4.3.2), satisfying the boundary conditions (4.3.5) and (4.3.6), indeed is a solution of the original problem of finding the maximal survival probability over the set of admissible trading strategies. We state their result.

Theorem 4.3.1 (Verification Theorem, Hipp and Plum (2000)). *Assume that there exists a solution $\delta^* : \mathbb{R}_+ \to [0,1]$ of the HJB-equation (4.3.2) with maximizing function $K^* : \mathbb{R}_+ \to \mathbb{R}$, with the following properties:*

$$\delta^*(0),\ \delta^{*\prime}(0) > 0,\ \delta^{*\prime}(0) = \frac{\lambda}{c}\delta^*(0),\ \text{and}\ \lim_{x\to\infty} \delta^*(x) = 1, \tag{4.3.7}$$

and δ^* is twice continuously differentiable on $\{x > 0\}$. Then $\delta^*(x) > 0$ for $x > 0$, and if $K \in \mathcal{K}$ is an arbitrary admissible trading strategy, then the corresponding survival probability, $\delta(x,K) : \mathbb{R}_+ \to [0,1]$, satisfies

$$\delta(x,K) \le \delta^*(x), \qquad \text{for } x \ge 0, \tag{4.3.8}$$

with equality for

$$K^*(t) = -\frac{a\delta^{*'}(Y(t-,x,K^*))}{b^2\delta^{*''}(Y(t-,x,K^*))}. \tag{4.3.9}$$

Remarks. Hipp and Plum omit the second boundary condition (4.3.6) in their original theorem. That this boundary condition is an essential part of verification can be seen in two different ways. We present one example with explicit solution, where it can be seen that there exists a family of solutions of the HJB-equation satisfying all the conditions from Hipp and Plum [17]. But only one of these solutions satisfies the second boundary condition and thus optimality. Then we shortly present, where the second boundary condition must appear in the proof of the Verification Theorem by Hipp and Plum, in order to make it fully work.

1. In Section 5 of Hipp and Plum [17], explicit solutions of the HJB-equation (4.3.2) are presented. For claims with an exponential distribution $dF(z) = e^{-z}dz$ and for $c = 2$, $\lambda = 3/2$, the general solution of the corresponding HJB–equation – which can be transformed into an equation for the derivative of the survival probability (see equation (4.3.15)) – is given by

$$u^\kappa(x) = 2\frac{e^{-x/2}}{2+\sqrt{1-\kappa}}\sqrt{\frac{1+\sqrt{1-\kappa}}{1+\sqrt{1-\kappa e^{-2x}}}}, \tag{4.3.10}$$

up to a multiplicative constant, where $u^\kappa = (\delta^\kappa)'$ is the derivative of the survival probability, and the parameter κ has to satisfy $0 \le \kappa \le 1$. The solution, which is presented by Hipp and Plum [17] in Section 5 of their paper, corresponds to

$$u^1(x) = e^{-x/2}\sqrt{\frac{1}{1+\sqrt{1-e^{-2x}}}}, \tag{4.3.11}$$

which is the only u^κ, $0 \le \kappa \le 1$, which satisfies the boundary condition $u(0) = 1$ (after normalization, see (4.3.12)).

2. In the proof of Theorem 2.1 from Hipp and Plum [17] we can see, e.g., for the second inequality after equation (2.3) on Page 218, that the condition $K^*(0) = 0$ is necessary, because otherwise the limit $\epsilon \to 0$ causes trouble. (We refer to the paper for the details.)

Since equation (4.3.2) determines δ^* up to a multiplicative constant, we will from now on use the boundary condition

$$\delta^{*'}(0) = 1 \text{ or equivalently } \delta^*(0) = \frac{c}{\lambda}. \tag{4.3.12}$$

We will also write λ instead of $\lambda b^2/a^2$ and c instead of cb^2/a^2, thus transforming the HJB-equation for δ^* into

$$\lambda \int_0^\infty [\delta(x-z) - \delta(x)]dF(z) + c\delta'(x) = \frac{1}{2}\frac{(\delta'(x))^2}{\delta''(x)}. \tag{4.3.13}$$

Integration by parts and the boundary condition (4.3.12) yield

$$-\lambda \int_0^x \delta'(x-z)\overline{F}(z)\,dz + c(\delta'(x) - \overline{F}(x)) = \frac{\delta'(x)^2}{2\delta''(x)}, \qquad \delta'(0) = 1. \tag{4.3.14}$$

Introducing $u(x) := \delta'(x)$, one obtains the following equivalent problem for u

$$-\lambda \int_0^x u(x-z)\overline{F}(z)\,dz + c(u(x) - \overline{F}(x)) = \frac{u(x)^2}{2u'(x)}, \qquad u(0) = 1. \tag{4.3.15}$$

Hipp and Plum [17] prove an existence theorem for equation (4.3.14), under the assumption of a locally bounded density of the claim size distribution function. We state their result.

Theorem 4.3.2 (Existence Theorem, Hipp and Plum (2000)). *Let F have a locally bounded density. Then there exists a solution $u^* \in C^1((0,\infty)) \cap C([0,\infty))$ of equation (4.3.14) satisfying $u^* > 0$, $u^{*\prime} < 0$ on $(0,\infty)$, and*

$$u^*(x) = 1 - \sqrt{\frac{x}{c}} + o(\sqrt{x}) \text{ as } x \to 0. \tag{4.3.16}$$

If moreover, $\overline{F}$ has a finite integral over $[0,\infty)$, then also u^* has a finite integral.

4.4 Auxiliary Results

In this section we will present some auxiliary results that will be needed for the proof of the Main Theorem 4.2.1.

Lemma 4.4.1. *Let* $\overline{F} : \mathbb{R}_+ \to \mathbb{R}_+$ *be of regular variation with index* $\rho < -1$ *and let* $u^* : \mathbb{R}_+ \to \mathbb{R}$ *satisfy equation* (4.3.15). *Then*

$$\lim_{x\to\infty} \frac{x^\delta u^*(x)}{\overline{F}(x)} = 0, \tag{4.4.1}$$

for some $\delta > 0$.

Proof. We first note that from Lemma 4.1.4 it follows that

$$\begin{aligned} \lim_{x\to\infty} x^{-\rho+\eta}\overline{F}(x) &= \infty, \\ \lim_{x\to\infty} x^{-\rho-\eta}\overline{F}(x) &= 0, \end{aligned} \tag{4.4.2}$$

for any fixed $\eta > 0$.

Since we will rely on parts of the proof of Theorem 3.1 of Hipp and Plum [17], we will split up our proof into two cases.

Case 1. $\rho > -1.49$.

Since we have assumed that $\overline{F}$ is of regular variation with index $\rho < -1$, it follows that $\overline{F}$ has a finite integral over $[0, \infty)$. Therefore, we can use Assertion (e) (iii) from Hipp and Plum [17] to prove our own assertion. We repeat their argument for the convenience of the reader.

Note that $\overline{F} \in \mathcal{R}_\rho$, for $\rho < -1$, implies that

$$\lim_{x\to\infty} x\overline{F}(x) = 0. \tag{4.4.3}$$

We will show that (see Hipp and Plum [17])

(i) $u^*(x) \to 0$,

(ii) $x\, u^*(x) \to 0$,

(iii) $x^{3/2}u^*(x) \to 0$.

In order to do so, we define the following operator $\Phi : C^1(0,\infty) \cap C[0,\infty) \to C[0,\infty)$

$$\Phi[u](x) := -\lambda \int_0^x u(x-z)\overline{F}(x)\, dx + c(u(x) - \overline{F}(x)). \tag{4.4.4}$$

Then, for u^* satisfying equation (4.3.15), we obtain that

$$-\Phi[u^*](x) = -\frac{u^*(x)^2}{2u^{*\prime}(x)} = \frac{1}{2}\frac{1}{(\frac{1}{u^*})'(x)}, \qquad \text{for } 0 < x < \infty. \tag{4.4.5}$$

To show (i), first observe that $u^* > 0$ and $u^{*\prime} < 0$ yield the existence of $u^*_\infty := \lim_{x\to\infty} u^*(x) \geq 0$ and of a sequence x_k, $k = 1, \ldots, \infty$, with $\lim_{k\to\infty} x_k = \infty$, such that $\lim_{k\to\infty} u^{*\prime}(x_k) = 0$. Assuming $u^*_\infty > 0$, we obtain from (4.4.5) that $\lim_{k\to\infty} \Phi[u^*](x_k) = -\infty$, which is a contradiction to the the fact that $\Phi[u^*]$ is bounded from below by

$$-\lambda\,(\sup_{x\in\mathbb{R}_+} u^*(x)) \int_0^\infty \overline{F}(z)\,dz - c\overline{F}(x) \geq -\lambda K - c, \tag{4.4.6}$$

where

$$K := \int_0^\infty \overline{F}(x)dx, \tag{4.4.7}$$

which is bounded by assumption. Therefore $\lim_{x\to\infty} u^*(x) = 0$ and assertion (i) is proved.

In order to prove (ii), we estimate – using that u^* and $\overline{F}$ are decreasing –

$$\begin{aligned} 0 < -\Phi[u^*](x) &\leq \lambda \int_0^{x/2} u^*(x-z)\overline{F}(z)\,dz + \lambda \int_{x/2}^{x} u^*(x-z)\overline{F}(z) + c\overline{F}(x) \\ &\leq \lambda \int_0^{x/2} u^*(x-z)\overline{F}(z)\,dz + \lambda \int_0^{x/2} u^*(z)\overline{F}(x-z) + c\overline{F}(x) \\ &\leq \lambda u^*(x/2)K + \lambda\overline{F}(x/2) + c\overline{F}(x), \end{aligned} \tag{4.4.8}$$

so that (i) and $\overline{F} \in \mathcal{R}_\rho$, for $\rho < -1$, imply that

$$\lim_{x\to\infty} \Phi[u^*](x) = 0. \tag{4.4.9}$$

But now we can apply de'l Hospital's rule, (i) and (4.4.5) to obtain

$$\lim_{x\to\infty} xu^*(x) = \lim_{x\to\infty} \frac{x}{\frac{1}{u^*(x)}} = \lim_{x\to\infty} \frac{1}{(\frac{1}{u^*})'(x)} = -2 \lim_{x\to\infty} \Phi[u^*](x) = 0. \tag{4.4.10}$$

Hence we have completed the proof of (ii).

In order to finally show (iii), we estimate similarly as in (4.4.8)

$$\begin{aligned} 0 &< -\sqrt{x}\Phi[u^*](x) \\ &\leq \lambda\sqrt{x} \int_0^{x-\sqrt{x}} u^*(x-z)\overline{F}(z)\,dz + \lambda\sqrt{x} \int_{x-\sqrt{x}}^{x} u^*(x-z)\overline{F}(z)\,dz + c\sqrt{x}\,\overline{F}(x) \\ &\leq \lambda\sqrt{x}u^*(\sqrt{x})K + \lambda x\overline{F}(x-\sqrt{x}) + c\sqrt{x}\,\overline{F}(x), \end{aligned} \tag{4.4.11}$$

so that (ii) and $\overline{F} \in \mathcal{R}_\rho$, for $\rho < -1$, yield that $\lim_{x\to\infty} \sqrt{x}\Phi[u^*](x) = 0$. But this, together with de l'Hospital's rule, provides us with the desired result

$$\lim_{x\to\infty} x^{3/2}u^*(x) = \lim_{x\to\infty} \frac{x^{3/2}}{\frac{1}{u^*(x)}} = \frac{3}{2} \lim_{x\to\infty} \frac{\sqrt{x}}{(\frac{1}{u^*})'(x)} = -3 \lim_{x\to\infty} \sqrt{x}\Phi[u^*](x) = 0. \tag{4.4.12}$$

To sum it up, we have shown that, if ρ is a number strictly greater than -1.5, then there exists a $\gamma > 0$ such that $\lim_{x\to\infty} x^\gamma u^*(x)/\overline{F}(x) = 0$, e.g., $\gamma = (1.5 + \rho)/2$.

Case 2. $\rho \leq -1.49$.

Since we use kind of an inductive method (as for the proof of **Case 1**), we split up the proof of **Case 2** into two steps.

Step 1. If

$$\beta \geq 0.98 \quad \text{and} \quad \alpha \geq \beta + \frac{1}{2} - \frac{1}{2\beta}, \tag{4.4.13}$$

as well as

$$\lim_{x\to\infty} x^{\beta} u^*(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} x^{\alpha}\overline{F}(x) = 0, \tag{4.4.14}$$

hold, then

$$\lim_{x\to\infty} x^{\beta+1/2} u^*(x) = 0. \tag{4.4.15}$$

Proof of Step 1. Similarly as for the proof of **Case 1**, we estimate $\Phi[u^*]$ in the following way

$$\begin{aligned} 0 &< -x^{\beta-1/2}\Phi[u^*](x) \\ &\leq \lambda x^{\beta-1/2}u^*(x^{1-1/2\beta})K + \lambda x^{\beta-1/2}x^{1-1/2\beta}\overline{F}(x - x^{1-1/2\beta}) + cx^{\beta-1/2}\overline{F}(x) \\ &\leq \lambda x^{\beta-1/2}u^*(x^{1-1/2\beta})K + \lambda x^{\beta+1/2-1/2\beta}\overline{F}(x - x^{1-1/2\beta}) + cx^{\beta-1/2}\overline{F}(x). \end{aligned} \tag{4.4.16}$$

Now since by assumption

$$\lim_{x\to\infty} x^{\beta} u^*(x) = 0, \tag{4.4.17}$$

the first term in (4.4.16) tends to zero as $x \to \infty$, because of

$$\lim_{x\to\infty} x^{\beta-1/2}u^*(x^{1-1/2\beta}) = \lim_{x\to\infty} (x^{1-1/2\beta})^{\beta} u^*(x^{1-1/2\beta}) = 0. \tag{4.4.18}$$

The second term also tends to 0, for $x \to \infty$, because we assumed that $\lim_{x\to\infty} x^{\beta+1/2-1/2\beta}\overline{F}(x) = 0$. The same holds for the third term, since $\beta - 1/2 \leq \beta + 1/2 - 1/2\beta$ for $\beta \geq 1/2$, and we assumed that $\beta \geq 0.98$. To sum it up,

$$\lim_{x\to\infty} x^{\beta-1/2}\Phi[u^*](x) = 0. \tag{4.4.19}$$

Now we can apply de l'Hospital's rule to finally arrive at

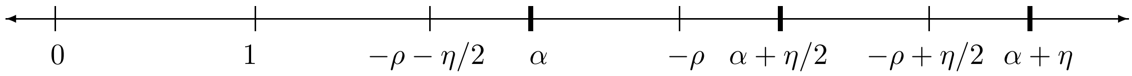
$$\begin{aligned} &\lim_{x\to\infty} x^{\beta+1/2}u^*(x) \\ = \; &\lim_{x\to\infty} \frac{x^{\beta+1/2}}{\frac{1}{u^*(x)}} \\ = \; &(\beta + \frac{1}{2}) \lim_{x\to\infty} \frac{x^{\beta-1/2}}{(\frac{1}{u^*})'(x)} \\ = \; &-(2\beta+1) \lim_{x\to\infty} x^{\beta-1/2}\Phi[u^*](x) \\ = \; &0. \end{aligned} \tag{4.4.20}$$

This concludes the proof of *Step 1.*

Step 2. From equation (4.1.8), we know that, for all $\eta > 0$, we can find an $\alpha \in (-\rho - \eta/2, -\rho)$ such that

$$\begin{aligned} \lim_{x\to\infty} x^{\alpha+\eta}\overline{F}(x) &= \infty \\ \lim_{x\to\infty} x^{\alpha}\overline{F}(x) &= 0. \end{aligned} \tag{4.4.21}$$

We illustrate this graphically



Now, let the map $T : \{y \in \mathbb{R}_+ : \ y \geq 1.48\} \to \mathbb{N} \cup \{0\}$ be defined by

$$T(y) := \inf\{n \in \mathbb{N} \cup \{0\} : \ y - \frac{n}{2} \leq 1.48\}. \tag{4.4.22}$$

Then since $\alpha + 1/2\alpha > 1.48$, for $\alpha \in (1, -\rho)$, we can apply T to $\alpha + 1/2\alpha$. As $T(\alpha + 1/2\alpha)$ is the infimum over all $n \in \mathbb{N} \cup \{0\}$ such that $\alpha + 1/2\alpha - n/2 \leq 1.48$, it follows trivially that $\alpha + 1/2\alpha - T(\alpha + 1/2\alpha)/2 \leq 1.48$, and therefore that

$$\beta := \alpha + \frac{1}{2\alpha} - \frac{T(\alpha + \frac{1}{2\alpha})}{2} - \frac{1}{2} \leq 0.98. \tag{4.4.23}$$

The fact $\rho \leq -1.49$ implies that

$$\lim_{x\to\infty} x^{\alpha}\overline{F}(x) = 0, \tag{4.4.24}$$

because $\alpha \in (-\rho - \eta/2, -\rho)$.

Now it follows from *Step 1* that

$$\lim_{x\to\infty} x^{\beta+1/2}u^*(x) = \lim_{x\to\infty} x^{\alpha+1/2\alpha - T(\alpha+1/2\alpha)/2}u^*(x) = 0. \tag{4.4.25}$$

If $T(\alpha + 1/2\alpha) = 0$, we obtain

$$\lim_{x\to\infty} x^{\alpha+1/2\alpha}u^*(x) = 0. \tag{4.4.26}$$

Using this and choosing η small enough in (4.4.21), we get the assertion of the Lemma with $\gamma = 1/(4\alpha)$. Thus we can assume without loss of generality (w.l.o.g.) that $T(\alpha + 1/2\alpha) \geq 1$.

From the definition of T as an infimum it follows that

$$\alpha + 1/2\alpha - T(\alpha + 1/2\alpha)/2 \geq 1.48 - \frac{1}{2} = 0.98. \tag{4.4.27}$$

Setting $\beta_0 = \alpha + 1/2\alpha - T(\alpha + 1/2\alpha)/2$, it follows that $\beta_0 \geq 0.98$. Moreover, for $T(\alpha + 1/2\alpha) \geq 1$, β_0 satisfies

$$\begin{aligned} & \beta_0 + \frac{1}{2} - \frac{1}{2\beta_0} \\ = \; & \alpha + (\frac{1}{2} - \frac{T(\alpha + 1/2\alpha)}{2}) + (\frac{1}{2\alpha} - \frac{1}{2\beta_0}) \\ \leq \; & \alpha, \end{aligned} \tag{4.4.28}$$

since $\beta_0 \leq \alpha$, for $T(\alpha + 1/2\alpha) \geq 1$.

Hence β_0 and α satisfy the conditions of *Step 1*. Applying *Step 1* we obtain that

$$\lim_{x\to\infty} x^{\beta_0+1/2} u^*(x) = 0. \tag{4.4.29}$$

If $T(\alpha + 1/2\alpha) - 1 = 0$, equation (4.4.29) is just equal to (4.4.26) and thus we obtain the desired result. If $T(\alpha+1/2\alpha)-1 \geq 1$, we can define $\beta_1 = \beta_0 + 1/2$ and proceed in exactly the same way as we did with β_0. In particular,

$$\begin{aligned} & \beta_1 + \frac{1}{2} - \frac{1}{2\beta_1} \\ = \; & \alpha + (\frac{1}{2} - \frac{T(\alpha + 1/2\alpha) - 1}{2}) + (\frac{1}{\alpha} - \frac{1}{\beta_1}) \\ \leq \; & \alpha, \end{aligned} \tag{4.4.30}$$

since

$$\begin{aligned} \beta_1 &= \alpha + \frac{1}{2\alpha} - \frac{T(\alpha + 1/2\alpha) - 1}{2} \\ &\leq \alpha + \frac{1}{2} - \frac{T(\alpha + 1/2\alpha) - 1}{2} = \alpha, \end{aligned}$$

where we used that $T(\alpha + 1/2\alpha) \geq 1$.

Applying *Step 1* to β_1 and α, we arrive at

$$\lim_{x\to\infty} x^{\beta_1+1/2} u^*(x) = 0. \tag{4.4.31}$$

Repeating this procedure we arrive (after a finite number of steps) at

$$\lim_{x\to\infty} x^{\alpha+1/2\alpha} u^*(x) = 0, \tag{4.4.32}$$

and hence at the desired result.

□

In the sequel, we want to apply a result from Luxemburg [24], which deals with the asymptotic behavior of convolution integrals. As this result is formulated for so called *admissible functions*, we have to make sure that we can apply the result in our case. This is done in the following easy lemma.

Lemma 4.4.2. *Let* $\overline{F}$ *be in* $\mathcal{R}_\rho$*, for* $\rho < -1$*. Then* $\overline{F}$ *is* admissible in the sense of Luxemburg*:* $\overline{F}$ *is continuous and strictly positive for all* $x > 0$ *and it satisfies*

(i) $\lim_{x\to\infty} \overline{F}(x+z)/\overline{F}(x) = 1$ *for all* $z > 0$*.*

(ii) There exists a constant $\lambda \geq 1$ *such that, for all* $x \geq 0$*,*

$$\max\{\overline{F}(y):\ x \leq y \leq 2x\} \leq \lambda \overline{F}(2x)$$

holds.

Proof. The tail distribution is trivially continuous and strictly positive.

To show (i), we use Theorem 4.1.5. The quotient $\overline{F}(x+z)/\overline{F}(x)$ can be written as

$$\frac{\overline{F}(x+z)}{\overline{F}(x)} = \frac{\overline{F}(\lambda(x,z)x)}{\overline{F}(x)}, \tag{4.4.33}$$

for $\lambda(x,z) := 1 + z/x \in (1,\infty)$. Since $\overline{F}(\lambda x)/\overline{F}(x)$ converges uniformly in λ to λ^ρ on $[1,\infty)$, we can deduce that

$$\lim_{x\to\infty} \frac{\overline{F}(\lambda(x,z)x)}{\overline{F}(x)} = \lim_{x\to\infty} \lambda(x,z)^\rho = 1. \tag{4.4.34}$$

To prove (ii), we note that $\overline{F}$ is monotonously decreasing, and therefore

$$\max\{\overline{F}(y):\ x \leq y \leq 2x\} = \overline{F}(x). \tag{4.4.35}$$

But from the definition of regular variation

$$\lim_{x\to\infty} \frac{\overline{F}(x)}{\overline{F}(2x)} = 2^{-\rho}. \tag{4.4.36}$$

Thus it is easy to find a constant $\lambda > 1$ such that $\overline{F}(x)/\overline{F}(2x) \leq \lambda$.

□

Lemma 4.4.3. *Let* $\overline{F}$ *be of regular variation with index* ρ*, where* $\rho < -1$*, and let* u^* *be the solution of equation* (4.3.15)*. Then*

$$\overline{F} * u^*(x) \sim \overline{F}(x) \int_0^\infty u^*(z)\,dz. \tag{4.4.37}$$

Remark. Note that we have normalized u^* such that $u^*(0) = 1$. Therefore $\int_0^\infty u^*(z)\,dz$ is no longer equal to 1.

Proof. First, we split up the convolution in the following way (see also the proof of Lemma 4.4.1)

$$\begin{aligned}\overline{F} * u^*(x) &= \int_0^{x/2} \overline{F}(x-z)u^*(z)\,dz + \int_0^{x/2} u^*(x-z)\overline{F}(z)\,dz \\ &= \overline{F}(x)\left(\int_0^{x/2}(p(x-z)-1)u^*(z)\,dz + \int_0^{x/2} u^*(z)\,dz\right) \\ &\quad + \int_0^{x/2} u^*(x-z)\overline{F}(z)\,dz, \qquad (4.4.38)\end{aligned}$$

where we have defined

$$p(y) := \frac{\overline{F}(y)}{\overline{F}(x)}. \qquad (4.4.39)$$

□

We claim that the first integral on the right hand side,

$$\int_0^{x/2}(p(x-z)-1)u^*(z)\,dz, \qquad (4.4.40)$$

tends to zero for $x \to \infty$, and thus that

$$\overline{F}(x)\int_0^{x/2}(p(x-z)-1)u^*(z)\,dz = o(\overline{F}). \qquad (4.4.41)$$

The argument is nearly the same as in the proof of Theorem 2.2 of LUXEMBURG [24]. For the convenience of the reader, we repeat it here.

Choose $\epsilon > 0$ and observe that $u^* \in L^1[0,\infty)$. This implies that there exists a constant c, such that

$$\int_c^{\infty} u^*(z)\,dz < \epsilon. \qquad (4.4.42)$$

Furthermore, we can use property (ii) of Lemma 4.4.2 to show that, for all $x > 2c$ (i.e. $x/2 > c$), and for all $x/2 \le y \le x - c$,

$$p(x-y) = \frac{\overline{F}(x-y)}{\overline{F}(x)} \le \lambda, \qquad (4.4.43)$$

for some $\lambda \ge 1$.

For all $x > 2c$, we can write

$$\begin{aligned}&\int_0^{x/2}(p(x-z)-1)u^*(z)\,dz \\ &= \int_0^{c}(p(x-z)-1)u^*(z)\,dz + \int_c^{x/2}(p(x-z)-1)u^*(z)\,dz. \qquad (4.4.44)\end{aligned}$$

Now the first integral on the right hand side of (4.4.44) tends to zero for $x \to \infty$, because for $0 \le z \le c$, the limit of $p(x-z)$ is one by property (i) of Lemma 4.4.2.

The second integral on the right hand side of equation (4.4.44) can be estimated by

$$\begin{aligned} & \int_c^{x/2} (p(x-z)-1)u^*(z)\,dz \\ \le\ & \sup_{z\in[c,x/2]} (p(x-z)-1) \int_c^{x/2} u^*(z)\,dz \\ \le\ & (\lambda+1) \int_c^{\infty} u^*(z)\,dz. \end{aligned} \tag{4.4.45}$$

Now we have chosen c such that

$$\int_c^{\infty} u^*(z)\,dz < \epsilon. \tag{4.4.46}$$

Since ϵ was arbitrary, it follows that

$$\lim_{x\to\infty} \int_0^{x/2} (p(x-z)-1)u^*(z)\,dz = 0, \tag{4.4.47}$$

just as we have claimed.

The second integral on the right hand side of equation (4.4.38) is of order $O(\overline{F})$, for $x \to \infty$, because u^* is integrable, i.e.

$$\overline{F}(x) \int_0^{x/2} u^*(z)\,dz = O(\overline{F}). \tag{4.4.48}$$

Let us now consider the third integral in (4.4.38),

$$\int_0^{x/2} u^*(x-z)\overline{F}(z)\,dz, \tag{4.4.49}$$

which we will denote by $I_3(x)$ in the following. Our aim is to show that $I_3 = o(\overline{F})$, i.e. that $I_3(x)/\overline{F}(x) \to 0$ for $x \to \infty$. We find

$$\begin{aligned} \frac{I_3(x)}{\overline{F}(x)} &= \frac{\overline{F}(x/2)}{\overline{F}(x)} \int_0^{x/2} \frac{u^*(x-z)}{\overline{F}(x/2)} \overline{F}(z)\,dz \\ &\le \frac{\overline{F}(x/2)}{\overline{F}(x)} \frac{u^*(x/2)}{\overline{F}(x/2)} \int_0^{x/2} \overline{F}(z)\,dz, \end{aligned} \tag{4.4.50}$$

because u^* is monotone decreasing.

Now,

$$\frac{\overline{F}(x/2)}{\overline{F}(x)} \tag{4.4.51}$$

can be bounded by some constant, because $\overline{F}$ is of regular variation (see the proof of property (ii) of Lemma 4.4.2).

The quotient

$$\frac{u^*(x/2)}{\overline{F}(x/2)} \tag{4.4.52}$$

tends to 0, for $x \to \infty$, because of Lemma 4.4.1. Since we have assumed that $\overline{F}$ has a finite integral, we obtain that $I_3(x)/\overline{F}(x) \to 0$, for $x \to \infty$.

Finally we arrive at

$$\begin{aligned} \overline{F} * u^*(x) &\sim \overline{F}(x) \int_0^{x/2} u^*(z)\, dz \\ &\sim \overline{F}(x) \int_0^{\infty} u^*(z)\, dz, \end{aligned} \tag{4.4.53}$$

which concludes the proof of the lemma.

□

Now we finally have all the necessary tools at hand to prove the Main Theorem 4.2.1.

4.5 Proof of the Main Theorem

For the reader's convenience we restate the main theorem in this place.

Theorem 1 (Main theorem – Tails of regular variation). *Consider the risk process defined in Section 1.1, and the investment opportunity, introduced in Section 1.4. Assume a positive safety loading,* $c - \lambda\mu > 0$. *Let* $\overline{F}$*, the tail distribution of the claim size, be of regular variation with index* $\rho < -1$. *Then, also the minimal ruin probability is of regular variation with index* ρ*, i.e.* $\Psi^*(x) \in \mathcal{R}_\rho$.

Proof of Main Theorem. We first recapitulate the fact that from $f \sim g$ and $f \in \mathcal{R}_\rho$ it follows that $g \in \mathcal{R}_\rho$ (Lemma 4.1.8).

Using this fact, Lemma 4.4.1 and Lemma 4.4.3 we get from equation (4.3.15) that

$$-\frac{u^{*2}}{u^{*\prime}} = \frac{1}{(\frac{1}{u^*})'} \in \mathcal{R}_\rho. \tag{4.5.1}$$

An elementary property of regularly varying functions gives

$$\left(\frac{1}{u^*}\right)' \in \mathcal{R}_{-\rho}. \tag{4.5.2}$$

Now using Karamata's Theorem (Theorem 4.1.6) yields

$$\frac{1}{u^*} \in \mathcal{R}_{-\rho+1} \tag{4.5.3}$$

or

$$u^* \in \mathcal{R}_{\rho-1}. \tag{4.5.4}$$

Again thanks to Karamata's Theorem we end up with

$$\Psi^* \in \mathcal{R}_{\rho}. \tag{4.5.5}$$

□

Corollary 4.5.1. *Let $\overline{F}(x) = x^\rho \ell(x)$, where $\rho < -1$ and $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ is a slowly varying function. Then the following holds*

$$\Psi(x) = 2\lambda \frac{b^2}{a^2} \frac{\rho - 1}{\rho} x^\rho \ell_1(x), \tag{4.5.6}$$

where ℓ_1 is a slowly varying function satisfying

$$\lim_{x\to\infty} \frac{\ell(x)}{\ell_1(x)} = 1, \tag{4.5.7}$$

and where a resp. b denote the drift resp. volatility of the stock price.

Proof. The proof of this corollary is just a matter of bookkeeping in the previous proof of this chapter's Main Theorem:

Remember that in the beginning, we replaced $c \cdot b^2/a^2$ resp. $\lambda \cdot b^2/a^2$ by c resp. λ. Thus when we look at equation (4.3.15), we see that it actually says

$$-\frac{1}{2} \frac{u^{*2}}{u^{*\prime}}(x) = \frac{b^2}{a^2} \lambda x^\rho \ell(x) + o(\overline{F}). \tag{4.5.8}$$

Proceeding in exactly the same way as before and using Karamata's Theorem (Theorem 4.1.6) in its full generality, we obtain the desired result.

□

Corollary 4.5.2. *Let $\overline{F}(x) = x^\rho \ell(x)$, where $\rho < -1$ and $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ is a slowly varying function. Then the optimal investment strategy K^* satisfies*

$$\lim_{x\to\infty} K^*(x) = \frac{a}{b^2} \frac{1}{1-\rho}. \tag{4.5.9}$$

Proof. According to Theorem 4.3.1 from Hipp and Plum [17], the optimal strategy is given by

$$K^*(x) = -\frac{a}{b^2} \frac{u^*(x)}{u^{*\prime}(x)}. \tag{4.5.10}$$

Now an easy application of Corollary 4.5.2 and the Monotone Density Theorem (Theorem 4.1.7) yields that $\lim_{x\to\infty} K^*(x) = a/(b^2(1-\rho))$.

□

Index of Notation

$(\Omega, \mathcal{F}, \mathbb{P})$	Complete probability space, page 11
a	Drift, page 18
(a, b, ν)	Characteristic triplet of a Lévy process, page 46
b	Volatility, page 18
$\mathcal{B}^d$	Borel sigma algebra of $\mathbb{R}^d$, page 45
c	Premium rate over time, page 11
$\mathcal{D}$	Class of dominatedly varying distributions, page 14
$\delta^*(x)$	Optimal survival probability, page 59
$\mathbb{E}_t$	Conditional Expectation, page 18
$\mathbb{F}, \mathcal{F}_t$	Filtration: exponential Lévy, page 49, and GBM, page 18
F	Distribution function of the claim size X, page 11
$f(K, r)$	Important function, page 21
$\hat{F}$	Moment generating function of X, page 13
$\overline{F}_I$	Integrated tail distribution, page 16
$\overline{F}$	Tail distribution function of the claim size X, page 12
$f(k, r)$	Important function, page 51
h	Moment generating function of X, shifted s.t. $h(0) = 0$, page 15
h	Truncation function
$\mathcal{H}$	Set of all heavy tailed distributions, page 13
$H(dt, dy)$	Joint distribution of τ and $Y(\tau-)$ conditional on ruin through a jump, page 28

i	Real interest force, page 25
K	Investment strategy in monetary units: exponential Lévy, page 49, and GBM, page 18
$k(x)$	Strategy in dependance of the level of wealth, page 31
$\hat{K}$	For exponential Lévy process: page 50, and for GBM, page 21
$\hat{K}^{(i)}$	$\hat{K}$ for positive real interest, page 25
$\mathcal{K}$	Set of admissible strategies: exponential Lévy, page 49, and GBM, page 18
K^*	Optimal strategy, page 19
k^*	Optimal strategy in dependence of level of wealth, page 33
$\tilde{Y}(t,x,K)$	$Y(t,x,K)$, stopped at the the time of ruin, page 22
ℓ	Slowly varying function, page 13
$\mathcal{L}$	Subclass of $\mathcal{H}$, page 13
λ	Intensity of the Poisson process N, page 11
$M(t,x,K,r)$	Exponential martingale introduced by Gerber, page 21
$\mathcal{B}_+$	Borel sigma algebra of $\mathbb{R}_+$, page 45
$\overline{M}^{(i)}$	Exponential martingale for positive real interest, page 25
$\tilde{M}^*$	Supremum process, page 28
$\tilde{M}(t,x,K,r)$	$M(t,x,K,r)$, stopped at the time of ruin, page 22
μ	Mean of the claim size, page 16
$\mu(.;dt,dx)$	Random measure, page 45
$\mu^X(,;(0,dt],dx)$	Random measure of jumps, page 46
$\mu^{X,c}(.;(0,dt],dx)$	Compensator of μ^X, page 46
N	Poisson process, page 11
ν	Lundberg exponent, refpage
$O(f),\ o(f)$	Landau symbols, page 56
ϕ	Characteristic function

$\Phi[u]$	Operator, page 62
$\Psi(x)$	Ruin probability without investment, page 11
$\Psi^*(x)$	Minimal ruin probability, page 19
$\Psi(x, K)$	Ruin probability, starting from x and using strategy K, page 18
R	risk process, page 11
$\mathcal{R}$	Class of distributions with regularly varying tails, page 13
$\hat{r}$	Lundberg coefficient with investment, page 21
$\hat{r}^{(i)}$	Lundberg exponent with investment with positive real interest, page 25
ρ	(Relative) safety loading, page 12
ρ_{ij}	Covariance matrix, page 40
r_∞	$h(r) \uparrow \infty$ for $r \uparrow r_\infty$, page 15
$\mathcal{R}_0$	Set of functions of slow variation, page 13
$\mathcal{R}_\rho$	Set of functions of regular variation with index ρ, page 13
S	Investment possibility: exponential Lévy, page 49, and GBM, page 18
$\mathcal{S}$	Class of subexponential distributions, page 13
$S_i, a_i, \ldots$	ith risky asset, drift of the ith asset, ..., page 40
T	Page 65
$\tau(x)$	Time to ruin without investment, page 11
$\tau(x, K)$	Time to ruin, starting from x and using strategy K, page 18
τ_z	First hitting time of the level z, page 34
$u^*(x)$	First derivative of $\Psi^*(x)$, page 61
W	Standard Brownian motion, page 18
$W_{a,b}$	Generalized Wiener process with drift a and standard deviation b, page 18
X	positive r.v. representing claim size, page 11
x	Initial reserve of insurer, page 11

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