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Contribution à l'étude de processus univariés et multivariés de la théorie de la ruine.

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par

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Je dédie cette thèse
à Ti Toy.

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Introduction générale

Introduction

La théorie de la ruine concerne la définition et l'étude de processus stochastiques introduits dans la modélisation de l'évolution de la richesse d'une compagnie d'assurances. L'objet de cette thèse est d'approfondir certains aspects mathématiques récemment développés dans ce domaine et de proposer quelques nouveaux concepts. Cette introduction vise à présenter et à remettre dans leur contexte cinq articles acceptés ou en cours de soumission, qui constituent les chapitres de cette thèse, et non à exposer une revue exhaustive du sujet. Les ouvrages suivants (dans l'ordre chronologique) : Gerber (1979); Grandell (1991); Rolski et al. (1999); Asmussen (2000) donnent un aperçu général des principaux résultats de cette théorie.
Le concept de *ruine* doit se comprendre dans ce travail comme la survenance d'un scénario défavorable, pouvant conduire à l'impossibilité, pour la compagnie, de faire face à certains de ses engagements, aussi bien envers ses assurés que ses actionnaires, voire à devoir cesser son activité pour cause d'insolvabilité.
Le but premier de la théorie de la ruine a donc logiquement été de modéliser l'évolution de la richesse de la compagnie par un processus stochastique, d'évaluer la probabilité de ruine, c'est-à-dire la probabilité que le scénario traduisant un échec se réalise, et d'estimer le niveau de réserve initiale pour rendre cette probabilité de ruine suffisamment faible.

Dans de nombreux modèles, on dispose d'expressions asymptotiques de la probabilité de ruine, quand le niveau de richesse initial est très élevé.Un des apports de cette thèse est de proposer des méthodes efficaces de calcul de probabilités de ruine, pour certains modèles, et quel que soit le montant de la réserve initiale.
Le modèle classique, fondé sur un processus de Poisson composé avec dérive, a considérablement évolué. L'ensemble des problèmes que l'on se pose s'est, lui aussi, fortement enrichi. Ainsi sont apparus des modèles fondés sur des processus de renouvellement, éventuellement modulés par un processus markovien décrivant l'état de l'environnement, qui peut être de nature économique, climatique, juridique, etc... Des stratégies de versements de dividendes, la prise en compte du taux d'intérêt ont également été introduites dans la modélisation. Les distributions des variables aléatoires représentant les montants de sinistres qui, dans les cas les plus simples, sont de type discret, exponentiel, gamma, ou même *phase-type*, peuvent être aussi des distributions générales de $\mathbb{R}^+$. Le concept de ruine a été étendu à l'évaluation de mesures de risques plus générales (sévérité de la ruine, fonctions de pénalité,...), au calcul d'espérances de montants (actualisés ou non) de dividendes versés aux actionnaires.
Dans cette voie, nous proposons un théorème de différentiation de fonctionnelles de processus de risque, qui permet d'introduire des nouvelles fonctions de pénalité et d'en obtenir les valeurs espérées.
Un des aspects importants de cette thèse est aussi de s'intéresser à l'évolution conjointe de processus représentant les niveaux de fonds propres de différentes branches d'activité d'une compagnie d'assurances. Outre le fait que, dans ces modèles multidimensionnels, il convient de bien définir

la dépendance stochastique entre les processus marginaux, ce changement de dimension amène à considérer de nouveaux concepts. Au delà du calcul de probabilités de ruine ou de sévérités moyennes de la ruine de la compagnie dans son activité globale, on s'intéresse aussi à la survenance d'événements défavorables, comme l'insolvabilité temporaire d'une (ou de plusieurs) des branches, ou encore le non paiement de dividendes pour cause de réemploi de l'excès de réserve d'une branche pour payer une pénalité due au déficit d'une autre. Il faut donc réfléchir à la pertinence des nouvelles définitions possibles du concept de mesure de risque multidimensionnelle. Dans ce cadre multidimensionnel, la deuxième partie est consacrée à la présentation de résultats obtenus sur l'allocation optimale des fonds propres initiaux dans les différentes branches associée à certaines mesures de risque, sur le calcul de certaines probabilités de ruine et de temps moyen de ruine, et sur le versement de dividendes et la situation financière des autres branches au moment où l'une d'entre elles est ruinée.
Dans la suite de cette introduction sont développés les points évoqués ci-dessus, en commençant par décrire le modèle classique, puis en présentant les généralisations et les nouveaux problèmes qui ont inspiré les travaux exposés dans les chapitres de cette thèse.

Le modèle classique de la théorie de la ruine

Le modèle classique de la théorie de la ruine représente le fonctionnement d'une compagnie d'assurance de la façon suivante.
On suppose que la compagnie d'assurance reçoit des cotisations de ses assurés, appelées primes, de façon déterministe et continue, à raison de c unités de compte par unité de temps. Elle dispose d'une réserve initiale u pour absorber un éventuel excès de sinistralité, et doit indemniser ses assurés pour les sinistres qui la concernent.
Le montant cumulé des sinistres au temps $t \geq 0$ est représenté par le processus stochastique

$$S(t) = \sum_{i=1}^{N(t)} W_i,$$

avec la convention selon laquelle la somme est nulle si $N(t) = 0$. Le nombre de sinistres survenus jusqu'au temps t, $N(t)$, est dans ce modèle décrit par un processus de Poisson de paramètre $\lambda > 0$. Le montant du $i^{\text{ième}}$ sinistre est modélisé par une variable aléatoire W_i à valeurs dans $\mathbb{R}^+$. Les $\{W_i, i \in \mathbb{N}^*\}$ forment une suite de variables aléatoires indépendantes, identiquement distribuées, et indépendantes du processus de Poisson $N(t)$. F_W désignera leur fonction de répartition, et leur espérance supposée finie sera notée μ. Le montant des réserves de la compagnie d'assurances au temps t est alors donné par le processus

$$R^u(t) = u + ct - S(t). \tag{1}$$

En absence d'ambiguïté, $R^u(t)$ sera le plus souvent noté $R(t)$. La probabilité de ruine en temps fini t avec réserve initiale u correspond à la probabilité que la réserve devienne strictement négative à un instant précédant t, et se note traditionnellement

$$\psi(u,t) = \mathbb{P}\left[\exists s \in [0,t], \quad R(s) < 0\right].$$

En temps infini, définissons la probabilité de ruine par

$$\psi(u) = \psi(u,+\infty) = \mathbb{P}\left[\exists s \geq 0, \quad R(s) < 0\right].$$

Les probabilités de non ruine correspondantes seront notées

$$\varphi(u,t) = 1 - \psi(u,t) \quad \text{et} \quad \varphi(u) = 1 - \psi(u).$$

Le chargement de sécurité est défini par

$$\rho = c - \lambda\mu.$$

Si $\rho > 0$, alors l'activité est dite rentable. En effet, la loi des grands nombres assure que, dans ce cas, le processus R_t tend vers $+\infty$ presque sûrement quand t tend vers l'infini. La probabilité de non ruine est alors non nulle. Si $\rho < 0$, alors R_t tend vers $-\infty$ presque sûrement quand t tend vers l'infini, et par conséquent $\psi(u) = 1$. Généralement, nous ferons l'hypothèse que l'activité est rentable.
La probabilité de non ruine $\varphi(u)$ est continue sur $\mathbb{R}_+$ et admet comme dérivées à gauche et à droite respectivement φ'_- et φ'_+, qui vérifient les équations intégro-différentielles respectives :

$$c\varphi'_+(u) = \lambda(\varphi(u) - \int_0^u \varphi(u-y)dF_W(y)) \tag{2}$$

$$c\varphi'_-(u) = \lambda(\varphi(u) - \int_0^{u-} \varphi(u-y)dF_W(y)) \tag{3}$$

Les équations 2 et 3 font intervenir à la fois une dérivée et une intégrale de φ. Il est possible de se ramener à une équation intégrale en intégrant 2 :

$$c\psi(u) = \lambda(\int_u^\infty \bar{F}_W(x)dx + \int_0^u \psi(u-x)\bar{F}_W(x)dx)$$

Les transformées de Laplace respectives $\hat{L}_\psi(s) = \int_0^\infty \psi(u)e^{-su}du$ et $\hat{L}_\varphi(s) = \int_0^\infty \varphi(u)e^{-su}du$ de ψ et de φ s'expriment alors en fonction de la transformée de Laplace $\hat{l}_W$ de W :

Proposition .1 *Pour tout $s > 0$,*

$$\hat{L}_\varphi(s) = \frac{c - \lambda\mu}{cs - \lambda(1 - \hat{l}_W(s))}$$

$$\hat{L}_\psi(s) = \frac{1}{s} - \frac{c - \lambda\mu}{cs - \lambda(1 - \hat{l}_W(s))}$$

On peut ensuite obtenir la *formule de Pollaczek-Khinchine* qui met en jeu des produits de convolution infinis de la fonction de survie intégrée F_W^s de W définie pour $x \geq 0$ par

$$F_W^s(x) = \frac{1}{\mu}\int_0^x \bar{F_W}(y)dy,$$

où $\bar{F_W}(y) = 1 - F_W(y)$ est la fonction de survie de W. Dans toute la suite, g^{*n} désignera le $n^{\text{ième}}$ produit de convolution d'une fonction g avec elle-même.

Proposition .2 (Formule de Pollaczek-Khinchine) *Pour tout $u \geq 0$,*

$$\psi(u) = (1 - \frac{\lambda\mu}{c}\sum_{n=1}^{\infty}(\frac{\lambda\mu}{c})^n(\bar{F_W^s})^{*n}(u)).$$

Cette formule fournit un premier moyen d'obtenir numériquement la probabilité de ruine en temps infini.
La première généralisation possible du modèle classique est le modèle dit de Sparre Andersen, dans lequel $N(t)$ n'est plus un processus de Poisson, mais un processus de renouvellement stationnarisé, donné par des temps inter-sauts Δ_i i.i.d., indépendants des W_i, et de fonction de répartition F_Δ. Ce cadre sera propice à l'introduction du modèle dual (voir page 18). Lorsque u est grand, on dispose dans ce modèle de renouvellement de résultats asymptotiques.

C'est en travaillant sur ces problèmes que l'école scandinave du début du vingtième siècle (Lundberg, Cramer,...) posera les premiers jalons de la théorie des grandes déviations. Ces résultats asymptotiques, qui portent maintenant le nom de Cramer-Lundberg, fournissent une information sur le comportement de la probabilité de ruine pour des réserves initiales élevées.

Propriétés asymptotiques dans le modèle de Sparre Andersen

Considérons le modèle de Sparre Andersen, toujours avec des montants de sinistres modélisés par des variables aléatoires indépendantes distribués comme U, et des temps inter-sinistres indépendants et distribués comme Δ, et indépendants des montants des sinistres. Supposons aussi que le chargement de sécurité est strictement positif. Le processus décrivant la réserve disponible au temps t s'écrit encore :

$$R(t) = u + ct - \sum_{i=1}^{N(t)} W_i.$$

Soit

$$\hat{m}_W(s) = \mathbb{E}\left[e^{sW}\right].$$

Pour les distributions de montants de sinistres à queues légères, l'équation en s

$$\hat{m}_W(s)\hat{l}_\Delta(cs) = 1$$

admet une solution non nulle γ (appelée le *coefficient d'ajustement*).
Soit $Y = W - c\Delta$, et

$$x_0 = \sup\{x, \quad F_Y(x) < 1\}.$$

Theorem .1 *Pour tout $u \geq 0$, on a l'encadrement*

$$b_- e^{-\gamma u} \leq \psi(u) \leq b_+ e^{-\gamma u},$$

où $0 \leq b_- \leq b_+ \leq 1$ vérifient :

$$b_- = \inf_{x\in[0,x_0[} \frac{e^{\gamma x}\bar{F}_Y(x)}{\int_x^\infty e^{\gamma y} d\bar{F}_Y(y)}, \qquad et \qquad b_+ = \sup_{x\in[0,x_0[} \frac{e^{\gamma x}\bar{F}_Y(x)}{\int_x^\infty e^{\gamma y} d\bar{F}_Y(y)}.$$

Si W suit une loi exponentielle de paramètre $1/\mu$, et si $N(t)$ est un processus de Poisson de paramètre λ, alors la propriété asymptotique devient une égalité vraie pour tout $u \geq 0$:

$$\psi(u) = (1 - \mu\gamma)e^{-\gamma u}.$$

Cette formule peut être généralisée à une famille de lois appelées lois *phase-type*, que nous introduirons plus tard, dont font partie entre autres les lois exponentielles et gamma.

Pour les distributions de montants de sinistres à queues lourdes, si la fonction de survie intégrée est sous-exponentielle, pour $E[W]$ fixée, le comportement asymptotique de la probabilité de ruine $\psi(u)$ ne dépend de W qu'à travers ses queues, et du temps inter-sinistres Δ seulement par rapport à son espérance :

Theorem .2 *Si $F_W^s \in \mathcal{S}$, alors*

$$\lim_{u\to\infty} \frac{\psi(u)}{\bar{F_W^s}(u)} = \frac{E[W]}{cE[\Delta] - E[W]}$$

Dans Rolski et al. (1999) page 267, on peut trouver des équivalents de $\psi(u)$ pour des montants de sinistres suivant des lois de Weibull, Pareto et lognormale.

Méthodes pour le calcul exact ou approché de probabilités de ruine

Le pendant de l'étude asymptotique est le calcul ou l'approximation de la probabilité de ruine en temps fini ou infini. Un des buts de cette thèse est de fournir des méthodes numériques pour calculer explicitement des probabilités de ruine en temps fini, ou d'évaluer des quantités liées à d'autres mesures de risque. La probabilité de ruine en temps fini avec des montants de sinistres à valeurs entières est l'objet du chapitre I.1. Dans ce cadre, le modèle à temps continu coïncide avec certains modèles à temps discret. En effet, pour déterminer si la réserve devient négative à un certain temps $t \in [0, T]$, il suffit de savoir si la réserve est négative à l'une des dates d'inventaires $0 < t_1 < \cdots < t_n < T$ judicieusement choisies.

Formule de Picard-Lefèvre

Pour calculer la probabilité de ruine en temps fini dans ce modèle, on dispose de la formule de Picard et Lefèvre (1997), et nous expliquons comment il est possible d'y adapter une formule de Seal (1969).
La formule de Picard-Lefèvre est basée sur les polynômes d'Appell généralisés. Ces polynômes sont définis récursivement par : $A_0 = 1$ et pour $n > 0$,

$$A_n' = \sum_{j=1}^{n} \lambda f_j A_{n-j} \tag{4}$$

$$A_n(v_n) = 0, \tag{5}$$

où

$$v_n = \max\left(\frac{n-u}{c}, 0\right), \tag{6}$$

et

$$f_j = \mathbb{P}[W_1 = j].$$

Soit

$$T_u = \inf\{t > 0, R^u(t) < 0\}$$

le temps de ruine, éventuellement égal à $+\infty$. Soit pour $t \in \mathbb{R}^+, n \in \mathbb{N}$,

$$P_n(t) = P(S_t = n, T_u > t). \tag{7}$$

En conditionnant par rapport au dernier instant de sinistre, Picard et Lefèvre (1997) montrent que $P_n(t) = 0$ quand $t < v_n$ et que

$$P_n(t) = e^{-\lambda t} A_n(t) \tag{8}$$

quand $t \geq v_n$, où les A_n sont des polynômes de degré n définis récursivement par (4).

Ces polynômes (voir Picard et Lefèvre (1997) pour une bibliographie sur ce sujet) peuvent aussi être obtenus grâce aux polynômes e_n, définis par la fonction génératrice formelle

$$\sum_{n=0}^{+\infty} e_n(t) s^n = e^{tg(s)} \tag{9}$$

où $g(s) = \sum_{j=1}^{+\infty} \lambda f_j s^j$. Les polynômes e_n s'obtiennent assez aisément à partir des convolutions successives de F_W. On peut alors exprimer les A_n en fonction des e_n et en déduire la formule suivante :

Theorem .3 *(Picard et Lefèvre (1997))*

$$\varphi(u,t) = e^{-\lambda t} \sum_{j=0}^{u} \left[e_j(t) + \sum_{n=u+1}^{\lfloor u+ct \rfloor} e_j\left(\frac{j-u}{c}\right) \frac{u+ct-n}{u+ct-j} e_{n-j}\left(t + \frac{u-j}{c}\right) \right] \tag{10}$$

La manipulation de ces polynômes présente l'avantage d'être généralisable au cas multidimensionnel (voir Picard et al. (2003a); Pommeret (2001)), auquel nous nous intéresserons pour étudier les modèles avec plusieurs branches de risque dans les chapitres II.1, II.2 et II.3. La formule "de type Seal" repose sur un résultat de Takács (1962), adaptation du problème du scrutin au processus de Poisson composé avec dérive.

Formules de Takács et de type Seal

Exposons maintenant les formules de Seal pour une distribution de W admettant une densité f_W, et dont la formule exposée dans Rullière et Loisel (2004) est une adaptation au cas de montants de sinistres à valeurs entières. Supposons que $P(W > 0) = 1$. Dans le cas d'une arrivée poissonnienne, cette hypothèse n'est pas restrictive. On peut toujours se ramener au cas $P(W > 0) = 1$, quitte à remplacer le paramètre λ par le paramètre $\lambda(1 - P(W = 0))$ et W par W', tel que $\forall k \geq 1, \quad P(W' = k)(1 - P(W = 0)) = P(W = k)$ et $P(W' = 0) = 0$. La fonction de répartition $F_{S(t)}$ de la variable aléatoire représentant le montant agrégé des sinistres jusqu'au temps t vérifie pour $x \geq 0$

$$F_{S(t)}(x) = e^{-\lambda x} + \int_0^x \tilde{f}_{S(t)}(y) dy,$$

où

$$\tilde{f}_{S(t)}(x) = \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} f_W^{*n}(y),$$

Cette fonction permet de calculer les probabilités $\varphi(u,x)$ de non ruine avant la date x avec réserve initiale u :

Théorème .1 (Formules de Seal)

$$\varphi(0,t) = \frac{1}{ct} E[R(t)_+] = \frac{1}{ct} \int_0^{ct} F_{S(x)}(y)dy$$

et pour $u > 0$ si W admet pour densité f_W,

$$\varphi(u,t) = F_{S(x)}(u+ct) - c\int_0^u \varphi(0,t-y)\tilde{f}_{S(y)}(u+cy)dy.$$

Quand les montants des sinistres suivent une loi exponentielle de paramètre $1/\mu$, on a les formules suivantes :

$$\psi(u,t) = 1 - e^{-u/\mu-(1+\alpha)\lambda t} g(u/\mu + \alpha\lambda t, \lambda t),$$

où $\alpha = c/(\lambda\mu)$, et où g est la fonction définie par

$$g(z,\theta) = J(\theta z) + \theta J'(\theta z) + \int_0^z e^{z-v} J(\theta v)dv - \frac{1}{\alpha}\int_0^{\alpha\theta} e^{\alpha\theta - v} J(zv/\alpha)dv$$

en notant

$$J(x) = \sum_{n\geq 0} \frac{x^n}{n!n!} = I_0(2\sqrt{x})$$

où I_0 est la fonction de Bessel modifiée

$$I_0(x) = \sum_{k=0}^{+\infty} \frac{(x/2)^{2k}}{k!k!}.$$

On peut aussi obtenir une formule équivalente :

$$\psi(u,t) = \frac{1}{\alpha} e^{-u\frac{\alpha-1}{\alpha\mu}} - \frac{1}{\pi} e^{-u/\mu-(1+\alpha)t} \int_0^{\pi} h(u/\mu, \lambda t, y)dy,$$

où h est définie par

$$q(w,\theta,y) = 2\sqrt{\alpha}\frac{e^{(2\sqrt{\alpha}\theta + w/\sqrt{\alpha})\cos y}}{1+\alpha-2\sqrt{\alpha}\cos y}(\sin y \sin(y + \frac{w}{\sqrt{\alpha}}\sin y)).$$

On trouvera dans Seal (1969), Takács (1962), et Rolski et al. (1999) les démonstrations et des extensions.
La formule classique de Takács (1962) est un outil clé pour obtenir notre formule de type Seal.

Lemme .1 *(Takács (1962)) Soit $Y_n = X_1 + .. + X_n$, où $X_i \in \mathbb{N}^*$ forment une suite de variables aléatoires indépendantes et identiquement distribuées, à valeurs dans $\mathbb{N}$. Pour $n \in \mathbb{N}^*$,$i \in \mathbb{N}$, $i \leq n$,*

$$\mathbb{P}\left[\{Y_r < r, r = 1, .., n\} \cap \{Y_n = n - i\}\right] = \frac{i}{n}\mathbb{P}\left[Y_n = n - i\right].$$

Le membre de gauche correspond à la probabilité que le candidat vainqueur d'un scrutin à deux candidats avec le score final n voix contre $n - i$ ait toujours été en tête lors du dépouillement. Cette probabilité s'écrit donc comme une fraction fois la probabilité d'arriver au même score final.
Grâce aux propriétés du processus de Poisson composé, la formule de Takács (1962) donne les deux résultats bien connus suivants dans le cas où W est à valeurs dans $\mathbb{N}$:

Théorème .2 *Takács (1962) Soit $n \in \mathbb{N}^*$.*

$$\begin{aligned}\mathbb{P}\left[S_{\frac{n}{c}} = i \cap R_t \geq 0, t \in [0, \frac{n}{c}]\right] &= \frac{n-i}{n}\mathbb{P}\left[S_{\frac{n}{c}} = i\right], i = 0, .., n \\ \varphi\left(0, \frac{n}{c}\right) &= \sum_{i=1}^{n} \frac{n-i}{n}\mathbb{P}\left[S_{\frac{n}{c}} = i\right] = \frac{1}{n}E\left(R^0_{n/c}\right)_+\end{aligned}$$

Théorème .3 *Pour $t \in \mathbb{R}^+$,*

$$\begin{aligned}\varphi(0,t) &= \frac{1}{ct}E\left(R^0_t\right)_+ \\ \varphi(0,t/c) &= \sum_{i=1}^{[t]}\left(1 - \frac{i}{t}\right)\mathbb{P}\left[S_{\frac{t}{c}} = i\right].\end{aligned}$$

En conditionnant par le dernier temps avant t où le processus $R(t)$ est égal à 0 en cas de ruine, le résultat précédent permettra d'obtenir au chapitre I.1 la formule (1.11) pour la probabilité de ruine en temps fini avec des montants de sinistres à valeurs entières.

Equivalence entre la formule de Picard-Lefèvre et celle de type Seal

Nous expliquons dans l'article Rullière et Loisel (2004), qui constitue le chapitre I.1 de cette thèse, l'équivalence entre ces deux formules. La démonstration s'appuie sur le résultat de Takács (1962) et sur le comportement des trajectoires, et fait intervenir ce que De Vylder (1999) a appelé des pseudo-distributions Poisson composées. Ces pseudo-distributions correspondent à des mesures signées ne permettant pas d'interprétation probabiliste immédiate, mais vérifiant des propriétés de convolution formelle qui permettent de conclure la démonstration, et qui fournissent toute une classe de formules avec un paramètre libre, dont la formule de type Seal et la formule de Picard-Lefèvre font partie. Pouvoir choisir l'une ou l'autre formule présente un intérêt non négligeable, car les temps de calcul sont très différents suivant le rapport entre la réserve initiale et les primes qui seront reçues jusqu'au temps T. Ce phénomène est très visible sur les figures 1.1 à 1.4, et analysé dans le chapitre I.1. De même, d'autres formules y sont proposées pour répondre à d'autres problématiques, comme la détermination de la loi du temps de ruine, ou de la sévérité de la ruine.
Cette notion tempère l'aspect dichotomique de la ruine, considérée plutôt comme un événement défavorable déclencheur d'une certaine pénalité, et dont on mesure la possible gravité à l'aide de mesures de risque.

Autres mesures de risques

Les mesures de risque couramment utilisées en théorie de la ruine sont tout d'abord l'instant de ruine et la sévérité de la ruine. On peut aussi considérer des fonctionnelles de l'instant de ruine, de la sévérité de la ruine et du niveau de richesse juste avant la ruine (comme par exemple les fonctions de pénalité de Gerber et Shiu (1997)). Mais il est également possible de s'intéresser au temps écoulé entre la ruine et le rétablissement (l'instant où la richesse devient positive), ou encore à l'aire comprise entre la courbe et zéro pour l'intervalle de temps séparant l'instant de

ruine du premier rétablissement.
Plus précisément, pour le processus de risque

$$R^u(t) = u + X_t,$$

où

$$X(t) = ct - S(t), \tag{11}$$

on a (voir Gerber (1988), Dufresne et Gerber (1988) et Picard (1994)) :

- le temps de ruine
$$T_u = \inf\{t > 0, u + X_t < 0\},$$
- la sévérité de la ruine
$$u + X_{T_u},$$
le couple $(T_u, u + X_{T_u})$,
- le temps passé en-dessous de zéro entre la première ruine et le rétablissement $T'_u - T_u$, où
$$T'_u = \inf\{t > T_u, u + X_t = 0\},$$
- la sévérité maximale
$$\left(\inf_{t>0} u + X_t\right),$$
- la sévérité agrégée de la ruine jusqu'au rétablissement
$$J(u) = \int_{T_u}^{T'_u} |u + X_t| dt.$$
- Enfin, Dos Reis (1993) a étudié le temps total passé en-dessous de zéro
$$\tau(u) = \int_0^{+\infty} 1_{\{u+X_t<0\}} dt$$
en utilisant les résultats de Gerber (1988).

On peut aussi s'intéresser aux probabilités

$$f(u, x, y) = \mathbb{P}(T_u < +\infty, -R^u(T_u) > x, R^u(T_u^-) > y).$$

Remarquons que pour $x = y = 0$, on retrouve la probabilité de ruine $\psi(u) = f(u, 0, 0)$. Pour un processus de Poisson composé avec distribution de montant de sinistre W, on a la formule explicite pour une réserve initiale nulle :

$$f(0, x, y) = \frac{\lambda}{c} \int_{x+y}^{+\infty} \bar{F}_W(v) dv$$

où $\bar{F}_W$ est la fonction de survie de W (voir par exemple Rolski et al. (1999) pour la preuve). Ce résultat peut être généralisé à une classe plus large de processus. Lin et Willmot (2000); Willmot et Lin (1998) ont étudié les propriétés et les moments du temps de ruine, du déficit juste après et du surplus juste avant la ruine et des fonctionnelles de ces quantités introduites par Gerber et Shiu (1997). Usábel (1999) donne des moyens pratiques d'approcher $f(u, x, y)$.
La distribution du temps de ruine et du temps passé en dessous de zéro est aussi étudiée par Picard (1994) , Frostig (2004b), Dos Reis (2000), et Gerber et Shiu (1998a).

Les mesures de risque en présence de plusieurs sources de risque et les problèmes d'allocation optimale de capital qui en découlent seront évoqués dans la partie de l'introduction consacrée aux modèles multidimensionnels (voir pages 25 et suivantes).
Dans l'hypothèse où la compagnie, en cas de ruine, a l'opportunité de combler sa dette sous la forme d'un emprunt tant que sa richesse algébrique est négative, l'aire entre le processus et zéro quand celui-ci est en-dessous de zéro (voir figure 1.1) peut être interprétée comme un multiple des intérêts à payer par la compagnie jusqu'au retour à la solvabilité. Le chapitre II.2 est consacré à l'étude de ce genre de mesures de risques. En particulier, un théorème de différentiation de fonctionnelles de processus de risque y est énoncé et démontré.
Ces mesures de risque sont définies par rapport à la richesse globale de la compagnie, en agrégeant les richesses de ses différentes branches. Dans le modèle multirisques, nous verrons des propositions d'extension de ces mesures de risque en présence de plusieurs sources de risque (voir page 25 et chapitre II.1 et II.2). Le théorème de différentiation nous donnera alors un éclairage sur la stratégie d'allocation optimale de réserve initiale.

Approfondissements du modèle classique

Le modèle classique de la théorie de la ruine, décrit précédemment page 4, peut être généralisé sous diverses formes, que l'on peut classer en deux rubriques, suivant que l'on s'intéresse aux hypothèses portant sur le passif de la compagnie (sur la modélisation des sinistres) ou sur son actif (sur les hypothèses modélisant la stratégie de versement de dividendes ou l'investissement de la réserve et la prise en compte du taux d'intérêt).

Hypothèses relevant du passif

Nombre de sinistres

Les premières consistent à remplacer le processus de Poisson modélisant l'arrivée des sinistres par un processus de renouvellement (on parle alors de modèle de Sparre Andersen), ou par un processus de Poisson modulé par un processus markovien supposé représenter l'évolution de l'état de l'environnement.
Ce modèle, introduit dans le domaine de la théorie du risque par Asmussen (1989), et étudié ensuite en particulier par Frostig (2004a) et Asmussen et Kella (2000), peut être étendu à un modèle multirisques, que nous introduirons ultérieurement (voir page 23). Soit n le nombre fini d'états possible de l'environnement. Soit J_t le processus markovien à temps continu représentant l'état de l'environnement au temps t, et pour chaque état de la nature $1 \leq i \leq n$, soient

$$X^i(t) = S^i(t) - c^i t$$

des processus de Lévy indépendants et d'exposants

$$\varphi^i(\alpha) = \ln\left(\mathbb{E}\left[e^{\alpha X^i(1)}\right]\right).$$

Ensuite on construit le processus $X(t)$ modulé par l'environnement comme suit :
Soit T_p l'instant du $p^{\text{ième}}$ saut du processus J_t. On peut alors définir $X(t)$:

$$X(t) - X(0) = \sum_{p\geq 1} \sum_{1\leq i\leq n} (X^i(T_p) - X^i(T_{p-1})\mathbf{1}_{\{J_{T_{p-1}}=i, T_p\leq t\}}$$

$$+\sum_{p\geq 1}\sum_{1\leq i\leq n}(X^i(t)-X^i(T_{p-1})\mathbf{1}_{\{J_{T_{p-1}}=i,T_{p-1}\leq t<T_p\}}.$$

Soit Q la matrice des taux de transition de J_t. Définissons

$$F(\alpha)=Q+\mathrm{diag}(\varphi^1(\alpha),\ldots,\varphi^n(\alpha)).$$

D'après le lemme 2.1 de Asmussen et Kella (2000), en notant pour $1\leq i\leq n$,

$$\widetilde{\underline{\mathbf{1}}}_i=(0,\ldots,0,\underbrace{1}_{i},0,\ldots,0),$$

$$M^W(t,\alpha)=e^{\alpha X(t)}\widetilde{\underline{\mathbf{1}}}_{J_t}e^{-F(\alpha)t}$$

est une martingale de dimension n (le nombre d'états) pour tout $\alpha\in\mathbb{C}$ tel que tous les $\varphi^i(\alpha)$ existent et pour toute distribution de $(X(0),J_0)$. Si $h(\alpha)$ est un vecteur propre (à droite) de $F(\alpha)$ pour la valeur propre $\lambda(\alpha)$, alors

$$N(t,\alpha)=e^{\alpha X(t)-\lambda(\alpha)t}h_{J_t}(\alpha)$$

est une martingale.
Cette modulation peut, par exemple, servir à incorporer dans le modèle l'impact de facteurs climatiques sur la survenance de catastrophes naturelles, ou l'impact de la répression et du climat sur le nombre d'accidents de la route et les coûts qu'ils engendrent. Une version discrète de ce modèle est utilisée par Lévi et Partrat (1989) pour modéliser les variations annuelles des nombres de cyclones.
La modulation par un environnement markovien permet en général de créer de la surdispersion, souvent observée dans la réalité. Mais le principal atout de ce modèle est qu'il permet, combiné à des chocs communs, de traduire de façon assez réaliste la dépendance entre plusieurs branches d'activité d'une compagnie d'assurances (voir chapitres II.1 et II.3, et page 23).

Montants de sinistres

En ce qui concerne les montants de sinistres, outre les cas les plus classiques de la distribution exponentielle, qui fournit une probabilité de ruine explicite dans le modèle classique, et les distributions à valeurs dans $d\mathbb{N}$, où $d\in\mathbb{R}^+$, qui permettent d'approcher toute distribution sur $\mathbb{R}^+$ et qui autorisent l'utilisation de formules de type Seal ou de Picard-Lefèvre, on peut rencontrer assez souvent les lois phase-type. Ces distributions correspondent à celles de temps d'atteinte d'un état absorbant par un processus de Markov à nombre d'états fini n, et sont donc décrites par une distribution initiale $\pi\in(\mathbb{R}^+)^n$ et une matrice de taux de transition élémentaires, de taille $n\times n$. Ces distributions présentent également l'avantage de constituer un sous-ensemble dense des lois sur $\mathbb{R}^+$, même s'il faut garder à l'esprit que le comportement asymptotique restera exponentiel, et que pour des lois à queues lourdes, la taille de l'espace d'état nécessaire pour obtenir une approximation correcte est bien souvent prohibitive. Toutefois, ce n'est pas toujours très grave car on dispose pour les modèles à queues lourdes d'autres méthodes, souvent de nature asymptotique (voir par exemple Frolova et al. (2002); Asmussen et Højgaard (1996)). De plus, il est parfois possible de démontrer des résultats pour des distributions de sauts de type *phase-type*, et d'utiliser ensuite un argument de densité et un passage à la limite pour obtenir le résultat pour une distribution de sauts quelconque (voir par exemple Asmussen et Kella (2000)). Le modèle Poisson composé *phase-type* et le modèle Poisson composé modulé avec sauts exponentiellement distribués peuvent être considérés comme voisins par passage au modèle dit dual (voir page 18, et dans Frostig (2004a)).

Hypothèses relevant de l'actif

Les modifications plutôt financières consistent à introduire un taux d'intérêt instantané (déterministe ou stochastique), et dans le processus des réserves une composante Brownienne ou de type Lévy. La plus importante de ces modifications prend en compte une politique de versements de dividendes aux actionnaires dès que la richesse devient assez élevée.

Taux d'intérêt

Dans le modèle avec taux d'intérêt instantané déterministe δ, le processus de risque devient régi par l'équation :

$$dR(t) = dX(t) + \delta R(t)dt$$

où $X(t)$ est défini par (11).
Dans le cas d'un modèle classique (Poisson composé) avec taux d'intérêt r constant, en notant δ le taux d'actualisation instantané, on a la formule suivante (voir par exemple Sundt et Teugels (1995)) pour la probabilité de ruine quand les montants des sinistres sont distribués selon une loi exponentielle de paramètre μ :

$$\psi(u) = \frac{\Gamma\left(\frac{\lambda}{\delta}, \frac{c}{\delta\mu} + \frac{u}{\mu}\right)}{\Gamma\left(\frac{\lambda}{\delta}, \frac{c}{\delta\mu}\right) + \frac{\delta}{\lambda}\left(\frac{c}{\delta}\right)^{\lambda/\delta} e^{-\frac{c}{\delta\mu}}},$$

où

$$\Gamma(v, w) = \int_w^{+\infty} t^{v-1}e^{-t}dt$$

désigne la fonction gamma incomplète.
Quand le chargement de sécurité est nul et le taux d'intérêt fixe, on peut trouver par exemple dans Segerdahl (1942) la formule

$$\psi(u) = -\ln\left(\frac{\int_u^{+\infty} e^{-\mu z}(1 + \frac{\delta z}{c})^{\frac{\lambda}{\delta}-1}dz}{\frac{c}{\lambda} + \int_0^{+\infty} e^{-\mu z}(1 + \frac{\delta z}{c})^{\frac{\lambda}{\delta}-1}dz}\right).$$

Konstantinides et al. (2002) obtiennent un encadrement asymptotique de la probabilité de ruine quand la distribution des montants de sinistres a une queue lourde en se ramenant au modèle classique sans taux d'intérêt. Yang et Zhang (2001) s'intéressent à la sévérité de ruine avec taux d'intérêt. Brekelmans et De Waegenaere (2001) approchent la probabilité de ruine en temps fini avec taux d'intérêt en passant en temps discret et avec une généralisation de la méthode de Panjer. Il est possible d'utiliser cela pour les applications numériques et le modèle multidimensionnel. Sundt et Teugels (1995, 1997) s'intéressent à des encadrements de la probabilité de ruine, et à la fonction d'ajustement qui donne un équivalent ou un encadrement d'équivalent de la probabilité de ruine en fonction de la réserve initiale et du taux d'intérêt.

La différence principale avec le modèle classique est l'existence d'un seuil de ce que Gerber appelle ruine définitive ou absolue. En effet, si la réserve devient inférieure à $-\frac{c}{\delta}$, même s'il ne se produisait aucun sinistre, la richesse continuerait sûrement à décroître vers $-\infty$, car les primes ne suffiraient pas à compenser les intérêts à payer. Même avec un chargement de sécurité positif, le processus de richesse ne tend plus vers $+\infty$ presque sûrement.

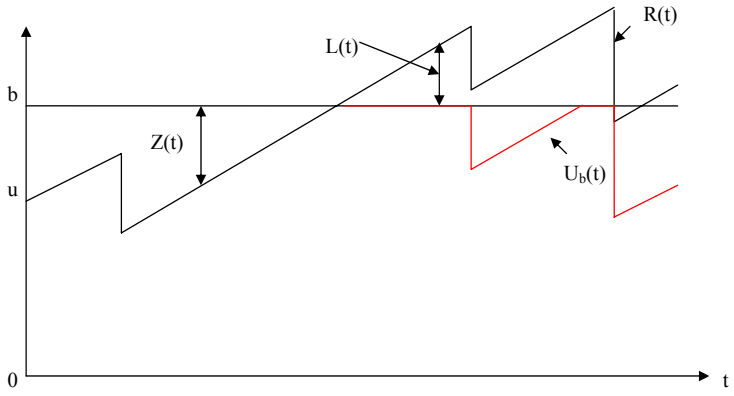

FIG. 1 – Illustration de $U_b(t)$, $Z(t)$ et de $L(t)$.

Versement de dividendes au delà d'une barrière

Outre le fait que le versement de dividendes aux actionnaires au-delà d'une barrière supérieure est un phénomène réaliste, il "présente l'énorme avantage" de conduire à une probabilité de ruine en temps infini égale à 1, même si le chargement de sécurité est positif. Cela permet de rapprocher la théorie de la ruine avec celle du risque de défaut, selon laquelle aucune compagnie n'est éternelle. La probabilité de ruine en temps infini est donc toujours 1 en risque de défaut, alors que la théorie de la ruine avait pour but initial de déterminer le niveau de réserve nécessaire pour rendre cette probabilité inférieure à ϵ.
On attribue traditionnellement les premières études de ce type de modèle à de Finetti (1957). Dès lors, on peut se poser des problèmes de politique de versement de dividendes optimale. On peut également ajouter une composante brownienne ou Lévy, et utiliser des méthodes du type Hamilton-Jacobi-Bellmann (Asmussen et al. (2000); Gerber et Shiu (1998a)), ou utiliser les méthodes du contrôle impulsionnel, c'est-à-dire réagir à un excès de capital par un versement de dividendes traduit par un saut (voir Sulem (2004)). La méthode la plus simple consiste à considérer une barrière horizontale de niveau $b \geq u$, et à verser instantanément aux actionnaires l'excès ou une fraction de l'excès de réserve par rapport à ce niveau b. Cela correspond soit à faire stagner le processus en b jusqu'au prochain sinistre (figure 1), soit à créer une sorte d'angle de réfraction lors de la traversée de la barrière en b (voir Gerber et Shiu (1998b); Asmussen et al. (2000)). Le processus $U_b(t)$ représentant la richesse de la compagnie au temps t avec cette stratégie de dividendes se déduit du processus classique $R(t)$. Soit $b > u$ et

$$X(t) = S(t) - ct.$$

La stratégie qui consiste à limiter supérieurement le surplus de la compagnie à la valeur b en versant sous forme de dividendes l'excès par rapport à b donne le processus $U_b(t)$. Le temps de ruine est alors $\tau = \inf\{t > 0, \quad U_b(t) < 0\}$. Le cumul des dividendes versés jusqu'au temps t est

$$L_t = -\inf_{0 \leq s \leq t} \{b - u + X(s)\}^-,$$

où $x^- = \min(x, 0)$.
Soit

$$Z(t) = b - u + X(t) + L(t).$$

Alors $U_b(t) = b - Z(t)$ est le processus recherché. Notons que

$$L_t = c \int_0^t \mathbf{1}_{U_b(s)=b} ds$$

quand $U_b(s) = b$, jusqu'au prochain sinistre, la richesse de la compagnie stagne en b, alors que l'accroissement de richesse avec une pente c est transféré aux actionnaires sous forme de dividendes. L augmente en t si et seulement si $U_b(t) = b$, ou encore $Z(t) = 0$. Ceci est illustré sur une trajectoire par la figure 1.
Asmussen et al. (2000) démontrent que, dans certains cas assez généraux, le modèle à barrière horizontal est un modèle optimal de versement de dividendes. Dans le cadre de ce modèle, on peut s'intéresser aux mesures de risque citées plus haut, mais aussi au montant total des dividendes distribués, actualisés ou non. Dans un premier temps, on peut s'intéresser à la probabilité qu'il y ait effectivement versement de dividendes, c'est-à-dire, dans les modèles considérés, que le processus partant de $u \in]0, b[$ atteigne la barrière supérieure b avant de croiser 0.
Dans l'article Rullière et Loisel (2004), qui constitue le chapitre I.2 de cette thèse, nous nous intéressons au calcul de cette probabilité, que nous appelons *win-first*, dans le modèle Poisson composé avec taux d'intérêt instantané δ. L'intérêt de cette quantité est de constituer une première étape potentielle dans le calcul de la moyenne des dividendes versés jusqu'à la ruine (voir Frostig (2004a)), mais aussi de représenter la probabilité de réalisation d'un objectif (atteindre b) sous contrainte de solvabilité.
Cela nous permet ensuite de calculer numériquement l'espérance des dividendes versés jusqu'à la ruine et ses dérivées avec une grande précision en suivant la méthode classique suivante.
Pour $b > u$, dans le modèle Poisson composé, Segerdahl (1970), puis Dickson et Gray (1984b) ont montré que la probabilité d'atteindre b avant la ruine en partant de u est donnée par

$$q_u = \frac{1 - \psi(u)}{1 - \psi(b)},$$

où $\psi(v)$ est la probabilité de ruine dans le modèle sans barrière supérieure. Ce résultat reste valable avec un taux d'intérêt fixe (voir le chapitre I.2). Plaçons-nous, l'espace d'un instant, sous la condition $u = b$. Soit Δ_1 l'instant du premier sinistre, et

$$\tau_1 = \inf\{t > \Delta_1, \quad U_b(t) = b\}.$$

τ_1 peut être infini si le chargement de sécurité est négatif, ce qui n'est pas exclu ici. Soit $q = \mathbb{P}(\tau < \tau_1)$ la probabilité en partant de b que la ruine survienne avant que $U_b(t)$ n'ait eu le temps de remonter en b, et $p = 1 - q$.
Revenons maintenant après ces définitions au cas $u < b$. Pour verser des dividendes avant la ruine, il faut déjà atteindre b avant la ruine, ce qui arrive avec probabilité q_u. Ensuite, la perte de mémoire à l'arrivée en b et la stationarité grâce au processus de Poisson nous assurent qu'une fois en b, le nombre de séjours en b avant la ruine (y compris le premier) suit une loi géométrique de paramètre q_u. De plus, le temps de chaque séjour du processus $U_b(t)$ en b est le temps entre l'arrivée en b et le prochain sinistre, et donc suit une loi exponentielle de paramètre λ. On a donc $EL_\tau = \frac{q_u c}{p\lambda}$. Dans le cas *phase-type*, on obtient q_u de façon explicite.

Frostig (2004a) obtient entre autres le temps moyen de ruine $E\tau$ et l'espérance du montant de dividendes versé jusqu'à la ruine EL_τ pour deux modèles, dont le modèle Poisson-phase-type, en utilisant des méthodes de martingales de Kella et Whitt (1992) et d'Asmussen et Kella (2000),

et le modèle fluide introduit page 18. Nous exposons ici brièvement sa démarche, car nous utiliserons une méthode similaire au chapitre II.3 dans un modèle multirisques, et parce que cette démarche peut aussi constituer une alternative à la solution numérique proposée au chapitre I.2 pour le calcul de probabilités *win-first* dans le cas $\delta = 0$, c'est-à-dire sans taux d'intérêt. En effet, on peut tenter d'approcher la loi du montant des sinistres par une loi phase-type, puis utiliser les formules explicites données entre autres par Frostig (2004a) pour la probabilité "win-first" et pour la moyenne des dividendes versés dans le cas de lois *phase-type*. Toutefois, il faut garder à l'esprit que le comportement asymptotique de ces lois sera toujours de type exponentiel, et que le nombre d'états (et donc le temps pour déterminer le bon fit) peut être très grand, et le fit délicat, en particulier dans le cas de queues épaisses.
Une méthode de martingales de Kella et Whitt (1992) permet de relier $E\tau$, $EZ(\tau)$ et $EL(\tau)$ (partie 2.1 p. 6 de Frostig (2004a)).
Soit

$$\varphi(\alpha) = \ln\left(Ee^{\alpha X(1)}\right) = -c\alpha - \lambda + \lambda M_V(\alpha),$$

où M_V est la transformée de Laplace du montant de sinistre V_1. $X(t)$ est un processus de Lévy d'exposant $\varphi(\alpha)$. De plus $Z(t) = Z(0) + X(t) + L(t)$ avec $Z(0) = b - u$. Le théorème 2 de Kella et Whitt (1992) assure que le processus

$$M(\alpha, t) = \varphi(\alpha) \int_0^t e^{\alpha Z(s)} ds + e^{\alpha Z(0)} - e^{\alpha Z(t)} + \alpha \int_0^t e^{\alpha Z(s)} dL(s)$$

est une martingale. $dL(t) = c\mathbf{1}_{Z(t)=0}$, ce qui implique que le dernier terme de $M(\alpha, t)$ est égal à $\alpha L(t)$:

$$M(\alpha, t) = \varphi(\alpha) \int_0^t e^{\alpha Z(s)} ds + e^{\alpha Z(0)} - e^{\alpha Z(t)} + \alpha L(t)$$

Comme $L(0) = 0$, pour tout $t \geq 0$,

$$EM(\alpha, t) = EM(\alpha, 0) = \varphi(\alpha).0 + e^{\alpha Z(0)} - e^{\alpha Z(0)} + \alpha L(0) = 0.$$

Asmussen et Kella (2000) utilisent le théorème d'arrêt de Doob pour obtenir

$$EZ(\tau) = Z(0) + \varphi'(0)E(\tau) + E(L(\tau)).$$

On peut obtenir $EL(\tau)$ facilement (voir section précédente). Il reste donc à obtenir $EZ(\tau)$ pour en déduire $E\tau$.
La méthode pour obtenir la distribution de $Z(\tau)$ (section 2.3 p.6 de Frostig (2004a)) permet aussi de retrouver $EL(\tau)$. En effet, $Z(\tau) = b + \xi$, où ξ est le déficit à la ruine et suit une certaine loi *phase-type*.
Ensuite, Frostig (2004a) utilise le modèle fluide, aussi appelé modèle du télégraphiste, qui lui permet de conclure.

Modèle dual et processus du télégraphiste

Ce processus correspond à un processus de Poisson composé dans lequel les sauts verticaux d'amplitude W sont remplacés par des descentes en pente douce (de pente -1) pendant un temps W. Le passage par ce modèle, fréquemment étudié en physique mathématique, permet de

passer du modèle classique ($c > 0$ et sauts vers le bas) au modèle dit dual ($c' < 0$ et sauts vers le haut). Les temps de ruine dans les deux modèles sont fortement liés, et l'étude du modèle dual permet, par l'équivalence expliquée par Mazza et Rullière (2004), de calculer facilement des probabilités de ruine dans le modèle non poissonien avec sauts d'amplitude distribuée selon une loi exponentielle, et de retrouver des résultats de Malinovskii (1998). Le lien entre les deux modèles considérés par Frostig (2004a), le modèle Poisson-composé-phase-type et le modèle Poisson-modulé-composé-exponentiel s'apparente à une situation de dualité. Comme nous utilisons dans le chapitre II.3 des méthodes similaires à celles de Frostig (2004a), qui utilise elle-même le processus du télégraphiste dans ce qu'elle appelle le modèle fluide, il nous a paru intéressant de rappeler brièvement la définition du modèle dual et les liens entre ce modèle et le modèle classique.

Dans le modèle classique, la richesse de la compagnie augmente linéairement en fonction du temps et chute brutalement à chaque sinistre qui survient. La richesse X est définie comme précédemment par :

$$X(t) = u + ct - \sum_{k=1}^{N(t)} W_k \tag{12}$$

où l'on fixe la somme nulle si $N(t) = 0$, et où c est une constante strictement positive. $N(t), t \geq 0$ est ici un processus de renouvellement défini par la loi Δ du temps inter-sinistres. Les W_k sont i.i.d. de même loi W et indépendants des temps inter-sinistres Δ_k. $u \geq 0$ est la réserve initiale et l'activité est supposée rentable.
Dans le modèle dual, l'événement que l'on continue à appeler sinistre est positif pour l'entreprise et occasionne donc un saut vers le haut de la richesse de la compagnie, qui diminue par ailleurs linéairement en fonction du temps. Le modèle dual peut correspondre au versement de rentes viagères par la compagnie, pour laquelle le décès d'un assuré est cyniquement un heureux événement, puisqu'il dispense l'assureur de verser les rentes suivantes. Les lois et grandeurs du modèle dual seront notées avec un '. La richesse X' est donc définie par :

$$X'(t) = u' - c't + \sum_{k=1}^{N'(t)} W'_k \tag{13}$$

où l'on fixe la somme nulle si $N'(t) = 0$, et où c' est une constante strictement positive. $N'(t), t \geq 0$ est ici un processus de renouvellement défini par la loi Δ' du temps inter-sinistres. Les W'_k sont i.i.d. de même loi W' et indépendants des temps inter-sinistres Δ'_k. $u' \geq 0$ est la réserve initiale et l'activité est supposée rentable. Un tel modèle sera dit de caractéristiques (W',Δ',c').

La différence principale entre les deux modèles est la suivante : dans le modèle classique, la ruine éventuelle intervient à l'instant d'un sinistre, avec une sévérité quelconque, correspondant à la différence entre le saut et la richesse de la compagnie juste avant le saut, alors que dans le modèle dual la ruine ne peut avoir lieu qu'entre les sauts, avec une sévérité nulle.
A l'aide de la théorie des ondes et du modèle du télégraphiste, Mazza et Rullière (2004) ont démontré une équivalence entre les modèles, en ce qui concerne la loi du temps de ruine. Cette équivalence correspond en fait à échanger les lois des montants et des temps inter-sinistres, et à modifier la réserve initiale et le temps.

Definition .1 *On dit qu'un modèle classique (W,Δ,c) et un modèle dual (W',Δ',c') sont de caractéristiques inverses si*

$$\begin{cases} W' = \Delta \\ \Delta' = W \\ c' = \frac{1}{c} \end{cases} \tag{14}$$

Le passage d'un modèle donné à un modèle de caractéristiques inverses est une involution et conserve le chargement de sécurité, et donc le caractère rentable ou non de l'activité. Soit T_v (resp. T'_v) le temps de ruine dans le modèle classique (resp. dual) avec réserve initiale v. Notons l'égalité en loi par $\stackrel{d}{=}$. Par convention, lorsque $v < 0$, $T_v \stackrel{d}{=} \delta_0$ suit une distribution de Dirac en 0. Le théorème suivant issu de Mazza et Rullière (2004) montre que l'on peut déduire la loi du temps de ruine à partir de celle du temps de ruine d'un modèle de caractéristiques inverses particulier.

Theorem .4 *Soit (W,Δ,c) et (W',Δ',c') deux modèles de caractéristiques inverses, Δ_0 et Δ'_0 deux variables aléatoires indépendantes de lois respectives Δ et Δ'. Supposons de plus que la réserve $u \geq 0$. Alors :*

$$cT_u \stackrel{d}{=} T'_{\frac{u}{c}+\Delta_0} - u \tag{15}$$

$$c'T'_u \stackrel{d}{=} u + T_{\frac{u}{c'}-\Delta'_0} \tag{16}$$

On peut utiliser cette équivalence pour retrouver $\Psi(u)$ dans le modèle classique non poissonnien mais où les montants sont exponentiels. On peut aussi obtenir une expression de la transformée de Laplace du temps de ruine récemment donnée par Malinovskii (1998). En effet, dans le modèle dual poissonnien $(Exp(\mu), W', c')$, pour $s > 0$,

$$E\left[e^{-sT'_u}\right] = e^{-R(s)u} \tag{17}$$

où $R(s)$ est l'unique solution strictement positive de l'équation

$$\mu + s - c'R = \mu E\left[e^{-RW'}\right]. \tag{18}$$

En revenant au modèle classique de caractéristiques inverses, d'après la définition .1 et (15), on aboutit au

Theorem .5 *Dans le modèle classique non poissonien $(\Delta, Exp(\mu), c')$ où les montants sont exponentiels, pour $s > 0$,*

$$E\left[e^{-\alpha T_u}\right] = \left(1 - \frac{R(\alpha/c) - \alpha}{c\mu}\right) e^{-\frac{u}{c}(R(\alpha/c)-\alpha)} \tag{19}$$

où $R(\alpha)$ est l'unique solution strictement positive de l'équation

$$c\mu + \alpha - R = c\mu E\left[e^{-R\Delta}\right]. \tag{20}$$

Le lien entre des mouvements aléatoires persistants et des processus de risque permet aussi de démontrer des propriétés des premiers grâce à des résultats connus de théorie de la ruine.
Un autre concept déjà évoqué est le problème de ruine à l'inventaire. La résolution de ce problème permet d'approcher la probabilité de ruine.

Definition .2 *Rappelons que $\Psi(u,t)$ et $\Psi'(u,t)$ sont les probabilités de ruine avant t avec réserve initiale u dans le modèle classique et dans le modèle dual. Soit*

$$\Psi_\delta(u,t) = \mathbb{P}\left[\exists i \in \mathbb{N}\,,\, 0 \leq i\delta \leq t\,, \quad u + X(i\delta) < 0\right] \tag{21}$$

la probabilité de ruine à l'inventaire aux dates multiples de δ. On définit de même $\Psi'_\delta(u,t)$ pour le modèle dual et les probabilités de non-ruine à l'inventaire $\Phi_\delta(u,t)$ et $\Phi'_\delta(u,t)$.

Bien sûr, être ruiné à l'inventaire implique être ruiné, ce qui fournit l'inégalité gauche de (22) et de (23). Dans le modèle classique, être ruiné avec une sévérité trop importante implique être ruiné à l'inventaire, car l'arrivée des primes jusqu'à l'inventaire ne compensera pas le déficit. Etre ruiné avec grande sévérité revient à être ruiné avec une réserve initiale plus élevée. Cet argument permet d'obtenir l'inégalité de droite de (22) et celle de (23). En effet, dans le modèle dual, être ruiné borne supérieurement la richesse à l'inventaire précédent, et implique donc la ruine à l'inventaire pour la réserve correspondante. On a donc pour $u, t \geq 0$ et $\delta > 0$:

$$\Psi_\delta(u,t) \leq \Psi(u,t) \leq \Psi_\delta(u - c\delta, t) \tag{22}$$

$$\Psi'_\delta(u,t) \leq \Psi'(u,t) \leq \Psi'_\delta(u - c'\delta, t) \tag{23}$$

On obtient les inégalités du même type renversées pour les probabilités de non-ruine.

Par souci de concision, nous omettons ici les travaux d'Orsingher (1999) sur les processus définis par intégrations successives du processus du télégraphiste, qui pourraient être reliées aux mesures de risques $E\left[I_{g,h}\right]$, déjà évoquées page 30, et qui seront définies au chapitre II.2 après avoir introduit les modèles multirisques.
Jusqu'ici, nous avons considéré la compagnie d'assurances comme un tout, et nous nous sommes exclusivement intéressés à sa richesse globale. Si la compagnie a plusieurs types d'activités, que nous appellerons branches (en anglais dans les articles *lines of business*), cela équivaut à sommer les richesses algébriques des branches. Nous allons maintenant prendre en compte l'évolution conjointe des différentes branches de la compagnie.

Modèle multi-branches

Il est évident que la somme des richesses est un indicateur important pour apprécier la solvabilité de l'entreprise. Toutefois, il est discutable que cet indicateur soit suffisant pour la décrire totalement. En effet, il semble logique que les actionnaires redoutent en cas de déséquilibre entre les branches, de devoir combler dans le futur les dettes appelées à empirer au détriment de versements de dividendes qui pourraient avoir lieu dans une configuration plus stable. Ce fut en particulier le cas pour une holding américaine dans le domaine du transport aérien, qui détenait une société florissante (extrapolons les réserves de cette société à 60 millions d'euros) et une autre société en difficulté (extrapolons la dette à 10 millions d'euros). Si l'on se borne à une approche de type additive, la valorisation boursière de la holding aurait dû être de l'ordre de 50 millions d'euros. En réalité, elle ne fut que de l'ordre de 20 millions d'euros[1].
De plus, les problèmes de gestion interne de la compagnie et les nouvelles réglementations sont susceptibles de créer des besoins d'analyse multidimensionnelle. En effet, certaines compagnies prennent en compte le besoin en capital économique créé par chaque branche pour déterminer

[1] Je remercie Jean-Paul Laurent pour cette anecdote historique.

leur implication dans chacune d'elles. Enfin, elles doivent autant que faire se peut présenter des modèles dans lesquels les branches sont censées rester à l'équilibre avec une probabilité élevée Ces considérations amènent à considérer l'évolution conjointe des richesses des $K \geq 1$ branches d'activité d'une compagnie d'assurance. Ces branches peuvent représenter des filiales différentes, des secteurs d'activité différents (assurance santé, habitation, automobile, responsabilité civile) ou encore des activités, différentes ou identiques, dans différents continents, pays ou régions.
Seuls Collamore (1998), avec des méthodes de grandes déviations, et Picard et al. (2003a) s'étaient jusqu'ici intéressés au processus de risque multidimensionnel. Comme nous reprenons dans le chapitre II.1 le formalisme de Picard et al. (2003a), il nous a paru nécessaire de rappeler brièvement leur méthode.

Résultats de Picard et al. (2003a) et Collamore (1998)

Picard et al. (2003a) considèrent un portefeuille, qui couvre K risques interdépendants, observé aux temps $t \in \mathbb{N}$. Picard et al. (2003a) notent le nombre de branches d'activités par n. Nous exposons ici leurs résultats en changeant n en K par souci de cohérence entre les chapitres. K désignera donc le nombre de branches, et n correspondra au nombre d'états possibles de l'environnement, introduit page 12. Le processus de risque est en fait une marche aléatoire dans $\mathbb{Z}^K$. Il est décrit par l'arrivée déterministe du vecteur des primes à chaque étape et par le vecteur des montants globaux des sinistres de chaque branche ou composante. La dépendance entre les sinistres des différentes branches est traduite sur le montant global des sinistres d'une période, et non sur la fréquence des sinistres comme dans la plupart des autres études, où l'on introduisait par exemple des lois Poisson-mélange. A chaque $t \in \mathbb{N}^*$ correspond une zone d'insolvabilité $D(t) \subset \mathbb{R}^K$ et la ruine est le fait d'atteindre une de ces zones. Dans cette partie, les lettres en gras désigneront des vecteurs dans $\mathbb{Z}^K$ ou $\mathbb{R}^K$. Plus précisément, les montants cumulés des sinistres pour les K risques durant la période $]t, t+1]$ pour $t \in \mathbb{N}$ constituent un vecteur $\mathbf{X}(t+1)$ à valeurs dans $\mathbb{N}^K$. Les $\mathbf{X}(t), t \in \mathbb{N}^*$ sont supposés indépendants et identiquement distribués de distribution $\{a_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^K\}$ avec $0 < a_{(0,\ldots,0)} < 1$. Dans la suite a_0 désignera $a_{(0,\ldots,0)}$. Au temps $t \in \mathbb{N}^*$, le montant cumulé des sinistres depuis le temps 0 est défini par $\mathbf{S}(t) = \mathbf{X}(1) + \cdots + \mathbf{X}(t)$. Dans la suite $\mathbf{v}^t$ désigne la transposée d'un vecteur $\mathbf{v}$, à ne pas confondre avec le temps t.
Pour $t \in \mathbb{N}$, $\mathbf{p} \in \mathbb{N}^K$,

$$P(\mathbf{S}(t) = \mathbf{p}) = a_0^t e_{\mathbf{p}}(t) \tag{24}$$

où chaque $e_{\mathbf{p}}$ est un polynôme de degré inférieur ou égal à $|\mathbf{p}|$. Ces polynômes s'obtiennent par convolutions successives d'une légère modification de la distribution de $\mathbf{X}(1)$, et leurs valeurs en $t > 2$ peuvent être calculées grâce à l'identité de type binômiale qu'ils vérifient (voir Picard et al. (2003a) , pp. 3-4). Le portefeuille dispose d'une réserve initiale $\mathbf{u} \in (\mathbb{R}^+)^K$. Juste après chaque temps $t \in \mathbb{N}$, il enregistre la recette déterministe $\mathbf{c}(t) \in (\mathbb{R}^+)^K$ destinée à couvrir les K sources de risque pendant la période $]t, t+1]$. L'inventaire est fait à la fin de chaque période. Le niveau du portefeuille au temps $t \in \mathbb{N}^*$ est donc donné par le vecteur

$$\mathbf{u} + \mathbf{c}(0) + \cdots + \mathbf{c}(t-1) - [\mathbf{X}(1) + \cdots + \mathbf{X}(t-1)]\,. \tag{25}$$

Les auteurs considèrent que la ruine intervient lorsqu'une des branches est ruinée. Si T est le temps de ruine, $T = \min\{T_1, \ldots, T_K\}$. Cela les conduit à définir des régions d'insolvabilité $D(t) \subset \mathbb{R}^K$ et leurs complémentaires $H(t)$ pour $t \in \mathbb{N}^*$. Ainsi la ruine au temps t équivaudra à $\mathbf{S}(t) \in D(t)$. Concrètement, le modèle retenu donne

$$H(t) = \{\mathbf{v} \in (\mathbb{R}^+)^K, \mathbf{v} \leq \mathbf{u} + \mathbf{c}(0) + \cdots + \mathbf{c}(t-1)\}. \tag{26}$$

où $\mathbf{v} \leq \mathbf{w}$ signifie $\forall i \in [1, K], \mathbf{v}_i \leq \mathbf{w}_i$. Procédant comme dans Picard et Lefèvre (1998), posons $v_{\mathbf{p}} = \max\{t, \mathbf{p} \in D(t)\}$ pour $\mathbf{p} \in D(0)$, $v_{\mathbf{p}} = 0$ sinon. C'est l'analogue de (6).
Soit $P_{\mathbf{p}}(t) = P[\mathbf{S}(t) = \mathbf{p}, T > t]$ pour $\mathbf{p} \in \mathbb{N}^K$, $t \geq 0$. Comme

$$[T > t] \Leftrightarrow [\mathbf{S}(1) \in H(1), \ldots, \mathbf{S}(t) \in H(t)], \tag{27}$$

on a aussi le théorème rappelant (8) et (4) :

Théorème .4 *Pour* $\mathbf{p} \in \mathbb{N}^K, t \geq 0$,

$$P_{\mathbf{p}}(t) = a_0^t A_{\mathbf{p}}(t) 1_{\{t > v_{\mathbf{p}}\}} \tag{28}$$

où les polynômes $A_{\mathbf{p}}$ *vérifient* $A_{\mathbf{p}}(v_{\mathbf{p}}) = \delta_{|\mathbf{p}|0}$ *(*$|\mathbf{p}| = \mathbf{p}_1 + \cdots + \mathbf{p}_K$*), et pour* $t \geq v_{\mathbf{p}}$,

$$A_{\mathbf{p}}(t) = \sum_{\mathbf{k}=0}^{\mathbf{p}} A_{\mathbf{k}}(v_{\mathbf{p}}) e_{\mathbf{p}-\mathbf{k}}(t - v_{\mathbf{p}}). \tag{29}$$

Dans le théorème précédent δ désigne bien sûr le symbole de Kronecker. Comme dans Picard et al. (2003b), il ne reste plus qu'à sommer les $A_{\mathbf{p}}$ correctement pour obtenir $\Phi(\mathbf{u}, t)$ et en se plaçant juste avant la ruine on peut obtenir la probabilité que les r premiers risques soient ruinés simultanément avec des sévérités de ruine W_i , $1 \leq i \leq r$. Notons $\mathbf{h}(t)$ le vecteur tel que

$$H(t) = \{\mathbf{v} \in (\mathbb{R}^+)^K, \forall i \in [1, K], \mathbf{v}_i \leq \mathbf{h}_i(t)\}.$$

Alors

Théorème .5 *Pour* $t \geq 0$,

$$P[T > t] = \Phi(\mathbf{u}, t) = a_0^t \sum_{\mathbf{p} \in H(t)} A_{\mathbf{p}}(t) \tag{30}$$

De plus, $P[T_1 = \cdots = T_r = t, T_{r+1}, \ldots, T_K > t]$

$$= \sum_{\substack{\mathbf{p} \in H(t-1) \\ \mathbf{k}_1 = \mathbf{h}_1(t)+W_1, \ldots, \mathbf{k}_r = \mathbf{h}_r(t)+W_r \\ \mathbf{p}_i \leq \mathbf{k}_i \leq \mathbf{h}_i(t), r+1 \leq i \leq K}} A_{\mathbf{p}}(t-1) a_{\mathbf{k}-\mathbf{p}} \tag{31}$$

Collamore (1998) considère aussi des marches aléatoires, mais à valeurs dans $\mathbb{R}^K$, et définit également la ruine comme l'entrée de la marche aléatoire à un certain temps t dans une zone d'insolvabilité modélisée par un sous-ensemble de $\mathbb{R}^K$ à chaque temps t. Il expose des méthodes de grandes déviations pour obtenir des résultats asymptotiques sur la probabilité de ruine, c'est-à-dire valables quand la réserve initiale est grande pour chaque branche d'activité.
Avant de se demander quels autres concepts de ruine et quelles autres mesures de risque proposer pour des processus multidimensionnels, il convient d'abord de bien prendre en compte et de bien modéliser la structure de dépendance entre les processus de richesse des différentes branches.

Modélisation de la corrélation entre les branches

Pour une durée d'exercice fixée T, la dépendance entre les variables aléatoires $S_1(T), \ldots, S_K(T)$, représentant les cumuls des sinistres pour les branches 1 à K pendant la durée T, est déterminée par un coupleur (*copula* en anglais). Les ouvrages de Nelsen (1999) et Joe (1997) fournissent un exposé général sur ce sujet (voir par exemple Charpentier et Denuit (2004) pour des applications à l'actuariat). Le concept de coupleur n'est pas très adapté aux processus dont les marginales sont du type Poisson composé ou similaire, car la corrélation change en fonction du temps, et il semble difficile d'obtenir des résultats avec les coupleurs. Comme nous l'avons vu, Picard et al. (2003a) et Collamore (1998) considèrent des marches aléatoires à accroissement indépendants, de loi jointe quelconque sur une sorte de sous-réseau de $\mathbb{Z}^d$ pour Picard et al. (2003a), et dans $\mathbb{R}^d$ avec des méthodes de grandes déviations pour Collamore (1998). Ces modèles sont intéressants et assez généraux, mais présentent le léger défaut de considérer que ce qui se produit pour la branche 2 à la période 1 est corrélé avec ce qui se passe pour la branche 1 à la période 1, mais indépendant de ce qui se passe pour la branche 1 à la période 2. Il y a en effet un déséquilibre entre l'indépendance temporelle et la totale généralité de la distribution multidimensionnelle des montants cumulés de sinistres sur une période.
Pour identifier la source de la corrélation entre les branches, on peut sélectionner deux phénomènes principaux :

- dans certains cas, l'influence du climat, de la répression et de la prévention routière, ou d'autres paramètres peuvent avoir un impact sur la fréquence de sinistres a priori indépendants les uns des autres une fois le climat et les autres paramètres influents fixés. Typiquement, une année particulièrement pluvieuse peut causer une augmentation du nombre d'accidents, le seul point commun entre deux accidents dans deux régions différentes étant la dégradation des conditions de circulation par la pluie. Cela correspond à la modulation du processus d'arrivée des sinistres par un processus représentant l'état de variables d'environnement.
- Dans d'autres cas, un événement unique peut induire des sinistres dans plusieurs branches d'activités différentes. Par exemple, pour un accident de voiture, il faut dans certains cas réparer la voiture (branche automobile), indemniser et réparer les dommages causés au tiers (responsabilité civile), et indemniser le conducteur en cas d'invalidité ou prendre en charge des frais de santé. Le modèle le plus classique pour prendre en compte cette éventualité est le modèle de chocs communs de Poisson.

La modélisation du premier phénomène a déjà été évoqué en dimension 1 page 12. En plus des travaux déjà cités sur ce sujet dans le modèle unidimensionnel, il faut mentionner l'article de Sumita et Masuda (1992), qui ont étudié des propriétés d'un processus de Poisson multidimensionnel modulé par un environnement markovien dans le domaine des files d'attente. De plus, les problèmes d'inférence statistique pour les paramètres de ce type de processus ont été examinés par Rydén (1994, 1996).
Lindskog et McNeil (2003) ont travaillé sur des problèmes de risque de crédit faisant intervenir des modèles à chocs communs. Frostig (2003) ordonne les probabilités de ruine pour des modèles avec arrivées de sinistres interdépendantes. Balu et Sabnis (1997) étudient la préservation de la structure de dépendance. Denuit et al. (1999) s'intéressent au processus somme de plusieurs risques dépendants. Sundt (2000) fournit une sorte de généralisation de la formule de Panjer pour le calcul de distributions Poisson composées multivariées.
Nous étudions, dans le chapitre II.1, un modèle avec des chocs communs, qui ont pour conséquences des sinistres pour les différentes branches de la compagnie avec des probabilités et des sévérités variables. Nous y généralisons alors la démarche de Picard et al. (2003a), présentée

page 21, à une version discrétisée de ce modèle.
Le modèle avec m types de chocs est un cas particulier du modèle général présenté et étudié dans le chapitre II.3. On peut incorporer dans ce modèle une influence de la situation financière des autres sur le comportement d'une branche d'activité de la compagnie. Cela permet de définir le processus modifié par la stratégie de barrière $(b_1, \ldots, b_K)$: quand une branche k parvient au niveau maximal b_k, les primes sont reversées sous forme de dividendes jusqu'au prochain instant de sinistre frappant la branche k. Il semble toutefois inconcevable de verser des dividendes aux actionnaires alors que les autres branches vont très mal. De plus, il est très probable que la situation financière des autres branches ait un impact, d'une façon ou d'une autre, sur chaque branche. Dans cet esprit, nous avons considéré dans le chapitre II.3 des processus pour lesquels les taux de cotisations sont fonctions de la position du processus. Ces modèles et leurs motivations étant décrits dans les articles constituant les chapitres II.1 et II.3, nous nous contentons de les replacer dans leur contexte. Dans le chapitre II.3, nous montrons qu'il est possible de calculer la moyenne des dividendes versés aux actionnaires et de ceux qui ont dû être utilisés pour payer des pénalités à cause de l'insolvabilité temporaire d'une autre branche. Nous calculons aussi le temps moyen de ruine de la branche principale et la loi de la richesse des autres branches à cet instant, ce qui donne une information sur la capacité des autres branches à faire face à la défaillance de la branche principale.
Examinons les mesures de risque envisageables, leurs avantages et leurs points faibles.

Mesures de risque multi-risques

En théorie de la ruine, la plupart des articles traitent les problèmes de théorie de la ruine classique en considérant le processus de richesse vu comme somme de processus interdépendants. D'une façon générale, les auteurs qui s'intéressent au modèle unidimensionnel agrégeant des risques de sources différentes observent que, dans la plupart des cas, la dépendance positive augmente la probabilité de ruine. Nous allons voir les différents concepts traditionnels de dépendance et les façons de modéliser cette dépendance pour des processus marginaux de type Poisson composé. Denuit et al. (2001a) montrent que la prime *stop-loss* est sur-additive pour des risques PCD (*positive cumulative dependent*). Le concept PCD est la généralisation à K risques du concept PQD (*positive quadrant dependence*) :

Definition .3 – *X_1 et X_2 sont PQD si et seulement si*

$$\forall (x_1, x_2) \in \mathbb{R}^2, P(X_1 > x_1, X_2 > x_2) \geq P(X_1 > x_1)P(X_2 > x_2)$$

– *K risques $X_1, \ldots, X_K$ sont PCD si et seulement si pour tout $\mathcal{K} \subset [1, K]$ et $j \notin \mathcal{K}$,*

$$S_{\mathcal{K}} = \sum_{k \in \mathcal{K}} X_k$$

et X_j sont PQD.

Ceci est une généralisation du résultat obtenu par Wang et Dhaene (1998) pour 2 risques PQD. Denuit et al. (2002) mesurent l'impact de la dépendance entre les survenances de sinistres dans le modèle individuel de risque.
Frostig (2001) compare des portefeuilles de risque dont les marginales sont $Y_k = I_k X_k$, où les I_k sont des variables aléatoires de Bernouilli, et les X_k des variables aléatoires à valeurs dans $\mathbb{R}_+^*$. Le vecteur aléatoire X est supposé indépendant de I. On étudie la corrélation entre les I_k,

et dans les exemples, les X_k sont indépendants ou échangeables. L'auteur s'intéresse au risque agrégé

$$\sum_{k=1}^{K} I_k X_k \tag{32}$$

et compare les primes stop-loss

$$E\{(\sum_{k=1}^{K} I_k X_k - d)_+\} \tag{33}$$

des différents portefeuilles à marginales fixées.

Barrieu et El Karoui (2004), Picard et al. (2003a) se sont intéressés à la probabilité que le processus entre avant un temps fini dans une zone d'insolvabilité, et principalement à la probabilité qu'au moins une des branches soit ruinée avant un temps fini. Il est possible de choisir d'autres formes de zones de solvabilité (voir Loisel (2004)), un peu à la manière des domaines d'acceptance définis par Artzner et al. (1999) ou Barrieu et El Karoui (2004). En effet, calculer uniquement la probabilité qu'au moins une branche soit ruinée avant un temps fini donné va à l'encontre du principe de recherches de risques plus ou moins compensatoires les uns des autres. Picard et al. (2003a) montrent en effet que si les branches sont positivement corrélées sous une certaine forme, alors les temps de ruine sont aussi positivement corrélés, ce qui diminue la probabilité de ruine ainsi définie.
Les auteurs s'intéressent ensuite à l'impact de la dépendance entre les risques sur la loi du temps de ruine. *Grosso modo,* plus les risques sont positivement dépendants, plus la ruine a de chances de survenir rapidement.

Definition .4 *Un vecteur aléatoire* $(Y_1, ..., Y_K)$ *est dit PLOD (resp. NLOD) (Positively (resp. Negatively) Lower Orthant Dependent) si*

$$\forall y_1, ..., y_K \in \mathbb{R},\ P[Y_1 \leq y_1, ..., Y_K \leq y_K] \geq (resp. \leq) \prod_{k=1}^{K} P[Y_k \leq y_k],$$

et PUOD (resp. NUOD) (Positively (resp. Negatively) Upper Orthant Dependent) si

$$\forall y_1, ..., y_K \in \mathbb{R},\ P[Y_1 > y_1, ..., Y_K > y_K] \geq (resp. \leq) \prod_{k=1}^{K} P[Y_k > y_k].$$

Ces deux définitions sont équivalentes quand $K = 2$ mais différentes dès que $K \geq 3$.

Théorème .6 *(Picard et al. (2003a))*
Si la distribution du montant global des sinistres $X(t)$ *est PLOD (resp. NLOD), alors la variable aléatoire des temps de ruine* $(T_1, .., T_K)$ *est PUOD (resp. NUOD).*

Definition .5 *Soit* $(Z_1, ..., Z_K)$ *et* $(Y_1, ..., Y_K)$ *deux variables aléatoires ayant les mêmes marginales. Y est dite plus petite que Z pour l'ordre de concordance inférieure (noté* $\leq_{lc}$*) si*

$$\forall y_1, ..., y_K \in \mathbb{R},\ P[Y_1 \leq y_1, ..., Y_K \leq y_K] \leq P[Z_1 \leq y_1, ..., Z_K \leq y_K],$$

et plus petite que Z pour l'ordre de concordance supérieure noté $\leq_{uc}$ *si*

$$\forall y_1, ..., y_K \in \mathbb{R},\ P[Y_1 > y_1, ..., Y_K > y_K] \leq P[Z_1 > y_1, ..., Z_K > y_K].$$

Encore une fois, ces deux définitions sont équivalentes quand $K = 2$ mais différentes dès que $K \geq 3$. Notons $(T_1^Y, ..., T_K^Y)$ les temps de ruine pour un processus de risque Y.

Théorème .7 *(Picard et al. (2003a))*
Si $(Y_1, ..., Y_K) \leq_{lc} (Z_1, ..., Z_K)$*, alors* $(T_1^Y, ..., T_K^Y) \leq_{uc} (T_1^Z, ..., T_K^Z)$

Revenons maintenant sur la notion de ruine avec plusieurs branches ou sources de risque. Picard et al. (2003a) ont considéré que la ruine intervient dès qu'une branche est ruinée. En pratique, même si les transferts de fond entre branches ne sont pas légalement autorisés, il semblerait que les pertes et profits des différentes branches puissent se compenser un minimum. On peut alors traduire de nouveaux critères de ruine plus généraux par de nouvelles régions $H(t)$ qui ne seront plus forcément des pavés (voir chapitre II.1 pour une discussion à ce sujet). Par exemple, si l'on poussait la compensation à l'extrême, on obtiendrait des régions de la forme $H(t) = \{v \in (\mathbb{R}^+)^K, v_1 + ... + v_K < h(t)\}$. Cela reviendrait alors à un modèle unidimensionnel avec le cumul des sinistres. Toutefois, il pourrait être intéressant d'utiliser d'autres critères intermédiaires plus réalistes, en faisant éventuellement intervenir la sévérité de ruine de chaque branche. Certains résultats de ce style pourraient certainement être également obtenus grâce à des méthodes semblables à celles des mathématiques actuarielles, par exemple pour se ramener à des groupes au premier décès[2].

Il est a priori plus logique de rechercher des risques négativement corrélés et donc d'adopter une mesure de risque qui les favorise. Toutefois, cette probabilité est intéressante comme renseignement complémentaire, par exemple de la probabilité de ruine classique du processus somme dans le modèle unidimensionnel, qui constitue un exemple de mesure de risque favorisant la corrélation négative entre les risques (voir Cossette et Marceau (2000)).
Jouini et al. (2005) a introduit récemment des mesures de risque vectorielles. Panjer (2001) et Panjer et Jing (2001) s'intéressent à la solvabilité à une unique date fixée d'une entreprise en présence de plusieurs risques décrits par des lois normales multivariées, et à l'allocation optimale de capital. Cette problématique, et les travaux de Dhaene et al. (2003), de Tsanakas et Barnett (2003) et de Finkelshtain et al. (1999) nous amènent naturellement à évoquer maintenant la question de l'allocation optimale de capital, ou de réserve.

Problèmes d'allocation optimale

Ces mesures de risques sont directement liées au problème d'allocation de capital économique ou d'allocation de réserve, à distinguer du capital à investir dans plusieurs branches.
Soit ρ une mesure de risque, et soit $(X_1, \ldots, X_K)$ un vecteur aléatoire représentant K risques p-intégrables (éventuellement à valeurs dans $\mathbb{Q}$ pour faciliter les preuves). On considère les portefeuilles de valeurs

$$X(u) = \sum_{k=1}^{K} u_k X_k$$

correspondant à une allocation u_k du capital total u à chaque risque k. ρ induit sur le portefeuille une mesure de risque ρ_B par $\rho_B(u) = \rho(X(u))$. Denault (2001) donne des conditions sur ρ et sur les risques pour que ρ_B soit une mesure de risque cohérente. Denault (2001), Delbaen (2002) et Fischer (2003) utilisent des résultats sur les jeux de coalition avec des joueurs fractionnels (notamment de Aubin (1979)) pour démontrer que l'allocation optimale peut être dans certains

[2]Je remercie Didier Rullière pour cette suggestion

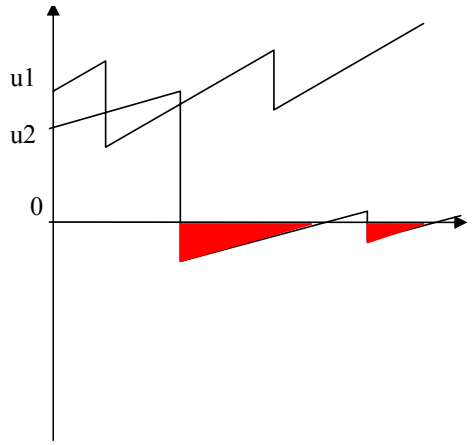

FIG. 2 – Première stratégie d'allocation de réserve (u_1, u_2)

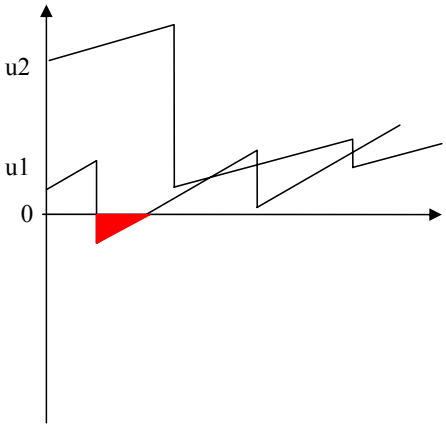

FIG. 3 – Une autre stratégie d'allocation de réserve (u_1, u_2)

cas obtenue avec la méthode du gradient : l'allocation par unité de capital optimale (u_k^*) est donnée par

$$\rho_B(u) = \sum_{k=1}^{K} u_k^* \frac{\partial \rho_B}{\partial u_k}(u).$$

Cette allocation de capital est différente du problème auquel nous nous sommes naturellement intéressé au chapitre II.2 : comment répartir la réserve initiale globale u entre les branches de façon à minimiser une certaine mesure de risque ? Il serait intéressant de disposer d'une mesure de risque neutre par rapport à la structure de dépendance entre les branches pour pouvoir ensuite déterminer si l'utilisation de la mesure de risque choisie favorise les branches positivement corrélées en leur allouant plus de capital, ou en tout cas si la dépendance a un impact significatif sur la stratégie d'allocation de capital. En fait, nous montrons dans le chapitre II.2 comment construire une telle mesure de risque à partir de la mesure de risque déjà évoquée, correspondant à la fonction de pénalité décrite par l'aire moyenne entre le processus et zéro lorsque celui-ci est en dessous de 0. On peut alors définir la mesure de risque donnée par la somme des aires espérées. Cette grandeur ne dépend pas de la structure de corrélation entre les branches. De plus, les théorèmes de différentiation, combinés à un lagrangien, permettent de déterminer l'allocation optimale $u_1 + \cdots + u_K = u$ telle que la somme des aires moyennes est minimale. Les figures 2 et 3 présentent deux trajectoires pour deux stratégies d'allocation différentes.

Elle consiste à ne pas affecter de réserve aux branches les plus sûres, et à répartir la réserve totale u entre les autres de façon à rendre égaux entre eux les temps moyens passés en-dessous de zéro par ces branches plus risquées. Ce résultat très parlant permet d'avoir une bonne conception de la stratégie d'allocation de capital avec cette mesure de risque.
Mentionnons que les contrats d'assurance étant souvent annuels, on ne peut s'en dégager aussi rapidement qu'une banque achète ou vend des titres. De plus, des règles comptables ou fiscales peuvent imposer de ne pas faire varier les réserves entre les branches trop souvent. C'est pour cette raison que nous ne nous sommes pas intéressés à des problèmes de contrôle optimal sur les capitaux des branches.
En économie du risque, tout ce qui correspond à de la monnaie est toujours regroupé dans une seule et même variable. Le seul cas où l'on considère des fonctions d'utilité à deux arguments est

celui où le deuxième bien ne s'échange pas avec de la monnaie, typiquement l'état de santé (voir les travaux de Courbage (2001) et Rey (2003)). Même si les richesses de deux branches d'une compagnie d'assurance sont toutes deux exprimées en monnaie, il est théoriquement interdit de combler le déficit de l'une avec l'autre. Dès lors, il semble intéressant de considérer une façon de représenter la satisfaction des actionnaires ou l'évaluation boursière de la compagnie grâce à une fonction d'utilité à K arguments, où K est le nombre de branches de la compagnie. Le transfert d'une partie de la réserve initiale de la branche 1 au profit de la branche 2 correspondrait alors au paiement d'une prime déduite de la richesse de la branche 1 pour réduire le risque supporté par la branche 2, ce qui correspond au type de problème étudié par Courbage (2001) et Rey (2003).

Principaux résultats

Cette thèse se décompose en deux parties. La première est consacrée à l'étude de processus en dimension 1. Dans la seconde, nous étudions des modèles multidimensionnels. Chaque chapitre correspond à un article.

1 Probabilité de ruine en temps fini

Le chapitre I.1 est formé par l'article *Another look at the Picard-Lefèvre formula for finite-time ruin probabilities.* Picard et Lefèvre (1997) donnent une formule pour calculer la probabilité de ruine en temps fini dans le modèle Poisson-composé, avec des montants de sinistres à valeurs entières. Leur résultat est fondé sur les polynômes d'Appell généralisés. Nous remarquons que la probabilité de ruine en temps continu est la même que celle en temps discret, pour des dates d'inventaire bien choisies. Nous expliquons comment obtenir une formule de type Seal, et comment passer d'une formule à l'autre. Les deux outils principaux sont un résultat classique de Takács (1962), et des convolutions formelles vérifiées par une classe de mesures signées. Ceci fournit une nouvelle démonstration de la formule de Picard-Lefèvre en considérant les trajectoires. Par ailleurs, les mesures signées que nous introduisons nous permettent d'obtenir toute une classe de formules englobant celle de Picard-Lefèvre et celle de type Seal. Il est intéressant de pouvoir choisir la formule adaptée, comme le montrent les figures 1.1 à 1.4 et l'analyse de la stabilité et de la complexité des algorithmes. Nous donnons aussi des formules adaptées au cas où l'on souhaite connaître la loi de l'instant de ruine.

2 Taux d'intérêt, double barrière, dividendes

Dans le papier intitulé *Hazard rates of a maximum-to-default distribution, and win-first probability under interest force,* constituant le chapitre I.2, nous nous intéressons à un problème de double barrière pour un processus stochastique affecté par un taux d'intérêt déterministe. Nous avons baptisé probabilité *win-first* la probabilité que le processus atteigne une barrière supérieure avant la ruine. Outre le fait de constituer un indicateur de risque et de profit, correspondant à la probabilité d'atteindre un objectif sous contrainte de solvabilité, cette probabilité est un élément de base à calculer quand on s'intéresse aux dividendes versés au-delà d'une barrière horizontale. Cette probabilité peut s'exprimer comme un quotient de probabilités de non ruine, pour lesquelles on dispose de méthodes dans des cas spécifiques, comme un taux d'intérêt très faible, ou des distributions d'amplitudes de sauts particulières. Nous proposons ici une méthode de calcul efficace qui ne requiert pas de telle hypothèse. Le principe est d'utiliser la fonction définie par le taux de hasard d'une variable aléatoire défective, correspondant au maximum du surplus partant de zéro jusqu'à la ruine. Cela nous permet d'obtenir une approximation très précise des probabilités *win first* et de ses dérivées par rapport à ses deux arguments jusqu'à un certain ordre. Nous

comparons ensuite notre méthode avec les méthodes du type de celle évoquée ci-dessus.

3 Différentiation de fonctionnelles de processus et allocation optimale

Dans l'article *Differentiation of some functionals of stochastic processes and optimal allocation,* qui constitue le chapitre II.2 de cette thèse, il nous a paru intéressant de considérer des fonctions de pénalité du type $I_T(u)$, où T est un temps fixé correspondant à un horizon de gestion. En effet, les actuaires et les organismes de contrôle et de régulation des assurances ont tendance à s'intéresser à des problèmes de ruine sur 5 ou 10 ans. Dans l'hypothèse où la compagnie, en cas de ruine, a l'opportunité de combler sa dette sous la forme d'un emprunt tant que sa richesse algébrique est négative, l'aire entre le processus et zéro quand celui-ci est en-dessous de zéro (voir figure 3.2) peut être interprétée comme un multiple des aggios, ou des intérêts à payer par la compagnie pour cause d'insolvabilité. Dans le modèle univarié, mais pour une grande classe de processus, nous observons que

$$u \to E\left(I_T(u)\right)$$

est dérivable par rapport à u sur $\mathbb{R}$, et que sa dérivée est égale à l'opposé du temps moyen passé en-dessous de zéro, à condition que ces grandeurs soient finies. Sous certaines conditions, dans le modèle Poisson-composé et en horizon infini, la dérivée du temps moyen passé en-dessous de zéro est égale à une constante (égale à c, la prime reçue par unité de temps) multipliée par la probabilité de ruine. Ceci permet, en particulier dans les cas où l'on connaît explicitement la probabilité de ruine, de retrouver le temps moyen passé en-dessous de zéro, déjà donné par Dos Reis (1993), et d'obtenir la moyenne de $I_\infty(u)$ par intégration par rapport à u. En effet, les quantités considérées sont nulles à l'infini. C'est le cas pour les modèles Poisson-exponentiel, et plus généralement Poisson-*phase-type*[3]. Enfin, ces théorèmes de différentiation peuvent être étendus à des fonctionnelles $I_{g,h}$, prenant en compte la notion de pénalité et de récompense, et s'orientant vers une approche économique liée aux fonctions d'utilité. La motivation principale, pour l'étude de cette aire, provient du modèle multirisques, car les théorèmes de différentiation permettent de déterminer l'allocation optimale de la réserve initialement répartie entre les branches. En particulier, si l'on cherche à minimiser la somme des "aires espérées en-dessous de zéro" pour chaque branche d'activité, nous obtenons la stratégie d'allocation optimale très rationnelle : elle consiste à ne rien allouer aux branches les moins risquées, et à répartir entre les branches les plus risquées la réserve initiale, de manière à rendre égaux entre eux les temps moyens passés par chaque branche d'activité en-dessous de zéro.

4 Autres modèles multidimensionnels

Dans les chapitres II.1 et II.3, nous introduisons et étudions des modèles prenant en compte l'évolution conjointe des différentes branches d'une compagnie d'assurances. Les processus multidimensionnels considérés correspondent à des processus de Poisson à chocs communs dans un environnement markovien. Ces deux phénomènes modélisent la dépendance entre les processus de cumul de sinistres jusqu'au temps t. Dans le chapitre II.3, nous étudions également des processus dont la position agit sur le niveau de cotisation reçu par les autres branches, et sur les dividendes versés par chaque branche, lorsqu'aucune autre branche n'est ruinée, dans la généralisation du modèle de dividendes du chapitre I.2. Dans le cas contraire, la compagnie est pénalisée et les

[3]Je remercie Esther Frostig pour cette remarque concernant la généralisation possible aux lois *phase-type*.

dividendes ne sont pas forcément entièrement reversés. Tous ces problèmes sont assez délicats, car ils font intervenir le comportement simultané des processus marginaux. Dans le chapitre II.1, nous étendons la formule de Picard-Lefèvre au modèle avec environnement markovien, et nous discutons du concept même de la ruine en dimension supérieure à 2 ainsi que de la pertinence des mesures de risque. Des inégalités reposant sur la structure de dépendance entre les risques et des résultats asymptotiques sont obtenus. Dans le chapitre II.3, nous montrons qu'il est possible de calculer la moyenne des dividendes versés aux actionnaires et de ceux qui ont dû être utilisés pour payer des pénalités à cause de l'insolvabilité temporaire d'une autre branche, en utilisant des méthodes de martingales introduites par Frostig (2004a) et par Asmussen et Kella (2000). En considérant une branche d'activité précise, notre approche consiste à discrétiser l'espace et à incorporer la position des autres branches dans l'ensemble des états de l'environnement. Nous calculons aussi le temps moyen de ruine de la branche principale et la loi de la richesse des autres branches à cet instant, ce qui donne une information sur la capacité des autres branches à faire face à la défaillance de la branche principale.

Première partie

Théorie de la ruine en dimension 1

1

Probabilité de ruine en temps fini Another look at the Picard-Lefèvre formula for finite-time ruin probabilities

In the compound Poisson risk model, with discrete claim size distribution, Picard et Lefèvre (1997) derived a formula to compute the finite-horizon ruin probability. Here, some alternatives to this formula are proposed : exact recursive formulas which provide the distribution of time to ruin at once and a Seal-type formula which only involve probabilistic quantities. Depending on the comparison between the initial reserve and the total premium up to the finite horizon, their different interests are discussed by comparing their performances. The numerical stability of the formulas is then investigated, and disagreements in the existing literature about the detection of critical values are explained.
Formal convolutions for pseudo-compound distributions are introduced, and a theorem is stated in order to switch between formulas based on Appell polynomials and Seal-type formulas. This also provides a derivation of the Picard-Lefèvre formula from sample path properties.

1.1 Introduction

Picard et Lefèvre (1997) gave a formula for the finite-time ruin probability where the claim amounts are integer-valued. This formula has been presented by De Vylder (1997) as a great advance in ruin theory. It has the advantage of introducing new tools which allow to use the law of cumulative claim amounts at some dates outside the interval $[0, t]$. This may be interesting for numerical purposes when the reserve is small compared to the cumulative premium up to time t. The formula involves what has been described by De Vylder (1999) as pseudo-compound Poisson probabilities, and the proof is based on Appell polynomials, which makes the generalization of the formula to a multidimensional framework possible Picard et al. (2003). However, these pseudo-probabilities may be very large in absolute value, which may be a problem for numerical analysis, when computations are made with a given number of significant digits. De Vylder (1999) detected critical values for the initial reserves after which numerical ruin probabilities may become inconsistent, whereas Ignatov et al. (2001), who also derived another formula, found no evidence of this problem using *Mathematica.*

This paper discusses several formulas that may be used to obtain exact finite-time ruin probabilities, and compares their speed and stability. We will so contribute to the existing debate about these numerical problems. More precisely, we will propose exact recursive schemes and a Seal-type formula to compute the finite-time ruin probability. Those formulas and the Picard-Lefèvre formulas have two main differences. Firstly, the Seal-type formula only involves probabilities which avoids the problem of large values. Depending on implementing standards and level of precision, we are able to obtain each result, which explains the appearing contradiction in the literature. Secondly, the formulas have very different complexities according to the initial reserve and the total premium received up to the finite horizon. Besides, the relative performances of the formulas depend on whether one wants to compute only the probability of ruin over a finite time horizon $[0, T]$, or the whole distribution of the time to ruin.
Another purpose of this paper is to show how pseudo-compound distributions are suitable for formal convolutions and may be used with sample path properties of compound processes to derive other formulas, to establish the link with the existing ones, and to shed a different light on the Picard-Lefèvre formula. In particular, we will give another proof of the Picard-Lefèvre formula, using formal convolutions and direct conditioning on the sample paths of the risk process. This provides new ways to get extensions to a multi-dimensional framework, or to compute other risk-measuring quantities.

In section 2, we briefly present a formula based on direct convolutions, and recursive formulas, which have the advantage of giving at once part of the distribution of the time to ruin. We

then recall the Picard-Lefèvre formulas, and state another formula for the finite-horizon ruin probability in which every term is a probability. In section 3, we show how pseudo-compound distributions enable us to switch between Appell-type formulas and Seal-type formulas. Section 4 gives a summary of the formulas and their complexities. Finally, in section 5, we compare computation times, values obtained by the different formulas and computation "sets" (the sets of points (j, τ) such that one needs to compute the law or conditional law of S_τ at point j for a given formula).

Let us model the reserves of an insurance company at time $t \geq 0$ as follows :

$$R_t = u + ct - S_t,$$

where u is the non-negative amount of initial reserves. The cumulative premium up to time t, $c(t)$, is any continuous function from $\mathbb{R}^+$ to $\mathbb{R}$. The cumulative amount of claims S_t is the compound Poisson process

$$S_t = \sum_{i=1}^{N_t} W_i,$$

where amounts of claims W_i, $i = 1, 2, ...$ are non-negative random integers. These amounts are mutually independent, identically distributed as a generic random variable W. The number of claims N_t until t is a homogeneous Poisson process with parameter λ. Claim amounts and arrival times are also independent.

For initial reserves $R_0 = u$, the time to ruin is defined as

$$T_u = \inf \{t, R_t < 0\},$$

with $T_u = +\infty$ if $R_t \geq 0$ for all $t \geq 0$. The probability of ruin over a finite time horizon $[0, t]$ is

$$\psi(u, t) = \mathbb{P}[T_u < t],$$

the probability of non-ruin being

$$\varphi(u, t) = \mathbb{P}[T_u \geq t] = 1 - \psi(u, t).$$

In the sequel, we denote the set of nonnegative integers by $\mathbb{N}$, the set of integers by $\mathbb{Z}$, and the set of positive integers by $\mathbb{N}^*$.

1.2 Direct convolutions and recursive formulas

1.2.1 Direct convolutions

Let us define the probability of ruin at inventory before time x as the probability that $R_t < 0$ for at least one time t belonging to a finite subset $\Gamma_x \subset [0, x]$ of inventory dates. Denoting by $\psi^*(u, x) = 1 - \varphi^*(u, x)$ that probability, it is obvious that if ruin is observed at an inventory date before x, then ruin is observed before time x, so,

$$\psi^*(u, x) \leq \psi(u, x) \quad \text{and} \quad \varphi^*(u, x) \geq \varphi(u, x).$$

We will see that the equality holds for a judicious choice of Γ_x.
Possible generalizations of the following formulas are shortly exposed at the end of this paragraph.

Denote the integer part of $x \in \mathbb{R}$ by $[x]$. Since $S_t \in \mathbb{N}$, the events $S_t \leq u+ct$ and $S_t \leq [u+ct]$ are equivalent. Define $m = [u] + 1$, $n = [u + cx] + 1$, $\tau_i = (i - u)/c$ for $i = m, ..., n - 1$, and $\tau_n = x$. Since S_t is non-decreasing in t, it follows

$$\mathbb{P}\left[S_t \leq u + ct, \forall t \in [0, x]\right] \quad = \quad \mathbb{P}\left[S_{\tau_i} < i, i = m, ..., n\right], \tag{1.1}$$

which may be rewritten as

$$\varphi\left(u, x\right) \quad = \quad \varphi^*\left(u, x\right) \text{ for } \Gamma_x = \left\{\tau_i, \quad i = m, ..., n\right\}.$$

Many algorithms may lead to an exact computation of ruin-at-inventory probabilities. For example, the following convolutions lead to the value $\varphi\left(u, x\right)$ where $x = \tau_n$ in the notation of formula (1.1).

Theorem I.1 *Let $j \in \mathbb{N}$, $i \geq m$ and $A_j\left(\tau_i\right) = \mathbb{P}\left[S_{\tau_i} = j \cap R_{\tau_k} > 0, k = m, \ldots, i\right]$. Let also $\tau_{m-1} = 0$ and $A_j\left(0\right) = \mathbf{1}_{j=0}$. Then*

$$A_j\left(\tau_i\right) = \mathbf{1}_{j<i} \cdot \sum_{k=0}^{j} A_{j-k}\left(\tau_{i-1}\right) \cdot \mathbb{P}\left[S_{1/c} = k\right], \; i \geq m,$$

$$\varphi(u, \tau_i) = \mathbb{P}\left[S_{\tau_k} < k, k = m, \ldots, i\right] = \sum_{j=0}^{i} A_j\left(\tau_i\right), \; i = m, \ldots, n.$$

For any couple (u, x), the survival function of T_u will be obtained on every inventory date of $[0, x]$, including x. A great advantage of such an approach is that other risk indicators, like the severity at ruin, will be easily computable. The probabilities $\mathbb{P}\left[S_{1/c} = k\right]$ can be calculated by Panjer's formula Panjer (1981).

This kind of approach is not limited to the homogeneous Poisson case, and it can be adapted to the case where the cumulative premium up to time t, $c\left(t\right)$, is different from $u + ct$.

Given a date $x > 0$, it is quite obvious that $\forall t \in \left[0, x\right], S_t \leq c\left(t\right)$ if and only if $\forall t \in \left[0, x\right], S_t \leq c_x^*\left(t\right)$, where $c_x^*\left(t\right) = \inf\left\{c\left(s\right), s \in \left[t, x\right]\right\}$ is non-decreasing in t. We can thus assume that c is an non-decreasing function without loss of generality. Define $m = 1 + [c(0)]$, $n = 1 + [c(x)]$, and $\tau_i = \sup\left\{t, t \in [0, x], c\left(t\right) < i\right\}$ for $i = m, ..., n$. As S_t is non-decreasing,

$$\mathbb{P}\left[R_t > 0, \forall t \in]0, x]\right] = \mathbb{P}\left[S_{\tau_i} < i, i = m, ..., n\right].$$

If, almost surely, no claim occurs during any finite set of dates, and $c(t)$ is a continuous, increasing function, then

$$\varphi\left(u, x\right) \quad = \quad \varphi^*\left(u, x\right) \text{ for } \Gamma_x = \left\{\tau_i, \quad i = m, ..., n\right\}. \tag{1.2}$$

If, besides, N_t is a (non-homogeneous) Poisson process, the following recursive formula gives the finite-time ruin probability :

Theorem I.2 *Let $j \in \mathbb{N}$, $i \geq m$ and $A_j\left(\tau_i\right) = \mathbb{P}\left[S_{\tau_i} = j \cap R_{\tau_k} > 0, k = m, \ldots, i\right]$. Let also $\tau_{m-1} = 0$ and $A_j\left(0\right) = \mathbf{1}_{j=0}$. Then*

$$A_j\left(\tau_i\right) = \mathbf{1}_{j<i} \cdot \sum_{k=0}^{j} A_{j-k}\left(\tau_{i-1}\right) \cdot \mathbb{P}\left[S_{\tau_i} - S_{\tau_{i-1}} = k\right], \; i \geq m,$$

$$\varphi(u, \tau_i) = \mathbb{P}\left[S_{\tau_k} < k, k = m, \ldots, i\right] = \sum_{j=0}^{i} A_j\left(\tau_i\right), \; i = m, \ldots, n.$$

1.2.2 The Picard-Lefèvre formula

Let us define for $j \in \mathbb{N}$ and $\tau \in \mathbb{R}$ the quantities $h_j(\tau)$, which may be computed by Panjer's recursive algorithm (see Panjer (1981), Wang et Panjer (1993)), as follows :

$$h_0(\tau) = e^{-\frac{\lambda\tau}{c}} \text{ and } h_j(\tau) = \frac{\lambda\tau}{cj} \sum_{i=1}^{j} i \cdot \mathbb{P}[W = i] \cdot h_{j-i}(\tau).$$

Define also for $j \in \mathbb{N}$ and $\tau \in \mathbb{R}^*$

$$\begin{aligned} H_j(\tau) &= \sum_{i=0}^{[j]} h_i(\tau), \\ \widetilde{H}_j(\tau) &= \sum_{i=0}^{[j]} h_i(\tau) \left(1 - \frac{i}{\tau}\right). \end{aligned}$$

One should keep in mind that, for $j > 0$, $H_j(\tau)$ and $\widetilde{H}_j(\tau)$ can be cumulated during the calculus of $h_j(\tau)$. So, computation times are similar for each of these three quantities. This also spares stocking memory. Let us define $\widetilde{H}_0(0) = 1$, according to the convergence of $\widetilde{H}_0(t)$ as t tends to 0.
For all $\tau > 0$, $h_j(\tau)$ is the probability that the cumulative claim amount at time τ/c is j :

$$\begin{aligned} h_j(\tau) &= \mathbb{P}\left[S_{\tau/c} = j\right], \\ H_j(\tau) &= \mathbb{P}\left[S_{\tau/c} \leq j\right], \\ \widetilde{H}_j(\tau) &= \frac{1}{\tau} E\left[\tau - S_{\frac{\tau}{c}} \mathbf{1}_{\{S_{\frac{\tau}{c}} \leq j\}}\right]. \end{aligned}$$

We will see later that $\widetilde{H}_{[\tau]}(\tau)$ may be interpreted as the probability that ruin has not occurred before time $\frac{\tau}{c}$.

Theorem I.3 *Picard et Lefèvre (1997) The probability of non-ruin over a finite time horizon* $[0, t]$ *is given by*

$$\varphi(u,t) = \sum_{j=0}^{u} \left[h_j(ct) + h_j(j-u) \sum_{i=u+1}^{[u+ct]} h_{i-j}(u+ct-j) \frac{u+ct-i}{u+ct-j} \right]. \tag{1.3}$$

To make computations faster, one may stock all $h_{i-j}(u+ct-j)$ during the computation of the last term $h_{[u+ct]-j}(u+ct-j)$ in the second sum of this formula. One may also use other versions of the formula, like in **?**. With a mere sum interchange, for $u \in \mathbb{N}$,

$$\mathbb{P}[T_u > t] = H_u(ct) + \sum_{i=0}^{u} h_{u-i}(-i) . \Delta\widetilde{H}_{i+1}^{i+[ct]}(i+ct),$$

$$\text{where} \quad \Delta\widetilde{H}_a^j(\tau) = \sum_{i=a}^{j} \left(1 - \frac{i}{\tau}\right) h_i(\tau) \quad \text{for } a \in \mathbb{N}^*.$$

This simple reordering is enough to compute compounds of $h_i(\tau)$ for increasing values of i, and thus saves time and space.
For $a \in \mathbb{N}^*$, values

$$\Delta\widetilde{H}_a^j(\tau) = \widetilde{H}_j(\tau) - \widetilde{H}_{a-1}(\tau)$$

are cumulated during the computation of $h_j(\tau)$.
In a paper on the moments of the time to ruin, Picard et Lefèvre (1998) used another version of their formula :

Theorem I.4 *Picard et Lefèvre (1998)*

$$\begin{aligned}\varphi(u,t) &= \sum_{i=0}^{u} h_{u-i}(-i) \sum_{j=0}^{i+[ct]} \left(1-\frac{j}{i+ct}\right) h_j(i+ct). \qquad (1.4)\\ \varphi(u,t) &= \sum_{i=0}^{u} h_{u-i}(-i)\cdot \widetilde{H}_{i+[ct]}(i+ct).\end{aligned}$$

Corollary I.1 *For $x>0$,*

$$\begin{aligned}\varphi(0,x/c) &= \widetilde{H}_{[x]}(x), \quad \text{and for } u>0, \qquad (1.5)\\ \varphi(u,x/c) &= \sum_{i=0}^{u} h_{u-i}(-i)\cdot\varphi\left(0,\frac{i+x}{c}\right).\end{aligned}$$

Let us also mention another formula derived by Ignatov et al. (2001), which allows dependence between claim amounts.

We may get the following formulas for $u\in\mathbb{R}^+$ from the original Picard-Lefèvre formula : in order not to be ruined, we just need to check that during time $c\varepsilon_u$, the cumulative claim amount does not exceed u, and that we are not ruined from an integer-valued reserve. This may be useful for numerical purposes, for example to inverse ruin probabilities. The formulas for $u\in\mathbb{R}^+$ are given by the following theorem :

Theorem I.5

$$\begin{aligned}\varphi(u,t/c) &= \sum_{j=0}^{[u]} h_j(j-u)\widetilde{H}_{[t+u-j]}(t+u-j). \qquad (1.6)\\ \varphi(u,t/c) &= \sum_{i=0}^{[u]} h_{[u]-i}(-i-\varepsilon_u)\widetilde{H}_{[t+i+\varepsilon_u]}(t+i+\varepsilon_u). \qquad (1.7)\end{aligned}$$

One of the advantages of the Picard-Lefèvre formula is that it can be generalized using Appell functionals (see Picard et al. (2003)). It is also more convenient for the case $u << ct$, as we will see later. However, these obtained formulas have some light drawbacks :

- Firstly, some quantities in these formulas (namely the $h_j(j-u)$ for $j<u$) are not probabilities and may take negative values. These quantities seem difficult to be interpreted and may be very large in absolute value, which may be a problem for numerical analysis, when computations are made with a given number of significant digits. For large values of u, it may require too high a precision level and strongly increase computation time and instability.
- Secondly, the proof is based on Appell and generalized Appell polynomials, and does not enable us to understand directly the formula from the properties of the sample paths.

Thanks to a result from Takács (1962), we are going to derive another formula (a particular case of the Seal formula) which has none of these drawbacks. The proof is simple and based on the sample path properties, and the formula only involves probabilities, which solves part of the numerical problems. This will be developed just after we state recursive formulas which are interesting if one wants to get the distribution of the time to ruin.

1.2.3 Recursive formulas

A first and direct application of simple convolutions of formula (1.2) provides an easy way to compute ruin probabilities. Let us first consider the case where we need to compute $\varphi(u, x/c)$ for fixed $u, x \in \mathbb{N}$. This restriction will be released in the following section.

Theorem I.6 *For $u \in \mathbb{N}$, $x \in \mathbb{N}$, the following, direct convolutions give the probability of non-ruin over a finite time horizon :*

$$\begin{aligned} A_k(0) &= \mathbf{1}_{k=0} \text{ and } h_k = h_k(1), \ k = 1, \ldots, u+x. \\ A_j\left(\frac{i}{c}\right) &= \mathbf{1}_{j<u+i} \sum_{k=0}^{j} A_{j-k}\left(\frac{i-1}{c}\right) h_k, \ i = 1, \ldots, x, \ j = 1, \ldots, u+i. \\ \varphi\left(u, \frac{i}{c}\right) &= \sum_{j=0}^{u+i} A_j\left(\frac{i}{c}\right), \ i = 1, \ldots, x. \end{aligned} \tag{1.8}$$

Remark I.1 *The last summation may be done while computing values $A_j\left(\frac{i}{c}\right)$.*

Other exact recursive formulas can be built. Let us define $\Omega_x = \{t, t \in]0, x], u + ct \in \mathbb{N}^*\}$.

A classical decomposition of the event $R_x \geq 0$, conditioning by last time t at which $R_t = 0$, gives for $x > 0$:

$$\mathbb{P}[R_x \geq 0] = \mathbb{P}[T_u > x] + \sum_{t \in \Omega_x} \mathbb{P}[R_t = 0] \, \mathbb{P}[T_0 > x - t].$$

Considering the more convenient date x/c with $x > 1$, it follows that

$$\mathbb{P}[cT_u > x] = \mathbb{P}\left[S_{x/c} \leq u + x\right] - \sum_{t \in \Omega_{x/c}} \mathbb{P}[S_t = u + ct] \, \mathbb{P}[cT_0 > x - ct], \ x > 1.$$

$S_t \in \mathbb{N}$ for all $t > 0$, so ruin does not occur before time x/c, with $x \leq 1$, if and only if $S_{x/c} \leq u$ and

$$\mathbb{P}[cT_u > x] = \mathbb{P}\left[S_{x/c} \leq u\right] \text{ with } x \leq 1.$$

Denote the integer part of $v \in \mathbb{R}$ by $[v]$. Denote also $\varepsilon_v = v - [v] \in [0, 1[$, $\overline{\varepsilon}_v = 1 - \varepsilon_v$. Define $\nu = [\varepsilon_u + \varepsilon_x]$ and $\varepsilon = \varepsilon_u + \varepsilon_x - \nu$.

Lemma I.1 *If the probability of ruin without initial reserve is known, the probability of non-ruin over a finite time horizon $[0, x]$, with initial surplus u, is given by the following recursive formula : for $n = 1, \ldots, [x]$,*

$$\begin{aligned} \varphi\left(u, \frac{\varepsilon_x}{c}\right) &= H_u(\varepsilon_x), \quad \text{and} \\ \varphi\left(u, \frac{n + \varepsilon_x}{c}\right) &= H_{u+n+\nu}(n + \varepsilon_x) - \sum_{k=1}^{n+\nu} h_{[u]+k}(k - \varepsilon_u) \, \varphi\left(0, \frac{n - k + \nu + \varepsilon}{c}\right). \end{aligned}$$

Proof. For all $i \in \mathbb{N}$, $u + c\left(\frac{i+\overline{\varepsilon}_u}{c}\right) \in \mathbb{N}$. Recalling that $\Omega_{x/c} = \{t, t \in]0, x], u + ct \in \mathbb{N}\}$, it follows $\Omega_{x/c} = \left\{\frac{i+\overline{\varepsilon}_u}{c}, \quad i = 0, \ldots, [x - \overline{\varepsilon}_u]\right\}$.
Putting $k = i + 1$, $\mathbb{P}[cT_u > x]$ may be expressed as

$$\mathbb{P}\left[S_{x/c} \leq [u + x]\right] - \sum_{k=1}^{[x]+\nu} \mathbb{P}\left[S_{(k-\varepsilon_u)/c} = [u] + k\right] \cdot \mathbb{P}\left[cT_0 > [x] - k + \varepsilon_u + \varepsilon_x\right],$$

and the result holds.

Lemma I.2 *The probability of non-ruin until time $n + \varepsilon$, with initial surplus 0, is given by the following recursive formula : for $n = 1, 2, \ldots, [x]$,*

$$\begin{aligned} \varphi\left(\frac{\varepsilon}{c}\right) &= H_0(\varepsilon), \quad \text{and} \\ \varphi\left(0, \frac{n+\varepsilon}{c}\right) &= H_n(n+\varepsilon) - \sum_{k=1}^{n} h_k(k)\, \varphi\left(0, \frac{n-k+\varepsilon}{c}\right). \end{aligned}$$

Proof. Direct application of previous lemma, for $u = 0$.

Theorem I.7 *(recursive computation) Let $x > 1$. Probabilities of non-ruin at times $n + \varepsilon_x$, $n = 1, 2, \ldots, [x]$, with initial surplus u, are given by following recursive scheme :*

- *Step 1 : $\varphi\left(0, \frac{\varepsilon}{c}\right) = H_x(\varepsilon)$ and $\varphi\left(u, \frac{\varepsilon_x}{c}\right) = H_{u+x}(\varepsilon_x)$.*
- *Step 2 : For n=1,...,[x],*

$$\varphi\left(0, \frac{n+\varepsilon}{c}\right) = H_n(n+\varepsilon) - \sum_{k=1}^{n} h_k(k)\, \varphi\left(0, \frac{n-k+\varepsilon}{c}\right).$$

- *Step 3 : For n=1,...,[x],*

$$\varphi\left(u, \frac{n+\varepsilon_x}{c}\right) = H_{[u]+n+\nu}(n+\varepsilon_x) - \sum_{k=1}^{n+\nu} h_{[u]+k}(k-\varepsilon_u)\, \varphi\left(0, \frac{n-k+\nu+\varepsilon}{c}\right).$$

Proof. Direct application of previous lemmas.

This recursive algorithm enables us to obtain the survival function of T_0 and T_u at $[x]$ points of the period $[0, x]$. If $u \in \mathbb{N}$, we have $\varepsilon_u = 0$, $\nu = 0$, $\varepsilon = \varepsilon_x$, and the supports of survival functions are identical for T_0 and T_u.

Corollary I.2 *For $u \in \mathbb{N}$, $x \in \mathbb{N}^*$, probabilities of non-ruin at times n, $n = 1, \ldots, x$, with initial surplus u, are given by the following recursive scheme :*

$$\begin{aligned} &\textit{Step 1} \quad : \quad \varphi(0,0) = 1 \text{ and } \varphi(u,0) = 1. \qquad (1.9) \\ &\textit{Step 2} \quad : \quad \varphi\left(0, \frac{n}{c}\right) = H_n(n) - \sum_{k=1}^{n} h_k(k)\, \varphi\left(0, \frac{n-k}{c}\right), \; n = 1, \ldots, x. \\ &\textit{Step 3} \quad : \quad \varphi\left(u, \frac{n}{c}\right) = H_{u+n}(n) - \sum_{k=1}^{n} h_{u+k}(k)\, \varphi\left(0, \frac{n-k}{c}\right), \; n = 1, \ldots, x. \end{aligned}$$

Proof : For $u \in \mathbb{N}$, we have $\varepsilon_u = 0$, $\nu = 0$, $\varepsilon = \varepsilon_x$. If, moreover, $x \in \mathbb{N}^*$, then $\varepsilon = \varepsilon_x = 0$, and the result is a direct application of the previous theorem.

1.2.4 Probabilistic equivalent of the Picard-Lefèvre formula

Let us start with a classical result from Takács (1962), which is a key lemma to get the other "Seal-type" formula.

Lemma I.3 *Takács (1962) Let $Y_n = v_1 + \cdots + v_n$, where the v_i are i.i.d., non-negative integer-valued random variables.*
For $n \in \mathbb{N}^*$ and $i \in \mathbb{N}$ such that $i \leq n$,

$$\mathbb{P}\left[\{Y_r < r, r = 1, \ldots, n\} \cap \{Y_n = n - i\}\right] = \frac{i}{n}\mathbb{P}\left[Y_n = n - i\right].$$

>From properties of compound Poisson processes, the application of the previous result leads to the two well-known following results :

Theorem I.8 *Takács (1962) Let $n \in \mathbb{N}^*$.*

$$\begin{aligned}\mathbb{P}\left[S_{\frac{n}{c}} = i \cap R_t \geq 0, t \in [0, \frac{n}{c}]\right] &= \frac{n-i}{n}\mathbb{P}\left[S_{\frac{n}{c}} = i\right], i = 0, \ldots, n.\\ \varphi\left(0, \frac{n}{c}\right) &= \sum_{i=1}^{n} \frac{n-i}{n}\mathbb{P}\left[S_{\frac{n}{c}} = i\right] = \frac{1}{n}E\left(R^0_{n/c}\right)_+.\end{aligned}$$

Theorem I.9 *Seal (1969), Takács (1962) More generally, for $t \in \mathbb{R}^+$,*

$$\begin{aligned}\varphi(0,t) &= \frac{1}{ct}E\left(R^0_t\right)_+. \qquad (1.10)\\ \varphi(0,t/c) &= \sum_{i=1}^{[t]}\left(1 - \frac{i}{t}\right)\mathbb{P}\left[S_{\frac{t}{c}} = i\right] = \widetilde{H}_{[t]}(t).\end{aligned}$$

This enables us to deduce the probability of ruin without initial reserve over a finite time horizon $[0, t]$ from the distribution of the cumulative claim amount at time t. Recall that ruin is here the same as ruin at inventory at regular dates. By using Takács' result and conditioning by the last time of ruin in case of ruin, we get the following theorem, only from sample path properties.

Theorem I.10 *Let $x > 1$. From formula $(I.7)$, probabilities of non-ruin at times $n + \varepsilon_x$, $n = 1, \ldots, [x]$, with initial surplus u, are given by*

$$\varphi\left(u, \frac{n + \varepsilon_x}{c}\right) = H_{u+n+\nu}(n + \varepsilon_x) - \sum_{k=1}^{n+\nu} h_{[u]+k}(k - \varepsilon_u)\,\widetilde{H}_{n-k+\nu}(n - k + \nu + \varepsilon)$$

In particular, for $n = [x]$,

$$\varphi\left(u, \frac{x}{c}\right) = H_{[u+x]}(x) - \sum_{k=1}^{[x]+\nu} h_{[u]+k}(k - \varepsilon_u)\,\widetilde{H}_{[x]-k+\nu}(x - k + \nu) \qquad (1.11)$$

This is a particular case of the Seal (1969) formula Gerber (1979). This formula only depends on quantities $h_j(\tau)$, like the Picard-Lefèvre formula. However, the involved couples (j, τ) belong to a set which is very different from that of the couples involved in the Picard-Lefèvre formula. All quantities $h_j(\tau)$, $H_j(\tau)$ and $\widetilde{H}_j(\tau)$ have here a probabilistic meaning and are thus between 0 and 1. For $u \in \mathbb{N}$, one may accelerate the computations by first computing $\alpha_k = h_{u+k}(k)$ and $\beta_k = \widetilde{H}_k(k)$ at the same time for $1 \leq k \leq n$.

1.3 Switching between Appell-type formulas and Seal-type formulas

Let us define for $t > 0$ and $m \in \mathbb{N}$ the formal quantities (which have no probabilistic meaning)

$$\tilde{P}(S(-t) = m) = e^{\lambda t} \sum_{k=0}^{m} \frac{(-\lambda t)^k}{k!} \mathbb{P}(X^{*k} = m).$$

Newton's binomial formula and definition of convolutions ensure that, formally, for all $m \in \mathbb{N}$,

$$\tilde{P}(S(t) + S(t') = m) = \tilde{P}(S(t + t') = m) = \sum_{k=0}^{m} \tilde{P}(S(t) = k)\tilde{P}(S(t') = m - k)$$

holds for all $t, t' \in \mathbb{R}$, with the convention $\tilde{P}(S(0) = m) = \delta_{m,0}$.
Moreover, if $t + t' > 0$, this corresponds to the real probability :

$$\tilde{P}(S(t + t') = m) = \mathbb{P}(S(t + t') = m).$$

With these definitions and basic properties, it is possible to derive the following formula.

Theorem I.11 *For $t \in \mathbb{N}^*$, $x \in \mathbb{N}$ and for any $z \in \mathbb{Z}$, $z > x - t$,*

$$\mathbb{P}(S(t) = x) = \sum_{j=0}^{x} \mathbb{P}(S(z + t - j) = x - j)\tilde{P}(S(j - z) = j)\frac{z + t - x}{z + t - j}. \tag{1.12}$$

Proof. Firstly, set $t' \in \mathbb{N}^*$, $x \in \mathbb{N}$, $x < t'$, and $j_0 \in \mathbb{N}$, such that $j_0 < t' - x$. By conditioning on the first time $t' - j_0 - k$ at which $S(t' - j_0 - k) = x - k$, and using Takács's result once again, we get

$$\mathbb{P}(S(t') = x) = \sum_{k=0}^{x} \left(h_{x-k}(t' - j_0 - k)\frac{t' - j_0 - x}{t' - j_0 - k} \right) h_k(k + j_0), \tag{1.13}$$

which is the required formula for $z = -j_0$ and $x - t < z \leq 0$. Now, set $z > -t$ and $i \in \mathbb{N}$. For $-z \leq i < t$,

$$h_{z+i}(t) = \mathbb{P}(S(t) = z + i) = \tilde{P}(S(z + t) + S(-z) = z + i).$$

$$h_{z+i}(t) = \sum_{j=0}^{z+i} \mathbb{P}(S(z + t) = z + i - j)\tilde{P}(S(-z) = j).$$

The application of equation (1.13) for $t' = z + t$ and $x = z + i - j$ is possible since $x < t'$. It gives, in the particular case $j_0 = j$,

$$h_{z+i}(t) = \sum_{j=0}^{z+i} \sum_{k=0}^{z+i-j} h_{z+i-j-k}(z + t - j - k)\frac{t - i}{z + t - j - k} h_k(k + j)\tilde{P}(S(-z) = j).$$

$$h_{z+i}(t) = \sum_{l=j+k=0}^{z+i} h_{z+i-l}(z + t - l)\frac{t - i}{z + t - l} \sum_{k=0}^{l} \mathbb{P}(S(l) = k)\tilde{P}(S(-z) = l - k).$$

$$h_{z+i}(t) = \sum_{l=0}^{z+i} h_{z+i-l}(z + t - l)\frac{t - i}{z + t - l}\tilde{P}(S(l - z) = l).$$

Replacing i with $x - z$ for $z > x - t$, gives the desired result, which may be easily extended to the case $t \in \mathbb{Z}$.

To keep it clear and concise, the proof of the equivalence of the two formulas is given here for $u \in \mathbb{N}$ and $c = 1$, but it may be easily adapted to the general case. Let us denote $n = [t]$. Derivation is direct for $t - n > 0$, and remains valid for $t = n$, with the convention $\widetilde{H}_0(0) = 1$. This is consistent with limit properties and the fact that $\mathbb{P}(T_u = t) = 0$ for all $t > 0$.
The Picard-Lefèvre formula is :

$$\varphi_{PL}(u,t) = \sum_{j=0}^{u}[h_j(t) + h_j(j-u)\sum_{i=u+1}^{u+n} h_{i-j}(u+t-j)\frac{u+t-i}{u+t-j}].$$

Thanks to theorem I.11,

$$\varphi_{PL}(u,t) = \sum_{i=0}^{u} h_i(t) + \sum_{i=u+1}^{u+n}\sum_{j=0}^{i} h_j(j-u)h_{i-j}(u+t-j)\frac{u+t-i}{u+t-j}$$

$$- \sum_{i=u+1}^{u+n}\sum_{j=u+1}^{i} h_j(j-u)h_{i-j}(u+t-j)\frac{u+t-i}{u+t-j}$$

$$= \sum_{i=0}^{u} h_i(t) + \sum_{i=u+1}^{u+n} h_i(t) - \sum_{i=u+1}^{u+n}\sum_{j=u+1}^{i} h_j(j-u)h_{i-j}(u+t-j)\frac{u+t-i}{u+t-j}.$$

With $i' = i - u$ and $k = j - u$, we get

$$\varphi_{PL}(u,t) = H_{u+n}(t) - \sum_{i'=1}^{n}\sum_{k=1}^{i'} h_{u+k}(k)h_{i'-k}(t-k)\frac{t-i'}{t-k}.$$

A simple sum inversion, with $l = i' - k$, gives

$$\varphi_{PL}(u,t) = H_{u+n}(t) - \sum_{k=1}^{n}\sum_{l=0}^{n-k} h_{u+k}(k)h_{l-k}(t-k)\frac{t-l}{t-k} = \varphi_{Alt.}(u,t).$$

We thus proved the equivalence between the two formulas. Note that, besides making the formula easier to interpret, Takács' result and basic convolution and summation properties enable us to give a new proof of the Picard-Lefèvre formula, based on sample path properties.

1.4 Complexity of each formula

The complexity of each algorithm depends on the quantities that must be computed. For example, for some $u \in \mathbb{N}$ and $x \in \mathbb{N}^*$, we prescribe to use different formulas to compute $\varphi\left(u, \frac{n}{c}\right)$, $n = 1, \ldots, x$, and to compute only $\varphi\left(u, \frac{x}{c}\right)$.

In order to simplify the comparison, we have chosen $u \in \mathbb{N}$ and $x \in \mathbb{N}^*$. Let us try to measure how many loops are needed to compute the values $\varphi\left(u, \frac{n}{c}\right)$, $n = 1, \ldots, x$ and the only value $\varphi\left(u, \frac{x}{c}\right)$. Denote by $n_u(x)$ the number of loops necessary to compute $\varphi\left(u, \frac{x}{c}\right)$, and by $N_u(x)$ the number of loops necessary to compute $\{\varphi\left(u, \frac{n}{c}\right), n = 1, \ldots, x\}$.
Very often, the only value $\mathbb{P}[T_u > x]$ is not enough, and we are interested in the whole distribution of T_u.

Here are summarized most of the exact formulas for the computation of ruin probabilities that we developed, for $u \in \mathbb{N}$ and $x \in \mathbb{N}^*$, the general case $u, x \in \mathbb{R}$ having been treated in previous sections for each formula. We give in each case approximations of $n_u(x)$ and $N_u(x)$ for large values of x or u. Computations of these numbers suppose that each algorithm is developed with as few loops as possible. In particular some probabilities are computed and stocked in order not to compute them several times. We consider that the number of elementary computations to get $h_j(n)$ is about $\frac{j^2}{2}$.

First Picard-Lefèvre's formula (1.3) :

$$n_u(x) \simeq \frac{u^3}{3} + \frac{ux(x+u)}{2}, \quad N_u(x) \simeq ux\left(\frac{u^2}{3} + \frac{ux}{4} + \frac{x^2}{6}\right).$$

$$\varphi\left(u, \frac{n}{c}\right) = \sum_{j=0}^{u} h_j(n) + h_j(j-u) \sum_{i=u+1}^{u+n} h_{i-j}(u+n-j) \frac{u+n-i}{u+n-j},\ n = 1, \ldots, x.$$

Other Picard-Lefèvre's formula (1.4) :

$$n_u(x) \simeq \frac{u^3}{3} + \frac{ux(x+u)}{2}, \quad N_u(x) \simeq ux\left(\frac{u^2}{3} + \frac{ux}{4} + \frac{x^2}{6}\right).$$

$$\varphi\left(u, \frac{n}{c}\right) = \sum_{i=0}^{u} h_{u-i}(-i) \cdot \widetilde{H}_{i+n}(i+n),\ n = 1, \ldots, x.$$

Direct, simple convolutions (1.8) :

$$n_u(x) \simeq \frac{x^3}{6} + \frac{ux(x+u)}{2}, \quad N_u(x) \simeq \frac{x^3}{6} + \frac{ux(x+u)}{2}.$$

$$\begin{aligned} A_j\left(\frac{n}{c}\right) &= \mathbf{1}_{j<u+n} \sum_{k=0}^{j} A_{j-k}\left(\frac{n-1}{c}\right) h_k(1),\ n = 1, \ldots, x,\ j = 1 \ldots u+n. \\ \varphi\left(u, \frac{n}{c}\right) &= \sum_{j=0}^{u+n} A_j\left(\frac{n}{c}\right),\ n = 1, \ldots, x. \end{aligned}$$

Recursive formulas (1.9) :

$$n_u(x) \simeq \frac{x^3}{3} + \frac{ux(x+u)}{2}, \quad N_u(x) \simeq \frac{x^3}{3} + \frac{ux(x+u)}{2}.$$

$$\begin{aligned} \varphi\left(0, \frac{n}{c}\right) &= H_n(n) - \sum_{k=1}^{n} h_k(k) \varphi\left(0, \frac{n-k}{c}\right),\ n = 1, \ldots, x. \\ \varphi\left(u, \frac{n}{c}\right) &= H_{u+n}(n) - \sum_{k=1}^{n} h_{u+k}(k) \varphi\left(0, \frac{n-k}{c}\right),\ n = 1, \ldots, x. \end{aligned}$$

Non-recursive alternative (1.11) :

$$n_u(x) \simeq \frac{x^3}{3} + \frac{ux(x+u)}{2}, \quad N_u(x) \simeq \frac{x^2}{2}\left(\frac{u^2}{2} + \frac{ux}{3} + \frac{x^2}{6}\right).$$

$$\varphi\left(u, \frac{n}{c}\right) = H_{u+n}(n) - \sum_{k=1}^{n} h_{u+k}(k)\, \widetilde{H}_{n-k}(n-k),\ n = 1, \ldots, x.$$

This suggests that the choice between the formulas depends on the comparison between u and ct, and on whether one aims at obtaining a description of the distribution of the time to ruin up to t, or at getting only one particular ruin probability. This will also be illustrated later with the so-called computation sets.

1.5 Numerical results

We do not suggest here that one formula is better than the others. The numerical results confirm that, depending on the case, different formulas are the most appropriate.

Here, numerical computations are made with standard programming language, namely *C++*, with standard precision level of 16 significant digits (without any variable-precision module) and with *Maple* with different levels of precision. For example Ignatov et al. (2001) show how to cope with numerical problems by using *Mathematica*, which adapts the number of internal digits to the numerical stability of the calculations (see also *Mathematica* Web site).

Without considering the numerical stability of the formulas, the Picard-Lefèvre formulas are well adapted to the case where u is small (compared to ct), and that the other formulas presented here are well adapted to the case where u is large.

We first checked the formulas we obtained with De Vylder's (1999) results. With our different formulas, we were able to get the same results as in his tables 1 and 2. In his table 3 De Vylder (1999), he tried to detect critical values : for deterministic claim amounts (W distributed as Dirac(1)), and for $\lambda = 1$, $c = 1.25$ and $t = 10$, he tried to determine until which initial reserve u the computation of $\psi(u, 10)$ seemed acceptable. He found a critical value for u around 22, after which ruin probabilities became irrelevant, whereas Ignatov et al. (2001) detected no critical value with *Mathematica*. If the precision level is fixed at 16 decimal digits, which corresponds with common implementing standards to a 52-bit mantissa, then 2^n equals $2^n + 1$ numerically for $n > 53$. In De Vylder (1999), the computation involves terms such as $h_0(0-u) = e^{\lambda u/c}$, which, for large u, become much larger than 2^{53} and are likely to generate numerical errors. Of course, for softwares which keep track of the needed precision level, this kind of phenomenon may be avoided. We show in table 1 that we can find results (column 3) similar to those of De Vylder (1999) (column 1) when precision is fixed at 20 digits. With a higher level of precision (column 3), we found back the results of Ignatov et al. (2001) (column 2). In column 3, we needed 200 digits to compute $\psi(120, 10)$. All results in the four columns are given by the Picard-Lefèvre formula. All this explains that the differences between the results of De Vylder (1999) and Ignatov et al. (2001) came from different levels of precision.

We have checked our results with the Seal-type formula and the other formulas. What is interesting is that when u is too large, the Picard-Lefèvre formula requires a very high precision level compared to the other ones.

Table 2 shows the behavior of the different formulas when t is larger (t=30), with $\lambda = 1.15$ and discretized exponential claim amount, with a fixed precision level of 19 digits. One can notice that the Picard-Lefèvre formula is much more unstable than the two other formulas in this case, and that it may lead to incorrect estimates of the initial reserve u needed to have $\psi(u, 30) = 0.005$. (The first column could let us think that u is between 45 and 48, although it is actually around 21.) The two other formulas give consistent results, and $\psi(u, 30)$ is correctly decreasing in u, which avoids the problem of finding the right u in this case.

Table 3 shows how fast one can get the initial reserve level needed to have $\psi(u, 30) = 0.005$

(with the same parameters as in Table 2) with the non-recursive alternative formula (1.11). To determine these values of u, we use an iterative parabolic interpolation of the logarithm of the probability of ruin, just because it is asymptotically linear.
Table 4 shows how fast the Picard-Lefèvre formula is when $u << ct$ compared to the other ones. In all cases, in this table, $c = 2$ and $\lambda = 1$. δ is the discretization step. We let the other parameters vary in order to confirm that using the good formula depends on the comparison between u and ct. All values are given as percentages of the computation time in case 3 with the Picard-Lefèvre formula (which represents less than 1 second). As computation times are similar for the different Picard-Lefèvre formulas, we took the average of the computation times for the three formulas. It is the same for alternative formulas.
This may be explained by figures 1 and 2 which respectively show the set of points (j, τ) such that one needs to compute $h_j(\tau)$ for the Picard-Lefèvre formula and for the "Seal-type" alternative formula. u is on the vertical axis, and τ on the horizontal axis. One can notice that the Picard-Lefèvre formula involves computations of $h_j(\tau)$ for some τ outside $[0, t]$ and that we have a half-square triangle of side u, plus only $[u]$ vertical lines of $h_j(\tau)$ to compute, instead of $[u + ct]$ vertical lines with the alternative formula. In figure 1, u is small compared to ct, which is in favor of the Picard-Lefèvre formula. However, the graphs for $u = 50$ give a different picture, and formula (1.11) would be more efficient (see figures 3 and 4).
There are thus differences of stability and rapidity depending on the formula one uses. More numerical analysis would be needed to determine on which sets of (j, τ) it is better to sum the $h_j(\tau)$ for all values.

Acknowledgment

We wish to thank the referee for the useful comments and suggestions.

Table 1 : Comparison of critical values of u for different formulas.

u	De Vylder (1998)	Ignatov et al. (2001)	Pic.-Lef. form. (19 digits)	Pic.-Lef. form. (up to 200 digits)
0	0.765864440648	0.7658644	0.7658644	0.7658644
5	0.039901595038	0.0399016	0.0399016	0.0399016
10	0.000692886838	6.928868×10^{-4}	6.928868×10^{-4}	6.928868×10^{-4}
15	0.000004740559	$4.74055872 \times 10^{-6}$	$4.74055872 \times 10^{-6}$	$4.74055872 \times 10^{-6}$
20	0.000000014254	$1.43380380 \times 10^{-8}$	$1.42088474805 \times 10^{-8}$	$1.43380380 \times 10^{-8}$
21	0.000000004408	$4.1128895951 \times 10^{-9}$	$4.4089798692 \times 10^{-9}$	$4.1128895951 \times 10^{-9}$
22	0.000000000187	$1.147486268 \times 10^{-9}$	$2.344247596 \times 10^{-10}$	$1.147486268 \times 10^{-9}$
23	0.000000003172	$3.115970161161 \times 10^{-10}$	$1.8896744167 \times 10^{-9}$	$3.115970161161 \times 10^{-10}$
24	−0.000000009229	$8.240887269 \times 10^{-11}$	$-4.39087866 \times 10^{-10}$	$8.240887269 \times 10^{-11}$
25	0.000000029732	$2.12406077199 \times 10^{-11}$	$9.3029175078 \times 10^{-9}$	$2.12406077199 \times 10^{-11}$
30	−0.000418906201	$1.675881883643 \times 10^{-14}$	$-2.706101609021 \times 10^{-6}$	$1.675881883643 \times 10^{-14}$
35	0.000000283956	$7.536921466955 \times 10^{-18}$	$4.371074791696747 \times 10^{-4}$	$7.536921466955 \times 10^{-18}$
40	−0.048800116564	$2.04232266789 \times 10^{-21}$	$-9.8650815650337854 \times 10^{-2}$	$2.04232266789 \times 10^{-21}$
50	5938.508301	$3.91429976066 \times 10^{-29}$	1601.85367765	$3.91429976066 \times 10^{-29}$
100	-4×10^{26}	$2.46817482667739799 \times 10^{-76}$	$-.703221452058 \times 10^{25}$	$2.46817482667739799 \times 10^{-76}$
120	−−	3.5×10^{-98}	$5.79908030 \times 10^{34}$	$3.484112512735 \times 10^{-98}$
150	−−	−−	$1.50547427295 \times 10^{48}$	$2.461597372394 \times 10^{-133}$

Table 2 : Comparison of values of $\psi(u, 30)$ for different formulas.

u	Picard-Lefèvre (1997-98)	Non-rec. (formula (1.11))	Convol. (form. (1.8))
10	0,112054303757283	0,112054303757294	0,112054303757292
15	0,0327177357095503	0,0327177357100829	0,0327177357100812
20	0,00830445394796087	0,00830445394710777	0,00830445394710599
30	0,000366709385616559	0,000366745838547566	0,000366745838545901
35	6,5754047682276E-5	6,51274901416476E-5	6,51274901399823E-5
40	-2,68966541625559E-5	1,04646136024344E-5	1,04646136007691E-5
45	0,00192612549290061	1,53362706611926E-6	1,53362706445392E-6
48	0,00671420246362686	4,64985359105441E-7	4,64985357329084E-7
50	0,041169099509716	2,06467681840294E-7	2,06467680063938E-7
60	89,8591194152832	2,97336189003516E-9	2,97336000265602E-9
80	-115695954	2,8110846983509E-13	2,79332112995689E-13

Table 3 : Search of the initial reserve u for which $\psi(u, 30) = 0.005$.

u	$\psi(u, 30)$ with non-rec. formula (1.11)
20	0.00830445394710777
30	0.00366745838547575
21.6261983515716	0.00518987000061200
21.7530269880689	0.00498677554876592
21.7446133197815	0.00499987086839471
21.7445305212707	0.00500000000139711
21.7445305221664	0.00500000000000000

Table 4 : Computation times of $\psi(u, t)$ for different parameter values.

	Case 1	Case 2	Case 3	Case 4	Case 5
u	20	10	10	20	50
t	25	10	5	5	10
δ	0.05	0.025	0.025	0.025	0.05
$\frac{u}{\delta}$	400	400	400	800	1000
$x = \frac{ct}{\delta}$	1000	800	400	400	400
Pic. - Lef.	339%	243%	100%	400%	672%
Alternatives	664%	403%	102%	243%	343%

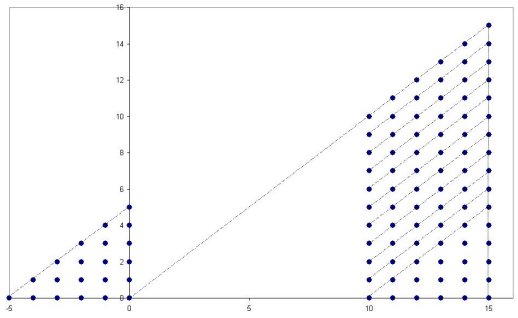

FIG. 1.1 – Picard-Lefèvre formula computation set ($u = 5, ct = 10$)

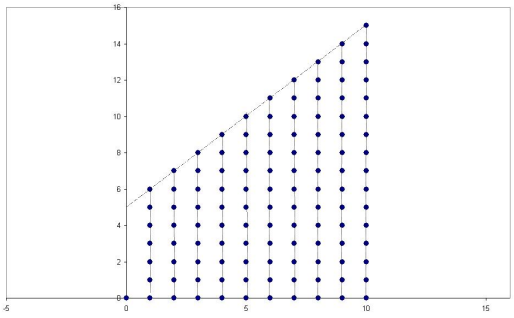

FIG. 1.2 – Rec. or non-recursive alternative method computation set.($u = 5, ct = 10$)

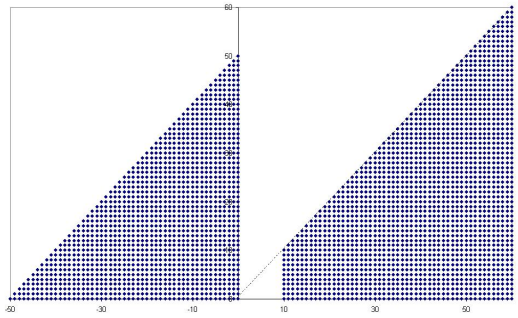

FIG. 1.3 – Picard-Lefèvre formula computation set($u = 50, ct = 10$)

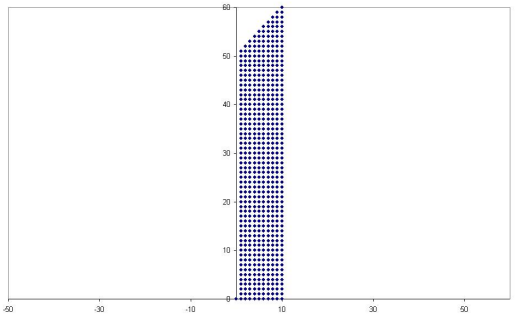

FIG. 1.4 – Rec. or non-recursive alternative method computation set.($u = 50, ct = 10$)

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2

Taux d'intérêt, double barrière, dividendes
Hazard rates of a maximum-to-default distribution, and win-first probability under interest force

In this paper, the surplus of a non-life insurance company affected by a constant interest force is considered. The win-first probability is defined as the probability that the surplus reaches an upper barrier before a lower barrier. This risk-return indicator satisfies a life-insurance type equation. Hazard rates of a maximum-to-default distribution, regarded as a remaining future lifetime distribution, are studied, and provide an alternative derivation of the win-first probability. After generalizing closed-form formulae, bounds and approximations for the win-first probability in special cases, we propose an algorithm to efficiently compute this risk-return indicator in the general case. Algorithms and numerical applications are detailed for practitioners. The efficiency of the proposed algorithm is compared with adaptations of other existing methods, and its interest is illustrated by the computation of the expected amount of dividends paid until ruin in a risk model with a dividend barrier strategy.

2.1 Introduction

In this paper, we propose a way to compute what we called win-first probabilities, *id est* probabilities that a risk process reaches an upper barrier (representing a goal or a threshold for a dividend policy) before crossing a lower barrier (representing the ruin of the company, or a threshold for insolvency penalties). Computing such probabilities is of real interest in modern ruin theory. For example, Frostig (2004), Gerber and Shiu (1998), considered the risk model with a dividend barrier, and computed the expected amount of dividends until time t and until ruin, or optimal dividend strategies. The win-first probability corresponds in this framework to the probability that the dividends are positive.

We consider the compound Poisson risk model with a constant instantaneous interest force δ. We first give properties of win-first probabilities, including a direct adaptation of a result of Segerdahl (1942). We thus obtain the win-first probability as a quotient of two non-ruin probabilities. A first way to tackle the problem of numerically compute win-first probabilities would be to use existing methods (Brekelmans and De Waegenaere (2001), Sundt and Teugels (1995, 1997), De Vylder (1999)) of computing ruin probabilities for some particular claim amount distributions, or for small δ, and to take the quotient. For exponential claim amounts, we can compute everything explicitly. For power-tailed or other types of claim amounts, special methods may be particularly adapted. In the general case, we show here an original way to tackle the problem, which is quite different from the problem of computing ruin probabilities, because we do not have to study the behavior of the process when it exceeds the upper barrier. The formulation of the problem, and the quotient of survival probabilities suggest the possibility to study a ruin-related survival function of some random variable θ, inspired from life-insurance theory. We study its hazard rate function and propose then an algorithm to compute the win-first probabilities and its derivatives, and a bound of the numerical error. A particular property of the hazard rates of θ (see theorem I.16) is the key argument which makes the method so efficient. We also compare our method to the one of Sundt and Teugels (1995).

2.2 The model

In the compound-Poisson risk model, under constant interest force δ, the surplus of an insurance company at time t may be modelled by the process R_t, where $R_0 = u$ and R_t satisfies the stochastic differential equation :

$$dR_t = cdt - dS_t + \delta R_t dt.$$

Here, u is the initial surplus, c the rate at which the premiums are received, and the aggregate claims process S_t is a compound Poisson process given by the Poisson parameter λ and the distribution function F_W of the individual claim amount W, with mean m. Assume that $c > \lambda m$.

Denote by T_u and T_u^v the respective times to lower or upper barrier, with initial surplus u,

$$T_u = \inf\{t, R_t < 0\} \quad \text{and} \quad T_u^v = \inf\{t, R_t \geq u + v\},$$

with $T_u = +\infty$ if $\forall t \geq 0, R_t \geq 0$ and $T_u^v = +\infty$ if $\forall t \geq 0, R_t < u + v$.

The non-ruin probability within finite time t is

$$\varphi_\delta(u, t) = \mathbb{P}(T_u > t),$$

and the eventual non-ruin probability and ruin probability are respectively

$$\varphi_\delta(u) = \mathbb{P}(T_u = +\infty) \quad \text{and} \quad \psi_\delta(u) = 1 - \varphi_\delta(u).$$

For exponentially distributed claim amounts, the probability of ruin under constant interest force is well-known (see Segerdahl (1942), or Sundt and Teugels (1995)). For general claim size distribution, bounds and Lundberg coefficients have been derived by Sundt and Teugels (1995, 1997), and several others.

As $c > \lambda m$, $ct - S_t \overset{a.s.}{\to} +\infty$ as $t \to \infty$. If $\delta = 0$ (no interest force), for any $(u, v) \in R^2$, T_v' is an almost surely finite stopping time and one can determine whether or not $T_u > T_u^v$.
However, if $\delta > 0$,

$$\mathbb{P}(R_t \to +\infty \text{ as } t \to +\infty) \neq 1,$$

because there exists a threshold $y < 0$ such that, if for some $t > 0$, $R_t < y$, then surely $\forall s > t$, $R_s < 0$. This corresponds to the definition of ruin under interest force of Gerber (1979). This phenomenon causes many generalizations of the classical risk model to fail.
Nevertheless, if for all $t \geq 0$, $R_t \geq 0$, then $R_t \overset{a.s.}{\to} +\infty$ as $t \to \infty$. This will be very important to compute the win-first probability with constant interest force.

In section 2.3, we study the win-first probability

$$\mathtt{WF}(u, v) = \mathbb{P}(T_u^v < T_u).$$

The lose-first probability satisfies $\mathtt{LF}(u, v) = \mathbb{P}(T_u \leq T_v')$. The lose-first probability corresponds to the probability of ruin in the presence of an absorbing upper barrier, already studied in the compound Poisson model without interest force by Segerdahl (1970), Dickson and Gray (1984a,b), Wang and Politis (2002). These probabilities provide risk and profit indicators with the same unit : subjectivity is reduced to the choice of the lower bound u, which represents the event "lose", and the upper bound v, which represents the event "win". One of the main drawbacks of the probability of ruin is that its minimization often prescribes the cession of the whole activity by the insurer to the reinsurer. Besides, it does not give any information about the possible profit, although non ruin often represents more than 99 percent of the cases. It is interesting to combine it with a return indicator. One of the simplest compromises is to consider the probability $\mathtt{WF}(u, v)$ to reach a certain level $u+v$ from initial surplus u before being ruined. It has the advantage not to require constrained optimization techniques. The lose-first probability $\mathtt{LF}(u, v) = 1 - \mathtt{WF}(u, v)$ is the probability to miss the objective because of early insolvency. From these probabilities, risk and return indicators with the same unit may be built. For example, the distance from the risk-return couple to a target provides a simple, interesting alternative to

traditional mean-variance type indicators.
Taking into account two barriers raises the subjective problem to choose their level. Nevertheless, it is always significant, from a practical point of view, to choose a lower barrier, which will correspond to the situation "the activity has failed", and an upper barrier, which will correspond to the situation "aims have been achieved".

Consider a confidence level ε, equal to 0.5% for example. Looking for a risk-return indicator having same unit as claim amounts, we can propose to compute the initial surplus required to avoid a failure, the objective level v being given, i.e. $u_\varepsilon(v) = \inf\{u, \mathtt{LF}(u,v) \leq \varepsilon\}$. Similarly, the initial surplus u being given, we can propose to compute the maximal objective level that is reasonably achievable, i.e. $v_\varepsilon(u) = \sup\{v, \mathtt{WF}(u,v) \geq 1-\varepsilon\}$. From these quantities, we may define the following risk-return indicators, which have a monetary meaning :

$$i_1 = (v, u_\varepsilon(v)) \quad \text{and} \quad i_2 = (v_\varepsilon(u), u)\,.$$

Of course, these profit-risk indicators are only examples, and a lot of propositions may be done to create synthetic indicators. Other quantities can be considered, such as $\mathbb{E}\left((T_u - T'_v)_+\right)$, $\mathbb{E}\left((T'_v - T_u)_+\right)$, $u_{1/2}$, $v_{1/2}$,...

The originality of this paper is to express this probability in subsection 2.3.2 as a function of hazard rates of the defective random variable

$$\theta = \sup\{R_t, t \leq T_0 \mid R_0 = 0\} \text{ if } T_0 < +\infty, \theta = +\infty \text{ otherwise,}$$

after observing that $\mathtt{WF}(u,v)$ satisfies a property of life-insurance type. We study the hazard rates of θ, and derive a formula for the win-first probability under constant interest force. For exponentially distributed claim amounts, we obtain a closed-form formula, and we get back some results of Wang and Politis (2002) in the special case $\delta = 0$. We use θ to propose a numerical algorithm to compute $\mathtt{WF}(u,v)$ in the general case. This kind of approach has in particular the advantage that the win-first probability may be derived from a function of one single variable, and that we may get successive derivatives of the win-first probability and of the hazard rate with a very high precision level, even for high values of δ.
The algorithm and reasons for someone to want to use it are detailed in section 2.4.

In section 2.5.2, numerical analysis and applications are proposed. In particular, we shall see that calculations involving win-first probabilities, like

$$\mathbb{E}[\mathtt{WF}(u - W, W)],$$

will occur in a natural way when studying, for example, the cumulative amount of dividends that may be paid when the process reach an upper barrier. This may constitute another interest to the study of such probabilities.

2.3 Win-first probability from a maximum-to-default distribution

In this section, we shall first adapt classical results of ruin theory to our framework. There is no new idea in subsection 2.3.1. This is the reason why we only state the results we shall need later. The proofs are similar as in the model without interest rate. We also state the close formulae we may get in special cases. We then introduce in subsection 2.3.2 the new method we propose to compute the win-first probabilities in the general case.

2.3.1 Adaptation of classical results and methods of ruin theory

Recall that the win-first probability is defined by

$$\texttt{WF}(u,v) = \mathbb{P}(T_u > T_u^v),$$

where

$$T_u = \inf\{t, R_t < 0\} \quad \text{and} \quad T_u^v = \inf\{t, R_t \geq u+v\}$$

are the respective time to ruin and time to reach $u+v$, with initial surplus $R_0 = u$. $T_u = +\infty$ if $R_t \geq 0$ for all $t \geq 0$, and $T_u^v = +\infty$ if $R_t \leq u+v$ for all $t \geq 0$.
Note that $\texttt{WF}(u,v)$ is nondecreasing with respect to u, nonincreasing with respect to v, and that

$$\begin{aligned}
\texttt{WF}(u,v) &= 0 \quad \text{for all } u < 0,\\
\texttt{WF}(u,v) &= 1 \quad \text{for all } u \geq 0,\ v \leq 0, \text{and}\\
\texttt{WF}(u,v) &> 0 \quad \text{for all } u \geq 0,\ v > 0.
\end{aligned}$$

Remark I.2 *In the special case $\delta = 0$, $R_t = u + ct - S_t$ corresponds to the classical compound Poisson risk process, and $R_t - R_0$ does not depend on $R_0 = u$. In this case, u is not necessarily the initial reserve, and $\texttt{WF}(u,v)$ corresponds to the probability that the surplus process R_t reaches $R_0 + v$ before reaching the barrier $R_0 - u$, and does not depend on R_0.*

Theorem I.12 *For $v \geq 0$, $w \geq 0$,*

$$\texttt{WF}(u, v+w) = \texttt{WF}(u,v) \cdot \texttt{WF}(u+v, w). \tag{2.1}$$

Proof : For $u \geq 0$, $v > 0$, $w > 0$, from stationarity and Markov property of R_t, earning $v+w$ before losing u may be decomposed into : earning v before losing u and then earning w before losing $u+v$. If $v = 0$ or $w = 0$ equality is obvious. For $u < 0$, both terms are equal to 0.

Theorem I.13 *For $u \geq 0$, $v > 0$,*

$$\frac{\partial}{\partial u}\texttt{WF}(u,v) - \frac{\partial}{\partial v}\texttt{WF}(u,v) = \frac{\lambda}{c+\delta u}\left(\texttt{WF}(u,v) - \mathbb{E}\left[\texttt{WF}(u-W, v+W))\right]\right), \tag{2.2}$$

$$\frac{\partial}{\partial u}\texttt{WF}(u,v) - \frac{\partial}{\partial v}\texttt{WF}(u,v) = \frac{\lambda}{c+\delta u}\texttt{WF}(u,v) \cdot \left(1 - \mathbb{E}\left[\texttt{WF}(u-W, W)\right]\right). \tag{2.3}$$

Initial conditions have been given above.

Proof : From Poisson process properties, we get

$$\texttt{WF}(u,v) = (1-\lambda\Delta t) \cdot \texttt{WF}\left(ue^{\delta\Delta t} + c\frac{e^{\delta\Delta t}-1}{\delta}, v - \left(u + \frac{c}{\delta}\right)\left(e^{\delta\Delta t} - 1\right)\right)$$

$$+\lambda\Delta t \cdot \mathbb{E}\left[\texttt{WF}\left(ue^{\delta\Delta t} + c\frac{e^{\delta\Delta t}-1}{\delta} - W, v - \left(u + \frac{c}{\delta}\right)\left(e^{\delta\Delta t} - 1\right) + W\right)\right] + o(\Delta t).$$

This heuristic argument shows that equation (2.2) may be derived with classical ruin theory tools. For $u = 0$, we take the convention that $\frac{\partial}{\partial u}\texttt{WF}(u,v)$ is the right derivative of $\texttt{WF}(u,v)$. Note

that in this case, the last term of equation (2.2) disappears.
Starting from (2.2), a direct application of (2.1) leads to

$$\mathtt{WF}\left(u-W,v+W\right)=\mathtt{WF}\left(u-W,W\right)\cdot\mathtt{WF}\left(u,v\right),$$

which provides the second equation.

Inequalities between win-first probabilities and some finite-time ruin probabilities may be derived.

Proposition I.1 *For any $u\geq 0$, $v\geq 0$, recalling $\varphi_\delta\left(u,v\right)=\mathbb{P}\left(T_u>v\right)$ and $\varphi_\delta\left(u\right)=\mathbb{P}\left(T_u=+\infty\right)$, we have*

$$\varphi_\delta\left(u\right)\leq\mathtt{WF}\left(u,v\right)\leq\varphi_\delta\left(u,\tau_\delta(u,v)\right), \tag{2.4}$$

$$\textit{where}\quad \tau_\delta(u,v)=\frac{1}{\delta}\ln\left(1+\frac{v}{u+c/\delta}\right)\quad \textit{if}\quad \delta>0\ ,\quad \textit{and}\quad \tau_0(u,v)=v/c.$$

Proof : For $u\geq 0$, $v\geq 0$, if $T_u=+\infty$ then the insurer reaches almost surely $u+v$ before 0, because $R_t\overset{a.s}{\to}+\infty$ as $t\to\infty$. It follows $\mathtt{WF}\left(u,v\right)\geq\mathbb{P}\left(T_u=+\infty\right)=\varphi_\delta\left(u\right)$.

Now, if the insurer earns v before losing u, time needed to earn v is necessarily greater than the solution $\tau_\delta(u,v)$ of equation in t :

$$ue^{\delta t}+\frac{c}{\delta}\left(e^{\delta t}-1\right)=u+v,$$

and $T_u>\tau_\delta(u,v)$. So, $\mathtt{WF}\left(u,v\right)\leq\mathbb{P}[T_u>\tau_\delta(u,v)]$.

Finally, considering $\lim_{v\to\infty}\mathtt{WF}\left(u,v\right)$, enables us to express $\mathtt{WF}\left(u,v\right)$ as a quotient of survival probabilities.

Theorem I.14 *For $u\geq 0$, $v\geq 0$,*

$$\mathtt{WF}\left(u,v\right)=\frac{\varphi_\delta\left(u\right)}{\varphi_\delta\left(u+v\right)}\ \textit{and}\ S\left(u\right)=\frac{\varphi_\delta\left(0\right)}{\varphi_\delta\left(u\right)}. \tag{2.5}$$

In the special case $\delta=0$, this result has been recently developed by Wang and Politis (2002), and had also been treated previously by Dickson and Gray (1984b) and Segerdahl (1970). The idea is here exactly the same, and we omit the proof of the extension, which is rather direct.

From equation (2.5), it is possible to derive an exact formula for $\mathtt{WF}(u,v)$ in the case of exponentially distributed claim amounts, and asymptotical equivalents and bounds for general claim size distribution.

For exponentially distributed claim amounts with parameter μ , the probability of ruin $\psi_\delta(u)$ is well-known (see Segerdahl (1942), or Sundt and Teugels (1995)) :

$$\psi_\delta(u)=\frac{\Gamma\left(\frac{\lambda}{\delta},\frac{c}{\delta\mu}+\frac{u}{\mu}\right)}{\Gamma\left(\frac{\lambda}{\delta},\frac{c}{\delta\mu}\right)+\frac{\delta}{\lambda}\left(\frac{c}{\delta}\right)^{\lambda/\delta}e^{-\frac{c}{\delta\mu}}},$$

$$\text{where}\quad \Gamma(v,w)=\int_w^{+\infty}t^{v-1}e^{-t}dt\quad \text{denotes the incomplete Gamma function.}$$

Corollary I.3 *For exponentially-distributed claim amounts with parameter μ,*

$$\mathtt{WF}(u,v) = \frac{A - \Gamma\left(\frac{\lambda}{\delta}, \frac{c}{\delta\mu} + \frac{u}{\mu}\right)}{A - \Gamma\left(\frac{\lambda}{\delta}, \frac{c}{\delta\mu} + \frac{u+v}{\mu}\right)},$$

where

$$A = \Gamma\left(\frac{\lambda}{\delta}, \frac{c}{\delta\mu}\right) + \frac{\delta}{\lambda}\left(\frac{c}{\delta}\right)^{\lambda/\delta} e^{-\frac{c}{\delta\mu}}.$$

If the safety loading is equal to zero, and if the interest force $\delta > 0$, the probability of ruin may be found for example in Segerdahl (1942) :

$$\psi_\delta(u) = -\ln\left(\frac{\int_u^{+\infty} e^{-\mu z}(1+\frac{\delta z}{c})^{\frac{\lambda}{\delta}-1}dz}{\frac{c}{\lambda} + \int_0^{+\infty} e^{-\mu z}(1+\frac{\delta z}{c})^{\frac{\lambda}{\delta}-1}dz}\right).$$

Corollary I.4 *For exponentially-distributed claim amounts with parameter μ, if the safety loading is equal to zero, and if the interest force $\delta > 0$,*

$$\mathtt{WF}(u,v) = \frac{1 + \ln\left(\frac{\int_u^{+\infty} e^{-\mu z}(1+\frac{\delta z}{c})^{\frac{\lambda}{\delta}-1}dz}{\frac{c}{\lambda}+\int_0^{+\infty} e^{-\mu z}(1+\frac{\delta z}{c})^{\frac{\lambda}{\delta}-1}dz}\right)}{1 + \ln\left(\frac{\int_{u+v}^{+\infty} e^{-\mu z}(1+\frac{\delta z}{c})^{\frac{\lambda}{\delta}-1}dz}{\frac{c}{\lambda}+\int_0^{+\infty} e^{-\mu z}(1+\frac{\delta z}{c})^{\frac{\lambda}{\delta}-1}dz}\right)}$$

Asymptotical results and bounds could also been exploited. Sundt and Teugels (1997) obtain bounds for the adjustment function. Konstantinides et al. (2002) obtain an asymptotical two-sided bound for heavy-tailed claim size distribution from generalizing results of the classical case $\delta = 0$ to the general case. It is possible to use these bounds to get a two-sided bound for the win-first probability with interest force with heavy-tailed claim size distribution. Many results may be possibly generalized to our framework. Let us cite for example Yang and Zhang (2001), who investigate the severity of ruin, and Brekelmans and De Waegenaere (2001), who approximate the finite-time ruin probability in the model with constant interest force, and references therein. We omit here these generalizations in the interest of conciseness. These methods provide a way to compute $\mathtt{WF}(u,v)$ in the general case. However, we do not need in our problem to compute ruin probabilities, and we shall now introduce a method which is adapted to the present framework, and which is more suitable in the general case and for general interest force δ than the method consisting in computing the two corresponding ruin probabilities.

2.3.2 Hazard rates of θ and applications

In this paragraph, we will try to use an interesting interpretation of $\mathtt{WF}(u,v)$. Let us change our notation for an instant and write

$${}_vp_u = \mathtt{WF}(u,v).$$

Property (2.1) can be written

$${}_{v+w}p_u =_v {}_vp_u \cdot_w {}_wp_{u+v},$$

and corresponds to a simple classical formula, expressed in International Actuarial Notation (see Actuarial Mathematics), stating that for a positive future lifetime θ at birth time,

$$\mathbb{P}(\theta \geq u+v+w|\theta \geq u) = \mathbb{P}(\theta \geq u+v|\theta \geq u) \cdot \mathbb{P}(\theta \geq u+v+w|\theta \geq u+v).$$

This formula, based on elementary conditioning, illustrates the fact that someone aged u survives $v+w$ years, if he first survives v years, and, being then aged $u+v$, survives w more years.
So, it seems logical to look for a nonnegative (defective) random variable θ such that $\mathtt{WF}(u,v) = \mathbb{P}(\theta \geq u+v|\theta \geq u)$. The survival function $S(x) = \mathbb{P}(\theta \geq x), x \in R^+$ will then be enough to determine $\mathtt{WF}(u,v)$ for all $u, v \geq 0$.

Theorem I.15 *Consider the random variable T_0 as described in the introduction. Let θ be the positive, defective random variable*

$$\theta = \sup\{R_t, t \leq T_0 \mid R_0 = 0\}.$$

For $u \geq 0$, $v \geq 0$, the win-first probability can be written as

$$\begin{aligned} \mathtt{WF}(u,v) &= \mathbb{P}(\theta \geq u+v|\theta \geq u) = S(u+v)/S(u), \\ \textit{with } S(x) &= \mathbb{P}(\theta \geq x) > 0,\ x \in \mathbb{R}. \end{aligned} \tag{2.6}$$

Proof : Let us first consider the case $u = 0, v \geq 0$. If $T_0 = +\infty$, $R_t \overset{a.s.}{\to} +\infty$ as $t \to \infty$, and upper barrier v is reached after an almost surely finite time $T_0^v < T_0$. In this case, given that $T_0 = +\infty$, $\mathtt{WF}(0,v) = 1 = \mathbb{P}(\theta \geq v)$. If $T_0 < +\infty$, upper barrier v is reached if and only if $\theta \geq v$, and $\mathtt{WF}(0,v) = \mathbb{P}(\theta \geq v)$. In every case $\mathtt{WF}(0,v) = \mathbb{P}(\theta \geq v)$, $v \geq 0$.

Consider now $u \geq 0$, $v \geq 0$. We have seen that $T_0 = +\infty$ implies $\theta \geq u$. So $\mathbb{P}(\theta \geq u) \geq \varphi_\delta(0) > 0$. Starting from property (2.1), we have $\mathtt{WF}(u,v) = \mathtt{WF}(0,u+v)/\mathtt{WF}(0,u) = \mathbb{P}(\theta \geq u+v)/\mathbb{P}(\theta \geq u)$. And the result is obvious since $v \geq 0$.

The random variable θ represents the defective bridge height of the reserve process from 0 until the process returns to 0. Its survival function, $S(u)$, is the probability that this height is greater than u. To understand what this bridge height represents, define $\theta_u = \sup\{R_t, t \leq T_u \mid R_0 = u\}$. Note that $\theta_0 = \theta$. Then

$$\mathtt{WF}(u,v) = \mathbb{P}(\theta_u \geq v) = \mathbb{P}(\theta \geq u+v|\theta \geq u).$$

θ_u may be considered as a residual height $\theta_u = (\theta - u)_+$. This variable θ_u has the same behavior as a classical residual future lifetime. θ and θ_u are illustrated by figure 2.1. We are now going to study θ and its hazard rates.

For $u \geq 0$, $v \geq 0$, let us define the hazard rate of the variable θ by

$$\mu_u(v) = -\frac{\partial}{\partial v} \ln \mathtt{WF}(u,v).$$

This rate is finite. It only depends of the sum $u+v$ and can be written $\mu_u(v) = \mu_{u+v}$.

In the case of discrete integer claims amounts, we will see that μ_u is continuous and derivable at each $u \in \mathbb{R}^+ \setminus \mathbb{N}$. For $u \in \mathbb{N}$, μ_u will be only right continuous and right derivable, so that we will take the convention that each derivative of μ is its right derivative. We will take the same convention for derivatives in u of $\mathtt{WF}(u,v)$.

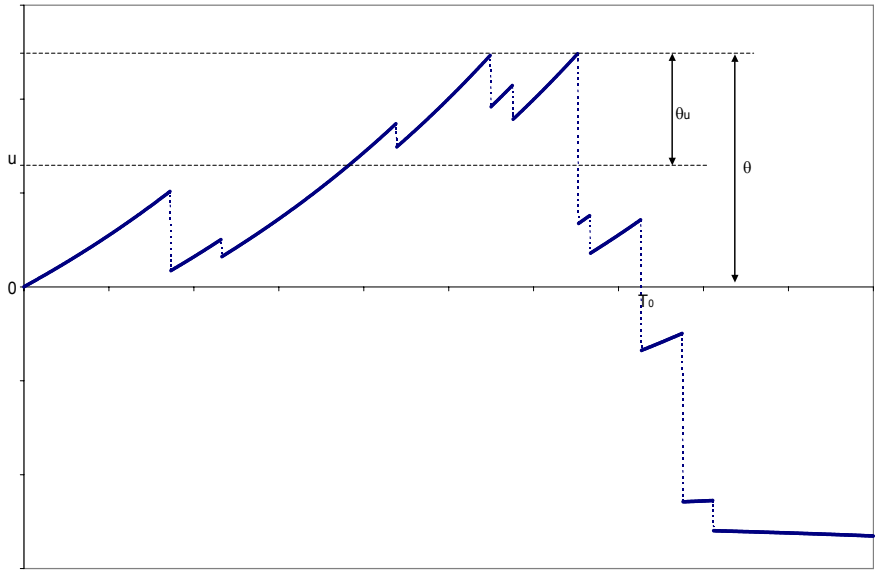

FIG. 2.1 – Sample path of R_t, with θ and θ_u ($\delta = 20\%$).

Given that $\theta \geq u$, the conditional density of θ is

$$f_{\theta_u}(x) = \frac{\partial}{\partial x}\mathbb{P}(\theta < u + x | \theta \geq u) = \mathtt{WF}(u, x) \cdot \mu_{u+x}.$$

Hence, for example, $\mathtt{LF}(u, v) = \mathbb{P}(\theta < u + v | \theta \geq u) = \int_0^v \mathtt{WF}(u, s)\, \mu_{u+s} ds$.
In the sequel, since we will use common actuarial tools, we will most often preferably write probabilities with standard actuarial notations, using ${}_tp_x$ instead of $\mathtt{WF}(x, t)$, and will also write :

$$\mu_u^{(i)} = \frac{\partial^i}{\partial u^i}\mu_u, \quad , S_u^{(i)} = \frac{\partial^i}{\partial u^i}S(u), \quad {}_tp_x^{(i)} = \frac{\partial^i}{\partial x^i}{}_tp_x, \quad {}_wp_{u-w}^{(i)} = \frac{\partial^i}{\partial u^i}{}_wp_{u-w}.$$

Note that, due to these definitions, we do not have an equality between $S_u^{(i)}$ and ${}_wp_{u-w}^{(i)}$ when $w = u$.

Proposition I.2 *For $u, v \geq 0$, and $k \geq 0$, we have*

$$\begin{aligned} \mathtt{WF}(u, v) &= \exp\left\{-\int_u^{u+v} \mu_s ds\right\}, && (2.7) \\ {}_tp_x^{(1)} &= {}_tp_x(\mu_x - \mu_{x+t}), \\ {}_tp_x^{(k+1)} &= \sum_{i=0}^{k} C_k^i\, {}_tp_x^{(i)}(\mu_x^{(k-i)} - \mu_{x+t}^{(k-i)}). && (2.8) \end{aligned}$$

Proof : From property (2.1), $\mu_u(v) = \mu_{u+s}(v - s) = \mu_{u+v}(0)$ for all s, $0 \leq s \leq u$. First equation holds directly from definition of $\mu_u(v)$, and differentiations are straightforward.

Proposition I.3 *A general link between unconditional survival function and hazard rate is given for $x \geq 0$, $k \geq 0$ by*

$$\begin{aligned} S^{(1)}(x) &= -\mu_x S(x), \\ S^{(k+1)}(x) &= -\sum_{i=0}^{k} C_k^i \mu_x^{(i)} S^{(k-i)}(x). \end{aligned} \tag{2.9}$$

Theorem I.16 *Due to particular properties of maximum-to-default distribution, the hazard rate of θ and its right derivatives are given for $u \geq 0$, $k \geq 0$ by :*

$$\begin{aligned} \mu_u &= \alpha_u(1 - \mathbb{E}({}_W p_{u-W})),\ \alpha_u = \lambda(c+\delta u)^{-1}, \\ \mu_u^{(k)} &= \alpha_u^{(k)} - \sum_{j=0}^{k} C_k^j \alpha_u^{(k-j)} \mathbb{E}({}_W p_{u-W}^{(j)}),\ \alpha_u^{(k)} = k!\lambda(-\delta)^k(c+\delta u)^{-(k+1)}. \end{aligned} \tag{2.10}$$

Proof : direct from (2.3) and from (2.7).

Note that previous relations could also be written :

$$\begin{aligned} \mu_u &= \frac{\lambda}{c+\delta u}\left(1 - \mathbb{E}\left[\mathbf{1}_{W\leq u} \exp - \int_{u-W}^{u} \mu_s ds\right]\right), \\ \mu_u &= \frac{\lambda}{c+\delta u}\left(\left(1 - F_W(u)\right) + \mathbb{E}\left[\mathbf{1}_{W\leq u} \cdot \mathtt{LF}(u-W, W)\right]\right). \end{aligned}$$

In particular, if W is a continuous random variable, the hazard rate μ_u is a continuous, decreasing function of u, such that $\mu_0 = \frac{\lambda}{c}$, $\lim_{u\to+\infty} \mu_u = 0$, $\mu_0' = -\frac{\lambda\delta}{c^2}$ and $\lim_{u\to+\infty} \mu_u' = 0$.

Proof : It is clear that $\mathtt{WF}(u-W, W) = 0$ if $W < u$. It follows from $(I.16)$ that $\mu_0 = \frac{\lambda}{c}$ and that $\forall u \geq 0$, $0 \leq \mu_u \leq \mu_0$. Since $\mathtt{WF}(0, v) = \exp\ -\int_0^v \mu_s ds > 0$ for each $v > 0$, $\mu_{+\infty} = \lim_{u\to+\infty} \mu_u = 0$. Furthermore, differentiation of μ_u follows immediately from (2.7) and $(I.16)$. The end of the proof is rather direct and omitted here.

Remark I.3 *For $\delta = 0$, differentiation of $\mathtt{WF}(u, v)$ makes sense, and computing $\mu_u, u > 0$ in terms of $\varphi_0(u)$ leads to*

$$\mu_u = \varphi_0'(u)/\varphi_0(u).$$

We also check that, in the special case $\delta = 0$, formula $(I.16)$ is a version of the classical risk theory formula

$$\varphi_0'(u) = \frac{\lambda}{c}\varphi_0(u) - \frac{\lambda}{c}\mathbb{E}\left[\varphi_0(u-W)\right].$$

2.4 Algorithm

The recursive determination of hazard rate μ_u and its derivatives, for successive values of u, gives a set of values of $S(u)$ and its derivatives up to a given order. Despite the purpose is here to find values of win-first probabilities, this will eventually give results on $\psi_\delta(u)$ when $\psi_\delta(0)$ is known.

One may use the algorithms that are proposed here for several reasons. First, the knowledge of several derivatives of μ_u and $S(u)$ helps understanding the behavior of θ and $\psi_\delta(u)$. Furthermore, all these quantities will be given for a set of successive value of u. This will simplify the computation of values of the whole function S, and will help giving different $\mathtt{WF}(u, v)$ in the

perspective of comparing such values. A third interest is that no assumption is made on claims amounts or interest force δ, making the context different from studies proposing approximations of $\psi_\delta(u)$ when δ is small (see Sundt and Teugels (1995) and section 2.5.3), and from the one using particular distributions for claims amounts (see Konstantinides et al. (2002) and Brekelmans and De Waegenaere (2001)). The proposed recurrence is given in infinite time, and is also different from the one of Cardoso and Waters (2003), which is based on successive values of time. Another point is that the current precision of the computer may be easily reached without spending too much time for calculations, and that each numerical error will be bounded. At last, quantities that are computed in the algorithms will appear in some studies, as we will see it in section 2.5.1.

The main assumption we will use for approximations is that μ_u will be considered locally polynomial of order r on sets of length ϵ. We will see further that even choice like $r = 2$ and $\epsilon = 0.5$ gives numerically quite good results (see section 2.5.2), and the precision of the algorithm increase rapidly for a better choice of these two parameters. In section 2.4.3, we will derive bounds for each approximated quantity in the algorithm, in order to ensure the numerical validity of this assumption.

2.4.1 Approximations

In the sequel, we will consider that W is a random variable taking values on $\mathbb{N}^*$. Define $\pi_i = \mathbb{P}(W = i)$, $i \in \mathbb{N}$.

This hypothesis is not so stringent : in practice, we may approach any continuous random variable by a discrete one, and the discretization step may be chosen as small as necessary. Instead of taking this step smaller than 1, we choose this step equal to 1 and change the monetary unit.

The restriction $\pi_0 = 0$ can be easily eliminated : if $\pi_0 > 0$, one may replace $\pi_0, \pi_1, \pi_2 ...$ with $0, \pi_1/(1-\pi_0), \pi_2/(1-\pi_0) ...$ and λ with $\lambda(1-\pi_0)$ (see De Vylder, 1999).

Let us define $S(x) = \mathtt{WF}(0, x)$. We have $\mathtt{WF}(u, v) = S(u+v)/S(u)$, $S(0) = 1$, and the problem is to determine $S(x)$ for all $x > 0$, and in particular for $x \in \mathbb{N}$.

From the r first derivatives of μ, under hypothesis H_r^ϵ that μ is locally polynomial of order r on intervals $[k\epsilon,\ (k+1)\epsilon[$, we get :

$$\begin{aligned}
\forall s \in [0, \epsilon[,\ \mu(x+s) &= \sum_{i=0}^{r} \frac{\mu^{(i)}(x)}{i!} s^i. \\
\text{For } x \in \epsilon\mathbb{N},\ S(x+\epsilon) &= S(x) \exp\left(- \int_0^\epsilon \sum_{i=0}^{r} \frac{\mu^{(i)}(x)}{i!} s^i ds \right), \\
S(x+\epsilon) &= S(x) \exp\left(- \sum_{i=0}^{r} \frac{\mu^{(i)}(x)}{(i+1)!} \epsilon^{i+1} \right). \qquad (2.11)
\end{aligned}$$

We can also derive $S(x+\epsilon)$ from derivatives of $S(x)$, but we choose to use the single hypothesis H_r^ϵ. In life insurance, the hypothesis of constant hazard rate is often considered for survival lifetimes, and corresponds here to the order $r = 0$.

In practice, it is possible to get on more derivative order of μ_u since computation of $\mu_u^{(r+1)}$ may be replaced with an approximation like $\mu_u'^{\text{left } (r)} = \left(\mu_u^{(r)} - \mu_{u-\epsilon}^{(r)}\right)/\varepsilon$. However, one should keep in mind that, if W takes values in $\mathbb{N}$, each $x \in \mathbb{N}$ is a point of discontinuity for function μ (see figure 2.4). So, using this approximation will give good results, except for $u \in \mathbb{N}$. Nevertheless, since numerical results are fine enough, and since the parameter r could be chosen, we did not use this approximation.

Proposition I.4 (approximation algorithm) *Under hypothesis H_r^ε, the following algorithm computes recursively the values of $S(u)$, μ_u, $\mathbb{E}[{}_Wp_{u-W}]$ and all their derivatives up to a given order r. With $S(0)=1$, and for $u\in\varepsilon\mathbb{N}$, $u\le u_{max}$, $k\in\mathbb{N}$, $k\le r$,*

$$\begin{aligned}
{}_wp_{u-w}^{(k)} &= \mathbf{1}_{\{k=0\}}\frac{S(u)}{S(u-w)}+\mathbf{1}_{\{k\ge1\}}\sum_{i=0}^{k-1}C_{k-1}^{i}\,{}_wp_{u-w}^{(i)}(\mu_{u-w}^{(k-1-i)}-\mu_u^{(k-1-i)}),w=1..[u],\\
\mu_u^{(k)} &= \alpha_u^{(k)}-\sum_{j=0}^{k}C_k^j\alpha_u^{(k-j)}\mathbb{E}({}_Wp_{u-W}^{(j)}),\\
S(u+\epsilon) &= S(u)\exp\left(-\sum_{i=0}^{r}\frac{\mu^{(i)}(u)}{(i+1)!}\epsilon^{i+1}\right).
\end{aligned}$$

Recall that $\alpha_u^{(k)}=k!\lambda(-\delta)^k(c+\delta u)^{-(k+1)}$, and note that $\mu_0^{(k)}=\alpha_0^{(k)}$ for all k. The previous algorithm gives derivatives of μ from order 0 to r. It also gives for each $u\le u_{max}$, $u\in\epsilon\mathbb{N}$, $S(u)$ and eventually $\psi_\delta(u)=1-\varphi_\delta(0)/S(u)$. To obtain derivatives of S, we can use following relation :

$$S^{(k+1)}(x)=-\sum_{i=0}^{k}C_k^i\mu_x^{(i)}S^{(k-i)}(x). \tag{2.12}$$

Let i be a positive integer. We have seen that $S(i)=\varphi_\delta(0)/\varphi_\delta(i)$. In the special case $\delta=0$, it is known that $\varphi_0(i)$ can be exactly computable by classical formulae (see Picard and Lefèvre (1997) and De Vylder (1999)) :

$$\begin{aligned}
\varphi_0(i) &= \left(1-\frac{\lambda m}{c}\right)\sum_{j=0}^{i}h_j(j-i),\\
\text{with } h_j(\tau) &= \frac{\lambda\tau}{cj}\sum_{k=1}^{j}k\pi_k h_{j-k}(\tau) \text{ and } h_j(0)=e^{-\frac{\lambda\tau}{c}}.
\end{aligned}$$

This formula has the advantage to give exact values if W is integer-valued. Nevertheless, the original algorithm presented in this section may be of interest, which we can check by comparing complexity of both algorithms.

In practice, computing $S(i)$ or $\varphi_\delta(i)$ for different values of i is far more interesting than computing only one value of $S(i)$ or of $\varphi_\delta(i)$: determining the initial surplus that is needed to avoid a ruin in most cases, drawing probabilities $\mathtt{WF}(u,v)$ for varying u and v, determining sets of (u,v) on which $\mathtt{WF}(u,v)$ is great enough, all these aims require the computation of a set of $S(i)$ or $\varphi_\delta(i)$.

Let us compare the number of loops involved in the computation of $S(i), i=1...x$ by algorithm $(I.4)$ and the number of loops involved in the computation of $\varphi_0(i), i=1...x$ by the Picard and Lefèvre (1997) algorithm. Computing $S(i), i=1...x$ implies $r^2/2$ loops for $i=1..x/\varepsilon$, $j=1...i\varepsilon$, so that the needed time is quite proportional to r^2x^2/ε. Computing $\varphi_\delta(i), i=1...x$ requires loops for $i=1...x$, $j=0...i, k=1...j$, so that the needed time is quite proportional to x^3.

To approximate a continuous distribution W by W_d taking values in $d\mathbb{N}$, time needed by algorithm $(I.4)$ is proportional to $r^2x^2/\left(d^2\varepsilon\right)$ against x^3/d^3 for the Picard and Lefèvre (1997)

formula. Recalling that hypothesis H_1^ε means that μ is linear on intervals of length $d\varepsilon$, one can use $\varepsilon = 1$ and $r = 1$ if d is small. As both formulae lead to an approximation of values obtained for a continuous W, algorithm $(I.4)$ may be of a practical interest even in the case $\delta = 0$. Moreover, we will see in section 2.4.4 that the complexity of the algorithm can be reduced in this case.

Another point with algorithm $(I.4)$ is that interesting values are computed at the same time as $S(i), i = 1...x$, like hazard rates μ_i and $\mu_i^{(k)}$, and that we could easily deduce values for any $\mathtt{WF}(u, v)$ and its derivatives with respect to u and v since :

$$\begin{aligned} {}_vp_u &= \mathtt{WF}(u,v) = S(u+v)/S(u), \\ \frac{\partial^k}{\partial v^k}{}_vp_u &= S^{(k)}(u+v)/S(u),\ k \geq 1, \\ \frac{\partial^k}{\partial u^k}{}_vp_u &= \sum_{i=0}^{k-1} C_{k-1}^i {}_vp_u^{(k-1-i)}(\mu_u^{(i)} - \mu_{u+v}^{(i)}),\ k \geq 1. \end{aligned}$$

2.4.2 Convergence for parameters r and n

The number of derivatives that are computed by the algorithm is r, and $n = 1/\epsilon$ is an integer that represent the number of sub-periods in one unit of time. The hypothesis in approximation algorithm is that, on each sub-period, μ_u is locally polynomial of order r.
The precision of the algorithm, at one step, is given by η, which represents the number of decimal digit that one aims to obtain. More precisely, $10^{-\eta}$ represent the error in the approximation of $\mu_{u+\epsilon}$ by the Taylor expansion of order r.
To improve the local precision of the algorithm, we can increase either n or r ; this may have different effects on the complexity of the algorithm. We only give here informal considerations for the choice of the couple (n, r) to minimize the complexity of the algorithm. This sections only aims at fixing ideas. It would be possible to get more rigorous results for that choice of parameters, but they are omitted here in the interest of conciseness.

Recall first that the remaining part in the Taylor expansion behaves like $\frac{\mu^{(r+1)}}{(r+1)!}\epsilon^{r+1}$. To simplify further calculation, take $u = 0$, since $\mu^{(r+1)}(0)$ is known, equal to $\alpha^{(r+1)}(0)$. In absolute value, the local error is then comparable to $\lambda(\frac{\delta}{cn})^{r+1}$. If this last quantity is set to be equal to $10^{-\eta}$, then a link appears between r and n :

$$r = \frac{\eta \ln(10) + \ln(\lambda)}{\ln(cn/\delta)} - 1,\ n = \frac{\delta}{c}(\lambda 10^\eta)^{1/(r+1)}.$$

Recall then that for a given u the complexity of the algorithm is proportional to

$$c(r) = nr^2 = \frac{\delta}{c}(\lambda 10^\eta)^{\frac{1}{r+1}} r^2.$$

Trying to find r_0 that minimizes $c(r)$, we find, in the case where $\ln(\lambda 10^\eta) \geq 8$

$$r_0 = \frac{1}{4}\left((\ln(\lambda 10^\eta) - 4 + \sqrt{\ln(\lambda 10^\eta)(\ln(\lambda 10^\eta) - 8)}\right),\ n_0 = \frac{\delta}{c}(\lambda 10^\eta)^{\frac{1}{r_0+1}}.$$

Since n_0 is here a real number, and should better be an integer greater than 1, and since r must also be an integer, we may choose the following parameters to ensure that required precision on μ is reached at the first point following $u = 0$:

$$n_{opt} = \max(2, [n] + (0 \text{ or } 1)) \text{ and } r_{opt} = \left[\frac{\eta \ln(10) + \ln(\lambda)}{\ln(cn_{opt}/\delta)}\right] + (0 \text{ or } 1).$$

As an example, take $\delta = 100\%$ so that we do not suppose that δ is close to 0. For $\lambda = 1$, $c = 1$ and $\eta = 12$ decimal digits, we get $n_{opt} = 9$ and $r_{opt} = 12$. When $\eta = 16$ decimal digits, we get $n_{opt} = 9$ and $r_{opt} = 16$, so that the complexity is multiplied by something less than 1.8 to reach 4 more decimal digits.

2.4.3 Bounds for μ_u, $\mathtt{WF}(u, v)$ and their derivatives

The algorithm makes only one approximation by changing μ_u by its Taylor expansion. Nevertheless, this approximation is used recursively, so that even if the error is locally bounded, we cannot ensure that the global result will be precise enough. For this reason, we must give exact bounds for the values we approximate.

Let us try to find the accuracy of the proposed algorithm : we aim at finding bounds for each quantity in the algorithm.

First, we will suppose that the computer makes computations with high enough a precision level, in order to get the precision due to approximations in the algorithm. In a second time, it will be possible to include numerical errors due to the arithmetic of the computer.

The method is here to reproduce quite the same algorithm, and to calculate at each step the lowest and the largest possible value of each quantity. This will give a quite large estimation of the error we may cause by using the algorithm, since we suppose here that successive errors all affect the result always in the worst way.

For a function f_u of u, we will use the following notations : ${f_u}^{[-1]}$ and ${f_u}^{[+1]}$ will be bounds of f_u, such that $f_u \in \left[{f_u}^{[-1]}, {f_u}^{[+1]}\right]$.

Define : $\sigma \in \{-1, +1\}$, $\mu_u^{(k)[\sigma]}$ such that $\mu_u^{(k)} \in [\mu_u^{(k)[-1]}, \mu_u^{(k)[+1]}]$. Let us define also : $\beta_u^{(k)} = \mathbb{E}\left[{}_W p_{u-W}^{(k)}\right]$ and $\beta_u^{(k)[\sigma]}$ such as $\beta_u^{(k)[\sigma]}$ such as $\beta_u^{(k)} \in [\beta_u^{(k)[-1]}, \beta_u^{(k)[+1]}]$. We will also use $S(u)^{[\sigma]}$, such that $S(u) \in \left[S(u)^{[-1]}, S(u)^{[+1]}\right]$.

For two bounded quantities a and b, we will use following arithmetic, that might be simplified when the signs of a and b are known.

$$\begin{aligned}
(a+b)^{[\sigma]} &= a^{[\sigma]} + b^{[\sigma]}, \\
(a-b)^{[\sigma]} &= a^{[\sigma]} - b^{[-\sigma]}, \\
(ab)^{[\sigma]} &= \max\{a^{[\sigma]}b^{[\sigma]}, a^{[-\sigma]}b^{[\sigma]}, a^{[-\sigma]}b^{[\sigma]}, a^{[-\sigma]}b^{[-\sigma]}\},\ \sigma \geq 0, \\
(ab)^{[-\sigma]} &= \min\{a^{[\sigma]}b^{[\sigma]}, a^{[-\sigma]}b^{[\sigma]}, a^{[-\sigma]}b^{[\sigma]}, a^{[-\sigma]}b^{[-\sigma]}\},\ \sigma \geq 0.
\end{aligned}$$

As it is shown hereafter, supposing first that the computer makes perfect calculation, it is possible to determine recursively each of previous quantity.

Note first that, when $u = 0$, $S(0)^{[+1]} = S(0)^{[-1]} = 1$.

From equation 2.8, we can bound $\beta_u^{(k)}$ from bounds of $\mu_u^{(j)}$ and $\beta_u^{(j)}$, $j < k$,

$$\begin{aligned}
{}_w p_{u-w}^{(0)\,[\sigma]} &= S(u)^{[\sigma]}/S(u-w)^{[-\sigma]}. \\
{}_w p_{u-w}^{(k)\,[\sigma]} &= \sum_{i=0}^{k-1} C_{k-1}^{i} \left({}_w p_{u-w}^{(i)} \left(\mu_{u-w}^{(k-1-i)} - \mu_u^{(k-1-i)}\right)\right)^{[\sigma]},\ k \geq 1.
\end{aligned}$$

From equation 2.10, we can also bound $\mu_u^{(k)}$ from bounds of $\beta_u^{(j)}$, $j \leq k$:

$$\mu_u^{(k)[\sigma]} = \alpha_u^{(k)} - \sum_{j=0}^{k} C_k^j \left(\alpha_u^{(k-j)}\beta_u^{(j)}\right)^{[-\sigma]}.$$

The knowledge of bounds of $\mu_{u+s}^{(k)},\ s<\epsilon$ will allow us to derive bounds of the derivative form of Taylor's remainder, and then bounds of $S(u)$.

We will now use for a function f_u of u the following notations : ${f_u}^{[-2]}$ and ${f_u}^{[+2]}$ will be bounds of f_u, such as $\forall s<\epsilon, f_{u+s} \in \left[{f_u}^{[-2]}, {f_u}^{[+2]}\right]$.

Note that, when $u=0$, $S(0)^{[+2]}=1$ and since S is decreasing, $S(0)^{[-2]} \leq S(\epsilon)^{[-1]}$. The sign of $\alpha_u^{(k)}$ is the same as the one of $(-1)^k$. Since $\alpha_u^{(k)}$ is thus either increasing or decreasing in u, depending on $k \bmod 2$, we remark also that

$$\begin{aligned}
\alpha_u^{(k)[\sigma]} &= k!\lambda(-\delta)^k(c+\delta u)^{-(k+1)},\ \sigma=-1,0,+1,\\
\alpha_u^{(k)[-2]} &= \mathbf{1}_{\{k \bmod 2=0\}}\alpha_{u+\epsilon}^{(k)} + \mathbf{1}_{\{k \bmod 2=1\}}\alpha_u^{(k)},\\
\alpha_u^{(k)[+2]} &= \mathbf{1}_{\{k \bmod 2=0\}}\alpha_u^{(k)} + \mathbf{1}_{\{k \bmod 2=1\}}\alpha_{u+\epsilon}^{(k)}.
\end{aligned}$$

We can then easily check that previous equations are also available for $\sigma \in \{-2,+2\}$

$$\begin{aligned}
{}_wp_{u-w}^{(0)\ [\sigma]} &= S(u)^{[\sigma]}/S(u-w)^{[-\sigma]},\\
{}_wp_{u-w}^{(k)\ [\sigma]} &= \sum_{i=0}^{k-1} C_{k-1}^i \left({}_wp_{u-w}^{(i)}\left(\mu_{u-w}^{(k-1-i)} - \mu_u^{(k-1-i)}\right)\right)^{[\sigma]},\ k\geq 1,\\
\mu_u^{(k)[\sigma]} &= \alpha_u^{(k)[\sigma]} - \sum_{j=0}^{k} C_k^j \left(\alpha_u^{(k-j)}\beta_u^{(j)}\right)^{[-\sigma]}.
\end{aligned}$$

For $s\in[0,\epsilon[$, μ_{u+s} is a continuous and $r+1$ times derivable function of s. we have

$$\mu_{u+s} = \sum_{k=0}^{r}\frac{\mu_u^{(k)}}{k!}s^k + R_{u,s}^{(r)},\ \text{with } R_{u,s}^{(r)} = \frac{\mu_{u^*}^{(r+1)}}{(r+1)!}s^{r+1},\ u^*\in[u,u+\epsilon[.$$

Since $\mu_{u*}^{(r+1)}$ is bounded, we can bound $R_{u,s}^{(r)}$:

$$\frac{\mu_u^{(r+1)[-2]}}{(r+1)!}s^{r+1} \leq R_{u,s}^{(r)} \leq \frac{\mu_u^{(r+1)[+2]}}{(r+1)!}s^{r+1}.$$

It follows that we can bound $S(u+\epsilon)$:

$$S(u+\epsilon)^{[\sigma]} = S(u)^{[\sigma]}\exp\left(-\sum_{k=0}^{r}\frac{\mu_u^{(k)[-\sigma]}}{(k+1)!}\epsilon^{k+1} + \frac{\mu_u^{(r+1)[-2\sigma]}}{(r+2)!}\epsilon^{r+2}\right),\ \sigma\in\{-1,+1\}.$$

The only difficulty to build the bounding algorithm is the following : since $S(u)$ is decreasing in u, a good lower bound for $S(x), x\in[u,u+\epsilon[$ is given by the lowest value of $S(u+\epsilon)$, so that we can propose $S(u)^{[-2]} = S(u+\epsilon)^{[-1]}$. Nevertheless, the calculation of $S(u+\epsilon)^{[-1]}$ from $S(u)^{[-1]}$ uses $\mu_u^{(r+1)[+2]}$, that is then calculated from $S(u)^{[-2]}$. Using such a bound gives then $S(u)^{[-2]}$ as a computable function of itself. We have built both a formal computation algorithm, in order

to get the root value of $S(u)^{[-2]}$, and also a fixed-point algorithm, starting from $S(u)^{[-2]} = 0$. Nevertheless, since the last term of Taylor expansion becomes very small for large values of r, such precise bounds of $S(u)^{[-2]}$ could be replaced with $S(u)^{[-2]} = 0$. The great acceleration resulting of this choice can be exploited to increase r or $n = 1/\epsilon$, for example, and then the precision of the algorithm. We will see with numerical figures that this approximation is sufficient to get very precise results. Indeed, it only changes bounds for the $r + 1^{th}$ derivative order of μ_u and has an impact comparable to $\mu_u^{(r+1)[+2]}\epsilon^{r+2}/(r+2)!$. The problem does not hold for $S(u)^{[+2]}$ since the better bound we can propose is $S(u)^{[+2]} = S(u)^{[+1]}$.

We shall recall here that, by construction, $S(u)^{[-1]}$ and $S(u)^{[+1]}$ give bounds for $S(u)$, not for its approximation $S(u)^{[0]}$, which can be outside the interval.

Proposition I.5 (bounding algorithm) *Bounds for μ_u, $\mathbb{E}[{}_W p_{u-W}]$, $S(u)$ and their derivatives up to order r are given by following algorithm, with initialization values $S(0)^{[-1]} = S(0)^{[+1]} = 1$. For $u = 0..u_{max}$ by step ϵ, for $k = 0..k+1$, and for $\sigma_0 = +1, +2$,*

$$
\begin{aligned}
&S(u)^{[+2]} = S(u)^{[+1]}, S(u)^{[-2]} = 0\\
&{}_w p_{u-w}^{(0)\,[\sigma]} = \frac{S(u)^{[\sigma]}}{S(u-w)^{[-\sigma]}}, \sigma = \pm\sigma_0\\
&{}_w p_{u-w}^{(k)\,[\sigma]} = \sum_{j=0}^{k-1} C_{k-1}^j \left({}_w p_{u-w}^{(k-1-j)}(\mu_{u-w}^{(j)} - \mu_u^{(j)})\right)^{[\sigma]}, u \geq 1, k \geq 1, \sigma = \pm\sigma_0\\
&\mu_u^{(k)[\sigma]} = \alpha_u^{(k)[\sigma]} - \mathbf{1}_{\{u\geq 1\}} \sum_{j=0}^{k} C_k^j \left(\alpha_u^{(k-j)} \mathbb{E}[{}_W p_{u-W}^{(j)}]\right)^{[-\sigma]}, \sigma = \pm\sigma_0\\
&S(u+\epsilon)^{[\sigma]} = S(u)^{[\sigma]} \exp\left(-\sum_{k=0}^{r} \frac{\mu_u^{(k)[-\sigma]}}{(k+1)!}\epsilon^{k+1} - \frac{\mu_u^{(r+1)[-2\sigma]}}{(r+2)!}\epsilon^{r+2}\right), \sigma = \pm 1.
\end{aligned}
$$

This algorithm is quite similar to the first one we proposed. Some remark can be done for its practical implementation.
First, we had better use only integer arguments, so that for $n = 1/\epsilon$,$n \in \mathbb{N}$, we preferably replace u with an index $i = 0..nu_{max}$, where i denotes nu.
Second, for each value of u, we do not use previous values ${}_w p_{u_0-w}^{(k)}$ and $\mathbb{E}[{}_W p_{u_0-W}^{(k)}]$, $u_0 < u$. In the algorithm, these quantities do not need to depend on u, and that spares stocking memory.
Third, many quantities, like $\mathbb{E}[{}_W p_{u-W}^{(k)}]$ or like Taylor integrated approximation in the exponential, can be computed in previous sums giving respectively ${}_w p_{u-w}^{(k)}$ and $\mu_u^{(k)}$.
At last, we may check at each step if the precision of the computer is high enough. If not, it is possible to change lower and upper bounds in order to include, at each step, the maximum numerical computer error.

Bounds for derivatives of S, are given for $\sigma \in \{-2, -1, 1, 2\}$ by

$$
S^{(k+1)[\sigma]}(x) = -\sum_{i=0}^{k} C_k^i \left(\mu_x^{(i)} S^{(k-i)}(x)\right)^{[\sigma]}.
$$

At last, we can finally deduce bounds for $\texttt{WF}(u,v)$ and its derivatives in u and v :

$$\begin{aligned}
{}_vp_u^{[\sigma]} &= \texttt{WF}(u,v) = S(u+v)^{[\sigma]}/S(u)^{[-\sigma]}, \\
\frac{\partial^k}{\partial v^k}{}_vp_u^{[\sigma]} &= S^{(k)}(u+v)^{[\sigma]}/S(u)^{[-\sigma]},\ k \geq 1, \\
\frac{\partial^k}{\partial u^k}{}_vp_u^{[\sigma]} &= \sum_{i=0}^{k-1} C_{k-1}^i \left({}_vp_u^{(k-1-i)}(\mu_u^{(i)} - \mu_{u+v}^{(i)})\right)^{[\sigma]},\ k \geq 1.
\end{aligned}$$

2.4.4 Further results and improved algorithm

We have seen that, given the survival function $S(x)$ for $x \in [0,u]$, it was possible to deduce exactly as many derivatives of μ_u and ${}_wp_{u-w}$ as wanted, and to get then an approximation of $S(u+\epsilon)$.

The previous algorithm was constructed on this idea. For k varying from 0 to a given derivative order r, let us recall here equations that are used in this differentiation step

$${}_wp_{u-w}^{(k)} = \mathbf{1}_{\{k=0\}}\frac{S(u)}{S(u-w)} + \mathbf{1}_{\{k\geq 1\}}\sum_{i=0}^{k-1} C_{k-1}^i {}_wp_{u-w}^{(i)}(\mu_{u-w}^{(k-1-i)} - \mu_u^{(k-1-i)}), w = 1..[u] \quad (2.13)$$

$$\mu_u^{(k)} = \alpha_u^{(k)} - \sum_{j=0}^{k} C_k^j \alpha_u^{(k-j)}\mathbb{E}({}_Wp_{u-W}^{(j)}), \quad (2.14)$$

$$\alpha_u^{(k)} = k!\frac{\lambda}{c+\delta u}\left(\frac{-\delta}{c+\delta u}\right)^k. \quad (2.15)$$

This step was of complexity proportional to ur^2. We will see here that it is sometimes possible to reduce this complexity to something proportional to $\ln(u)r^2$. To do so, we shall denote by Ω_ν a random variable distributed as $W^{*2^\nu} = W_1 + ... + W_{2^\nu}$, with $\Omega_0 = W$. The law of Ω_ν can be easily constructed for integer claim amount W, since for $k \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{P}[\Omega_0 = k] &= \mathbb{P}[W = k], \\
\mathbb{P}[\Omega_{\nu+1} = k] &= \sum_{i=0}^{k} \mathbb{P}[\Omega_\nu = i]\mathbb{P}[\Omega_\nu = k-i],\ k \in \mathbb{N}, \nu \geq 0.
\end{aligned}$$

Remark also that if S is given on $[0,u]$, we can easily deduce ${}_{\Omega_\nu}p_{u-\Omega_\nu}$ from S. We will see that since $W \geq 1$ we will only need law of Ω_ν when $\Omega_\nu \leq u$, i.e. for $\nu \leq \frac{\ln(u)}{\ln(2)}$.

We previously gave derivatives for almost all relations, except a very important one :

Proposition I.6 *By derivation of actuarial property of win-first probabilities,*

$$\begin{aligned}
{}_{s+t}p_x &= {}_sp_x\,{}_tp_{x+s}, \\
{}_{s+t}p_x^{(k)} &= \sum_{j=0}^{k} C_k^j {}_sp_x^{(j)}\,{}_tp_{x+s}^{(k-j)}.
\end{aligned} \quad (2.16)$$

Consider first the case $\delta = 0$. In this case, $\alpha_u^{(k)} = \frac{\lambda}{c}\mathbf{1}_{\{k=0\}}$. Recalling ${}_wp_{u-w} = 0$ when $w > u$ and injecting equation 2.14 into equation 2.13,

$${}_{W_1}p_{u-W_1}^{(k+1)} = \frac{\lambda}{c}\sum_{i=0}^{k} C_k^i {}_{W_1}p_{u-W_1}^{(i)}\mathbb{E}_{W_2}[{}_{W_2}p_{u-W_2}^{(k-i)}] - \frac{\lambda}{c}\sum_{i=0}^{k}\mathbb{E}_{W_2}\left[C_k^i {}_{W_2}p_{u-W_1-W_2}^{(k-i)}\,{}_{W_1}p_{u-W_1}^{(i)}\right].$$

Using proposition I.6, we get then the following theorem, reducing the complexity of differentiation step to something proportional to $\ln(u)r^2$:

Theorem I.17 *When $\delta = 0$, and if $S(x)$ is given on $x \in [0, u]$, then all derivatives of ${}_{\Omega_\nu}p_{u-\Omega_\nu}$ are given by the following recursion : for ν from $[\ln u/\ln 2]$ downto 0, for k from 0 to r,*

$$\begin{aligned} \beta_{u,\nu}^{(k+1)} &= \mathbf{1}_{\{2^\nu \le u\}} \left(-\frac{\lambda}{c}\beta_{u,\nu+1}^{(k)} + \frac{\lambda}{c}\sum_{i=0}^{k} C_k^i \beta_{u,\nu}^{(i)}\beta_{u,\nu}^{(k-i)} \right), \; k \ge 0, \delta = 0 \\ \text{with } \beta_{u,\nu}^{(k)} &= \mathbb{E}\left[{}_{\Omega_\nu}p_{u-\Omega_\nu}^{(k)} \right]. \end{aligned}$$

Consider now the case $\delta > 0$. In this case, note that

$$C_k^j \alpha_u^{(j)} \alpha_u^{(k-j)} = \alpha_u^{(0)} \alpha_u^{(k)}, \text{ with } \alpha_u^{(0)} = \frac{\lambda}{c + \delta u}. \tag{2.17}$$

Assume that u is given and define $\tilde{\mu}_x^{(k)} = \mu_x^{(k)}/\alpha_u^{(k)}$ and ${}_t\tilde{p}_x{}^{(k)} = {}_tp_x^{(k)}/\alpha_u^{(k)}$. Using previous relations on $\alpha_u^{(k)}$, we easily see that equations in the differentiation step become

$${}_w\tilde{p}_{x-w}^{(k+1)} = \gamma_k \sum_{i=0}^{k} {}_w\tilde{p}_{x-w}^{(i)} \left(\tilde{\mu}_{x-w}^{(k-i)} - \tilde{\mu}_x^{(k-i)} \right), \text{ with } \gamma_k = \frac{-\lambda}{(k+1)\delta}, \tag{2.18}$$

$$\tilde{\mu}_x^{(k)} = \frac{\alpha_x^{(k)}}{\alpha_u^{(k)}} \left[1 - \alpha_x^{(0)} \sum_{j=0}^{k} \left(\frac{\alpha_u^{(j)}}{\alpha_x^{(j)}} \mathbb{E}({}_W\tilde{p}_{x-W}^{(j)}) \right) \right], \tag{2.19}$$

and the actuarial property may now be rewritten as

$${}_{s+t}\tilde{p}_x^{(k)} = \frac{\lambda}{c + \delta u} \sum_{j=0}^{k} {}_s\tilde{p}_x^{(j)} {}_t\tilde{p}_{x+s}^{(k-j)}. \tag{2.20}$$

The simplified equations 2.18, 2.19 and 2.20 could be useful for computations and for further analysis, since they avoid to compute binomial terms in the recurrence or factorials in the Taylor's expansion :

$$\sum_{k=0}^{r} \frac{\mu_u^{(k)}}{(k+1)!}\epsilon^{k+1} = \frac{\lambda\epsilon}{c + \delta u} \sum_{k=0}^{r} \frac{\tilde{\mu}_u^{(k)}}{k+1} \left(\frac{-\delta\epsilon}{c + \delta u} \right)^k.$$

This improvement being quite simple, the resulting algorithms for approximation and bounds are omitted here.

Other extensions may be found for $\delta > 0$ by similar arguments as in the case $\delta = 0$. Injecting equation 2.19 into equation 2.18, and using the actuarial property, we can get

$${}_{W_1}\tilde{p}_{u-W_1}^{(k+1)} = \gamma_k \left[\sum_{j=0}^{k} \alpha_u^{(0)} {}_{W_1}\tilde{p}_{u-W_1}^{(j)} \sum_{i=0}^{k-j} \mathbb{E}[{}_{W_2}\tilde{p}_{u-W_2}^{(i)}] - \mathbb{E}\left(\frac{\alpha_{u-W_1}^{(0)}}{\alpha_u^{(0)}} \sum_{l=0}^{k} {}_{W_1+W_2}\tilde{p}_{u-W_1-W_2}^{(l)} \right) \right].$$

However, the presence of the ratio $\alpha_{u-W_1}^{(0)}/\alpha_u^{(0)}$ does not allow to get a direct iterative algorithm for $\beta_{u,\nu}^{(k)}$. Since this ratio is bounded, there exists $x_\delta \in [\frac{c+\delta u}{c+\delta(u-1)}, \frac{c+\delta u}{c}]$, with $x_0 = 1$, such that

$$\begin{aligned}\tilde{\beta}_{u,\nu}^{(k+1)} &= \mathbf{1}_{\{2^\nu \le u\}} \frac{\lambda}{\delta(k+1)} \sum_{i=0}^{k} \left(x_\delta \tilde{\beta}_{u,\nu+1}^{(i)} - \frac{\lambda}{c+\delta u} \tilde{\beta}_{u,\nu}^{(i)} \sum_{j=0}^{k-i} \tilde{\beta}_{u,\nu}^{(j)} \right), \\ \text{with } \tilde{\beta}_{u,\nu}^{(k)} &= \mathbb{E}[_{\Omega_\nu} \tilde{p}_{u-\Omega_\nu}^{(k)}],\ k \ge 0,\ \delta > 0.\end{aligned}$$

This result allows us to derive bounds for $\tilde{\beta}_u = \beta_{u,0}^{(k)}$, $\tilde{\mu}_u^{(k)}$ and $\mu_u^{(k)}$. The complexity of the differentiation step is here proportional to $\ln(u)r^2$ instead of ur^2, but the obtained bounds are less precise than in previous bounding algorithm. Nevertheless, this last equation might be useful when looking for analytic bounds of $\tilde{\beta}_{u,\nu}^{(k)}$.

2.5 Applications and numerical results

2.5.1 An example of application : paiement of dividends

Let us now modify our process R_t by an horizontal dividend barrier strategy. Starting from u, if the surplus reaches the upper barrier $u+v$, all the premium income and the interest (at rate δ) is paid as dividends until the next claim, i.e. after an exponentially distributed time Δ, with parameter λ.

We will show here that it is possible to determine the total amount of dividends that will be paid until the process reaches the lower barrier 0, and that this cumulative amount of dividends depends on win-first probabilities and on quantities which are computed in the previous algorithm. The total expected amount of dividends is given here as a simple example, and depending on the purpose of the study, one may introduce either a discounting factor or other parameters. We shall keep in mind that the total dividend amount might be here represented by a defective random variable.

Denote by D_i the cumulative amount of dividends that is paid during the i^{th} period of paiement, distributed as $D = \int_0^\Delta e^{\delta s} ds$, where is N the number of paiement periods, and T is the total amount of dividends $T = \sum_{i=0}^{N} D_i$. We denote by B_Z the Bernouilli random variable, equal to 1 if the process reaches $u+v$ before reaching 0. We also use N_0 and T_0, the random variables distributed as N and T given that $N > 0$.

For any random variable X, we will denote respectively by F_X and f_X its distribution and density function.

Recalling that $\beta_u = \mathbb{E}[\mathtt{WF}(u-W, W)]$. We easily show that :

$$\begin{aligned}\mathbb{P}[N=0] &= 1 - \mathtt{WF}(u,v), \\ \mathbb{P}[N=k] &= \mathtt{WF}(u,v)\beta_{u+v}^{k-1}(1-\beta_{u+v}), k \ge 1,\end{aligned}$$

From now on, we will consider that $\lambda > \delta$, in order to get $\mathbb{E}[D] < \infty$ and $\mathbb{E}[T] < \infty$. In this case, we have the following result :

Proposition I.7 *If $\lambda > \delta$ and $\beta_{u+v} < 1$, the mathematical expectation of the cumulated amount of dividends is given by*

$$\mathbb{E}[T] = \frac{\mathtt{WF}(u,v)\beta_{u+v}}{(1-\beta_{u+v})(\lambda-\delta)}.$$

The expected value only depends on λ, δ, $\mathtt{WF}(u,v)$ and β_{u+v}, which we are able to compute with as fine precision as necessary. From now on, we will suppose that u and v are given. In order to simplify the presentation of this example, we will use the following notation : $\omega = \mathtt{WF}(u,v)$ and $\beta = \beta_{u+v}$.

Let us try to find properties of the distribution of the cumulative amount of dividends. We can give a first way to determine the distribution of T.

Proposition I.8 *The distribution of cumulative dividends is given by*

$$\begin{aligned}\mathbb{P}[T \leq x] &= \mathbb{P}[N=0] + \sum_{n=1}^{\infty} \mathbb{P}[N=n]\mathbb{P}[D^{*n} \leq x].\\ \mathbb{P}[T \leq x] &= (1-\omega) + \omega(1-\beta)\sum_{k=1}^{\infty} \beta^{k-1} F_{D^{*k}}(x). \qquad (2.21)\end{aligned}$$

Note first that $N_0 - 1$ is a geometric random variable with parameter β_{u+v}, and that $T \equiv B_Z T_0$ with $T_0 = \sum_{i=1}^{N_0} D_i$. Due to the continuous version of Panjer's recursions, or by a renewal argument, we get

$$\begin{aligned}T_0 &\equiv B_\beta T_0 + D,\\ T &\equiv B_Z T_0.\end{aligned}$$

Consider a Bernouilli random variable B_β of parameter β_{u+v}. We get then the following defective renewal equation :

$$\mathbb{P}[T_0 \leq x] = (1-\beta_{u+v})F_D(x) + \beta_{u+v}\mathbb{P}[D + T_0 \leq x],$$

which can also be written

$$A(x) = a(x) + A * G(x),$$

where $a(x) = (1-\beta_{u+v})F_D(x)$ and $G(x) = \beta_{u+v}F_D(x)$, $G(\infty) < 1$.

The resolution of this defective renewal equation only gives back equation (2.21), but helps giving asymptotical results.

Proposition I.9 *If there exists R such that $\int_0^\infty e^{Rx} dG(x) = 1$, i.e. $\mathbb{E}[e^{RD}] = 1/\beta$, then we can get, by application of Smith's theorem,*

$$\lim_{x\to\infty} \mathbb{P}[T \leq x] = (1-\omega) + \omega \lim_{x\to\infty} e^{-Rx} \frac{\int_0^\infty \bar{a}(y)dy}{\int_0^\infty (1-\bar{G}(y))dy},$$

with $\bar{a}(y) = (1-\beta)e^{Ry}F_D(y)$, $d\bar{G}(y) = \beta e^{Ry} f_D(y)$, $F_D(y) = 1-(1+\delta y)^{-\frac{\lambda}{\delta}}$, and $f_D(y) = \lambda(1+\delta y)^{(-\frac{\lambda}{\delta}-1)}$.

This simple example shows results on the cumulative amount of dividends that use both quantities $\omega = \mathtt{WF}(u,v)$ and $\beta = \mathbb{E}[\mathtt{WF}(u+v-W,W)]$ in a natural way.

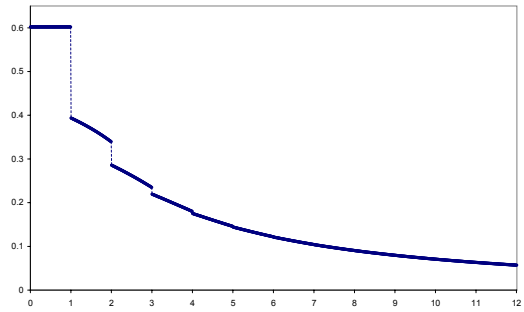

FIG. 2.2 – Aspect of μ for integer-valued W and $\delta = 0$.

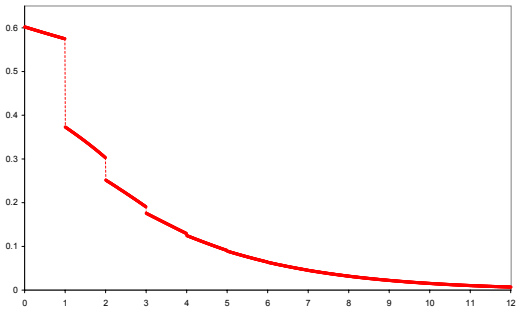

FIG. 2.3 – Aspect of μ for integer-valued W and $\delta = 0.05$.

2.5.2 Numerical results

The results presented hereafter have been obtained for $\lambda = 1$ and $c = 1.05$. W is first exponentially distributed with parameter 1, and then discretized with $F_d(id)$ defined on each interval $[id, id + d[$, such that :

$$F_d(id) = \frac{1}{d} \int_{[id,id+d[} F_W(x)\, dx.$$

We have taken $d = 1$. As explained in section 2.4.1, in order to cancel $\pi_0 = \mathbb{P}(W = 0)$, the Poisson parameter λ has been modified into $\lambda(1 - \pi_0)$, and the π_i have been changed too. This discretization procedure if fully described in De Vylder (1999). This explains values for $x = 0$ in figures 2.2 and 2.3.

The observation of the evolution of $\mu_x, x > 0$ for integer-valued claim amounts confirm that it is nonincreasing, but not continuous. This is a classical fact in ruin theory, and it explains that we usually observe discontinuity points that really exist, even if exact computations are carried out.

Let us explain, for example, that if $\delta = 0$, $\mu_x = \mu_0$ for each $x < 1$. Recall $\theta = \sup\{R_t, t \leq T_0 \mid R_0 = 0\}$. Starting from 0, the random variable θ keeps growing, as a survival lifetime, until the first claim. If the claim occurs before θ reaches the value 1, then ruin occurs since the claim amount is a positive integer. As long as $\theta < 1$, for $\delta = 0$, the probability that θ stops growing is directly linked with the hazard rate of the time of the first claim, which is constant and equal to the modified λ. If θ has reached $x > 1$, situation is more complex, since the probability that θ stops growing will also depend on the claim amount.

The analysis of derivatives of hazard rates (see figures 2.4 and 2.5) may be important to understand approximations that are made in the proposed algorithm. Replacing $\mu_{i\varepsilon}^{(k+1)}$ with the approximation $\mu_{i\varepsilon}^{\text{left }(k)} = \left(\mu_{i\varepsilon}^{(k)} - \mu_{(i-1)\varepsilon}^{(k)}\right)/\varepsilon$ gives good results for $i\varepsilon \notin \mathbb{N}$, but it must be done keeping in mind the discontinuity of μ_x and of its derivatives on atoms of the distribution of the claim amount (see tables 2.5.2 and 2.5.2).

Despite discontinuities of hazard rates of θ (see figures 2.2 and 2.3), survival function $S(x)$ is continuous (see figure 2.6), and tends to $\varphi_\delta(0)$ as $x \to +\infty$. This function is sufficient to obtain all values of $\texttt{WF}(u, v) = S(u + v)/S(u)$, $u, v > 0$.

Of course, in the special case $\delta = 0$, computation of probabilities of ruin and non-ruin are already well-known, and may be computed for example with classical formulae (see Picard

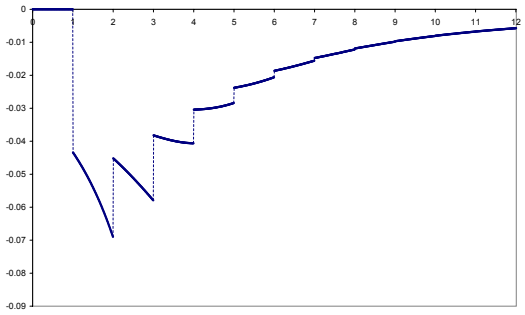

FIG. 2.4 – Aspect of derivative function of hazard rate μ'_x, $x \notin \mathbb{N}$ and $\delta = 0$.

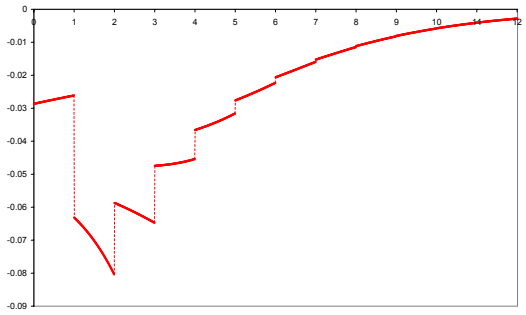

FIG. 2.5 – Aspect of derivative function of hazard rate μ'_x, $x \notin \mathbb{N}$ and $\delta = 0.05$.

$i\varepsilon$	$\mu^{(1)}_{i\varepsilon}$	$\mu'^{\text{left}}_{i\varepsilon}$
4.95	-0.02858	-0.02860
4.96	-0.02854	-0.02856
4.97	-0.02850	-0.02852
4.98	-0.02846	-0.02848
4.99	-0.02842	-0.02844
5	**-0.02380**	**-0.17041**
5.01	-0.02378	-0.02379
5.02	-0.02376	-0.02377

Table 1 Some values of derivatives of μ for $\delta = 0$ and $\epsilon = 0.01$.

$i\varepsilon$	$\mu^{(1)}_{i\varepsilon}$	$\mu'^{\text{left}}_{i\varepsilon}$
4.95	-0.03185	-0.03188
4.96	-0.03179	-0.03182
4.97	-0.03173	-0.03176
4.98	-0.03166	-0.03169
4.99	-0.03160	-0.03163
5	**-0.02763**	**-0.16936**
5.01	-0.02758	-0.02761
5.02	-0.02753	-0.02756

Table 2 Some values of derivatives of μ for $\delta = 0.05$ and $\epsilon = 0.01$.

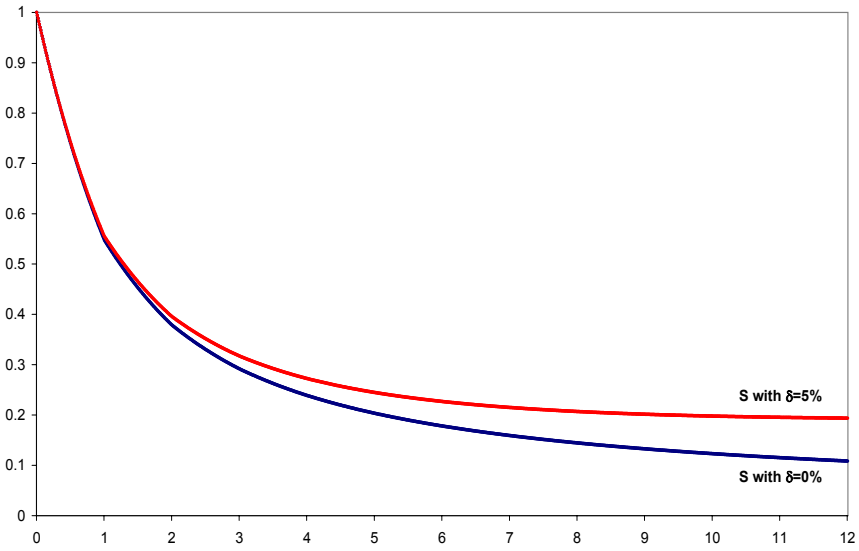

FIG. 2.6 – Aspect of survival function $S(u) = \texttt{WF}(0, u)$ for $\delta = 0$ and $\delta = 0.05$.

u	$\varphi_0(u)$	$\varphi_0(0)/\varphi_0(u)$	$S(u)_{(n=2,r=2)}$	$S(u)_{(n=2,r=7)}$
0	0.047619048	1	1	1
1	0.086942973	0.547704386	.5477043856	.5477043856
2	0.125654634	0.378967699	.3789347571	.3789676986
3	0.163135685	0.291898413	.2918589855	.2918984132
4	0.199174553	0.239081985	.2390475932	.2390819852
5	0.233726482	0.203738350	.2037113041	.2037383494
6	0.266813025	0.178473475	.1784528790	.1784734745
7	0.298480705	0.159538110	.1595224001	.1595381102
8	0.328784306	0.144833700	.1448214757	.1448336999
9	0.357780267	0.133095791	.1330859961	.1330957906
10	0.385524138	0.123517681	.1235095719	.1235176811

TAB. 2.3 – Exact values of $\varphi_0(u)$ by the Picard-Lefèvre formula and approximations of $S(u)$ ($\delta = 0$).

u	$S(u)$	μ_u
0	1.	.602019579836721598480453552227180126242084637112 60
1	.555367531438989480337047316237963515019377563588 98	.372917090402665365479622441325825768317614125740 29
2	.395710616616572908087399086810111444892971033053 12	.251518629064295922019391267947879953076266012470 85
3	.317177966431731246726440363879537691015317760764 77	.175617216292754199323962852686276205866361250284 27
4	.272419498642806659165916557603019815526365883918 55	.124630742076172059878293962152275887389563124578 40
5	.244757282698191919476064786081132301397790822287 07	.088983882528775672523830328545016100434528133542 912
6	.226841510146420030465670153187898652616806133840 77	.063489602807296175860368607790255643020687513766 155
7	.214926405707724066047622199961190937084032470135 15	.045055660722929856034785799235531431265782502796 578
8	.206895279938524674592829721729310793840705366604 47	.031695402749427521457236730930928679745933711961 554
9	.201457627512475514972168459370900375985324545312 47	.022050837923035677891605896212785430522573468958 071
10	.197782021460327243980880079777997470093056730791 75	.015147834698460297368806746234185068943442859266 538

TAB. 2.4 – Exact decimal digits of $S(u)$ and μ_u by bounding algorithm ($\delta = 0.05$).

and Lefèvre (1997) or Rullière and Loisel (2004)). We retrieve $S(u)$ by computing the ratio $\varphi_0(0)/\varphi_0(u)$ (see table 2.3). We shall remember that, for $u > 0$, although the computation of $\varphi_0(u)$ is exact, it does not use previous computations of $\varphi_0(x), x = 1, ..u-1$. This implies, especially if discretization of W is really accurate, a computation time that could be important. It might be interesting to propose another way to determine $\mathtt{WF}(u, v)$, that would help to understand the structure of θ. In table 2.3, we see that approximation algorithm gives quite precise results for small values of convergence parameters n and r, and that precision increase rapidly when r becomes larger.

To give an idea of the convergence of the bounding algorithm, we have taken convergence parameters $n = 2$ and $r = 100$. Keeping $c = 1.05$, $\lambda = 1$, we obtain quantities in tables 2.4, 2.5, 2.6, 2.7 and 2.8, in both cases $\delta = 0.05$ or $\delta = 1.2$. Rather than proposing very near bounds for each quantity, we preferred showing only decimals that were in common in lower and upper bounds. The great number of correct digits shows that the algorithm gives very thin bounds when r becomes large. It may help measuring quality of analytical approximations, and also helps comparing precision of the algorithm with the existing one in the literature. This last point will be developed in section 2.5.3.

In table 2.6, we gives bounds for the 10 first order derivatives of S and μ. When r is sufficiently large, bounds stay very thin also for these derivatives, and are far much precise than the one that could be obtained by successive finite differences on thin intervals of length ε. This result

u	$S(u)$	μ_u
0	1.	.602019579836721598480453 55
1	.669335178799902610915664 93	.162075563152058955336495 87
2	.597309764420937736031183 02	.054417223495799818214328 19
3	.574943530778403546647774 59	.019386890805402392329501 03
4	.567256831175426532958872 68	.007037901027429777131590 14
5	.564520414463911195854457 02	.002570564679201732338310 39
6	.563533102986143853905224 42	.000940699172496845884764 97
7	.563174867566306107112550 72	.000344449553776472217330 89
8	.563044535775480056405951 34	.000126147948618162087925 51
9	.562997046967372385900465 41	.000046203398066839249917 26
10	.562979726095201898814091 16	.000016924066229861253005 14

TAB. 2.5 – Exact decimal digits for $S(u)$ and μ_u by bounding algorithm ($\delta = 1.2$).

k	$S(u)^{(k)}$	$\mu_u^{(k)}$
0	.24475728269819191947606478608113230139779082228 7	.08898388252877567252383032854501610043452813354 2
1	-.02177945329167824813985334396450402226477300005 1	-.02762948367335636932495066499521290291446255969 7
2	.00870053765949241688934832175055661910148213495 7	.00478175978065090114117854092081576773992184850 9
3	-.00314808824973118243611859168353157099928892443 0	.00118405617472855563306678266059799280769578529 0
4	.00102392917413696142500095722865608280907176945 3	.00007918428192102922509243358732848437939068795 3
5	-.00060488523630982027936190199793822518850161856 8	-.00040262354541525280062331773267984559612764452 4
6	.00034996096997569529364338922047310802003519753 3	-.00038637000045138273164584648953158154012033840 0
7	-.00009869021012622389353700612319327053899731156 7	-.00017918022548445207955148173107475015987450513 5
8	.00016201342998874836973343519327907149169263805 9	.00007672264083679168853309869610526698998221044 5
9	-.00007550614830280871975968875474022219114078255 4	.00026499126223433548048753229043888318037307187 8
10	-.00000870145697072955458398250782997172296834880 9	.00028570419210252047195241176465885864902146306 2

TAB. 2.6 – Exact decimal digits of right derivatives $S(u)^{(k)}$ and $\mu_u^{(k)}$ ($u = 5$, $\delta = 0.05$).

comes directly from the fact that equation I.16 gives at point x exact values of these derivatives, when S is given on $[0, x]$. Only approximations on S, that are numerically very precise, have an impact on these derivatives.

At last, we have bounded, for $u = 5$ and $v = 4$, terms appearing in differential equation 2.3 that is recalled here :

$$\frac{\partial}{\partial u}\mathtt{WF}(u,v) - \frac{\partial}{\partial v}\mathtt{WF}(u,v) = \frac{\lambda}{c+\delta u}\mathtt{WF}(u,v)\cdot(1-\mathbb{E}[\mathtt{WF}(u-W,W)])$$

Only decimal that are in common in lower and upper bounds are written in tables 2.7 and 2.8. Recalling λ was modified to eliminate the mass $P[W = 0]$, we easily verify this differential equation, in both cases $\delta = 0.05$ and $\delta = 1.2$. Convergence parameters $n = 2$ and $r = 100$ give in both cases, for the equality, a better precision than the 120 decimal digits we used for calculations. An interesting result of the algorithm is that it also gives all derivatives up to a given order, with respect to u or in v, of $\mathtt{WF}(u, v)$.

quantity	value for $u = 5$, $v = 4$, $\delta = 0.05$
$\mathtt{WF}(u,v)$	.82309145326184702980537190114869824277841420062710 74...
$\frac{\partial}{\partial u}\mathtt{WF}(u,v)$	.05509201695577856324789329595515527720176742845166 02...
$\frac{\partial}{\partial v}\mathtt{WF}(u,v)$	-.01814985623171288470150842179406380159358909721724867...
$\mathbb{E}[\mathtt{WF}(u-W,W)]$	.81699844171848461222353476005300514855910196532408671833 4...
$\lambda/(c+\delta u)$	.48624658371427513723421248449118394811860682228325551191 7...

TAB. 2.7 – Exact decimal digits of quantities in differential equation by bounding algorithm ($\delta = 0.05$).

quantity	value for $u = 5$, $v = 4$, $\delta = 1.2$
$\mathtt{WF}(u, v)$	.997301483777189077761083 8...
$\frac{\partial}{\partial u}\mathtt{WF}(u, v)$	.002517549251265514816121 93...
$\frac{\partial}{\partial v}\mathtt{WF}(u, v)$	-.000046078717447606893398087...
$\mathbb{E}[\mathtt{WF}(u - W, W)]$	.971330657205712950601312 80...
$\lambda/(c + \delta u)$	.089662490613979812539642018416814 06...

TAB. 2.8 – Exact decimal digits of quantities in differential equation by bounding algorithm ($\delta = 1.2$).

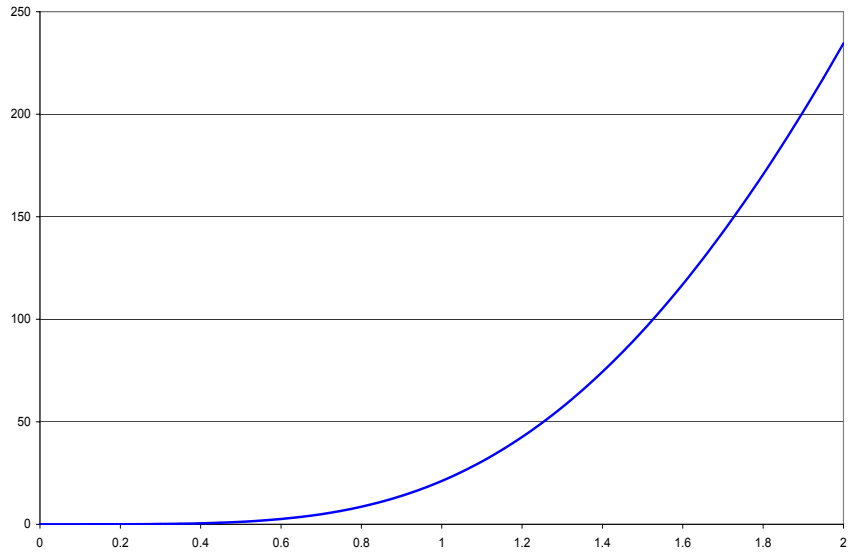

FIG. 2.7 – Average cumulative dividends as a function of premium rate c ($\delta = 0.05$).

To give a concise illustration of section 2.5.1, figure 2.7 draws the evolution of average cumulative dividends that may be paid each time the process reaches the upper barrier without having reached the lower one. This simple, natural example is based on quantities computed in approximation or bounding algorithm. It is given here in a simplified environment, and introduction of other economical parameters, such as a discounting factor, would require further analysis.

2.5.3 Comparison with other methods

Sundt and Teugels (1995) proposed several methods to compute $\psi_\delta(u)$. Each one is based on the value of $\psi_\delta(0)$. If these methods are used to compute $S(u) = \mathtt{WF}(0,u) = \frac{1-\psi_\delta(0)}{1-\psi_\delta(u)}$, then the result obtained depends on the value of $\psi_\delta(0)$. Sundt and Teugels (1995) proposed for example a recursive algorithm that we rewrite with our notations :

$$\begin{aligned}\varphi_\delta(hk)^{[-1]} &= \gamma^- \left(c\varphi_\delta(0) + \sum_{j=1}^{k} \varphi_\delta(h(k-j))^{[-1]} f_j^+ \right), \\ \varphi_\delta(hk)^{[+1]} &= \gamma^+ \left(c\varphi_\delta(0) + \sum_{j=1}^{k-1} \varphi_\delta(h(k-j))^{[+1]} f_j^- \right).\end{aligned}$$

with $\gamma^- = \frac{1}{c+\delta hk}$, $\gamma^+ = \frac{1}{c+\delta hk - f_1^+}$, and, in the special case of integer-valued claims amounts, $f_k^+ = \delta h + \lambda h \mathbb{P}[W \geq hk]$, $f_k^- = f_{k+1}^+$, $h \leq 1$. $f_1 = (\lambda+\delta)h$.

Note that the corresponding formulae for this quantities in Sundt and Teugels (1995) (top of page 12) have to be switched.

Let us try to minimize

$$\Delta_{\varphi_k^h} = \varphi_\delta(hk)^{[+1]} - \varphi_\delta(hk)^{[-1]}$$

under the (very) optimistic hypothesis that for $j < k$, $\Delta_{\varphi_{k-j}} = 0$.

Note that

$$\varphi_\delta(u)F(u) > \mathbb{E}\left[\varphi_\delta(u-W)\mathbf{1}_{W\leq u}\right].$$

Hence, after some omitted computations, with $u = hk$,

$$\Delta_{\varphi_k^h} > \frac{\left(\varphi_\delta(hk) - \varphi_\delta(0)\right)\left(\lambda(1-F(hk)) + \delta\right)}{c+\delta hk} h, \tag{2.22}$$

As an example, in the case $\delta = 0$, we can get from table 2.3 values for the right member of (2.22). With same numerical parameters, the minoration of $\Delta_{\varphi_k^h}$ changes from values 10^{-3} to 10^{-5} when $hk \in [1,10]$. To get the same precision level $10^{-\eta}$ as in table 2.4, would require an h smaller than $10^{-\eta+5}$, and a much higher complexity in $1/h^2$ than with our method. One must add to this problem the error possibly made in $\varphi_\delta(0)$, which was supposed to be avoided, and the propagation error due to $\Delta_{\varphi_{k-j}^h}, j \leq k$.

Other methods might not be more efficient, except in the case where $\varphi_\delta(0)$ is precisely bounded, for as well small or large values of δ.

Besides, the adjustment functions do not in general provide directly two-sided bounds for ruin probabilities. They are particularly adapted to the case of large initial reserve, which does not correspond to the assumption made here.

This shows that the method consisting in taking the quotient of two non-ruin probabilities computed with methods efficient for that problem, is not adapted to our framework, and that the algorithm proposed in section 2.4 is more convenient to this problem.

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Deuxième partie

Théorie de la ruine en dimension finie

1

Modèle multidimensionnel, dépendance entre branches
Finite-time ruin probabilities in the Markov-Modulated Multivariate Compound Poisson model with common shocks, and impact of dependence

Following Picard et al. (2003), a multi-dimensional process is proposed to model the simultaneous evolution of the surpluses of $K \geq 1$ lines of business of an insurance company. It is based on common Poisson shocks and Markov modulation. Different ruin concepts are compared. In the framework of Picard et al. (2003), who derived a formula for the probability that at least one line of business gets ruined before a finite time t in a model with stationary and independent increments, a recursive formula to compute ruin probabilities are given. The impact of dependence and reinsurance is studied. An asymptotical result for the optimal reserve allocation is exposed. Finally, properties and interests of some proposed multi-component risk measures are discussed. Numerical results are given.

1.1 Introduction

We consider here the case of an insurance company with $K \geq 1$ lines of business. Some authors, like Cossette and Marceau (2000), and many others, considered multi-risk models. However, in most cases, they focus on the unidimensional risk process representing the total wealth of the company.
From now on, we consider a fixed accounting time horizon T, which may be infinite, and study the evolution of the wealths of the lines of business of the company between times 0 and T. With three lines of business (three kinds of activities), for example liability, disablement and driving insurance, it is not the same situation to have $(1M, 2M, -2.8M)$ (*id est* 1 million euros for the first branch, 2 million euros for the second one and to be short of 2.8 million euros for the last branch), or to have $(0, 0.1M, 0.1M)$. Considering only the total wealth (0.2 million euros) does not reflect the situation of the company very well. A few years ago, a holding company mainly had two large airlines companies. The first one was doing well, say its wealth was 10M dollars. The second one was undergoing a bad period, with a debt of, say, 2M dollars. Even if the subcompanies were collateralized, the holding company was estimated 4M dollars, instead of 8M dollars, by the market. The reason was that analysts expected the healthy line of business to be penalized by the other one, which was in the red. To be able to detect such penalties, and to compute probabilities of such unfavorable events, the multi-dimensional process has to be studied as an additional indicator. There is the same phenomenon for a single line of business during different time periods : managers tend to smooth results over time artificially. Of course, it is possible to smoothen results between the lines of business and over time up to a certain point, but it is interesting to try to take this into account for the choice of risk measures. Some of them should so reward good, smooth results better than a mix of excellent and bad results corresponding to equivalent global results. Our purpose here is not to claim that everybody should use multidimensional ruin probabilities. As most companies have collateralized lines of business, the total wealth process, and the risk measures associated with this process remain the main risk indicators one has to consider. However, it is not enough to understand what may happen in the future, and how the market evaluates the global company. Multidimensional ruin probabilities can bring the additional piece of information which is needed.

To tackle such a problem, it is even more important than in the classical case to describe and take into account the dependence structure between the lines of business correctly. In Picard et al. (2003), the reserves of the n lines of business of an insurance company are modelled by a process with stationary, independent increments with discrete distributions. The full freedom of choosing the space correlation (correlation between claims of the lines of business during the same period) is counterbalanced by the independence of the increments, allowing no time-

correlation. We propose here a model with more balance in time and space correlation, inspired from dependence concepts which are well-known in the univariate case, namely common Poisson shocks, and Markovian environment. These concepts seem logical to model dependence between the different lines of business of an insurance company. In the univariate model, Markovian environment has been introduced by Asmussen (1989), and studied by many others. Cossette et al. (2003) studied recently the compound Markov binomial model (see references therein). Cossette and Marceau (2000) studied a discrete time risk model with correlated lines of business and considered the total wealth of the company. Asmussen and Højgaard (1996) considered the Markovian environment model with heavy tails. Many others studied this kind of model : (Jasiulewicz (2001), Xiang and Liu (2003), Snoussi (2002), Bäuerle (1996), Rydén (1994, 1996) and Lehtonen and Nyrhinen (1992)).
Sumita and Masuda (1992) studied multivariate Markov modulated Poisson processes for queuing theory.

The common shock model is now well-known. Some recent work has been done by Lindskog and McNeil (2003) (applications to insurance and credit risk modelling) , Frostig (2004), Balu and Sabnis (1997) (preservation of dependence structure) Denuit et al. (1999) (sums of dependent risks),Sundt (2000) (multivariate compound Poisson distributions) among many others...
This list is very far from being exhaustive.

The first paper to consider ruin as an event depending on the multi-dimensional process was Collamore (1998), and is based on large deviations. The only paper exposing computations of multi-dimensional ruin probabilities is Picard et al. (2003). They derived a formula for the probability that at least one line of business gets ruined before a finite time t. We generalize this work to derive ruin probabilities for other concepts of ruin and to estimate other risk measures. We show in section 1.4 that minimizing the multi-dimensional probability of ruin, defined as in Picard et al. (2003), is in favor of positive correlation between the lines of business. This implies that it should not be considered as a good global risk measure for the company. However, it remains interesting, because it represents the probability of an unfortunate event, which will have consequences on the evaluation of the holding company by the market, as in the case of American airlines companies. Some other multi-component risk measures may be of interest, particularly what we call in section 1.4 multi-utility based risk measures.

1.2 Multi-dimensional claim model

We consider the process modelling the wealth of the K lines of business of an insurance company. Typical lines of business are driving insurance, house insurance, health, incapacity, death, liability,... Two main kinds of phenomena may generate dependence between the aggregated claim amounts of these lines.

- Firstly, in some cases, claims for different lines of business may come from a common event : for example, a car accident may cause a claim for driving insurance, liability and disablement insurance. Hurricanes might cause losses in different countries. This should correspond to simultaneous jumps for the multivariate process. The most common tool to take this into account is the Poisson common shock model.
- Secondly, there exist other sources of dependence, for example the influence of the weather on health insurance and on agriculture insurance. In this case, claims seem to outcome independently for each branch, depending on the weather. This seems to correspond rather to models with modulation by a Markov process which describes the evolution of the state of the environment.

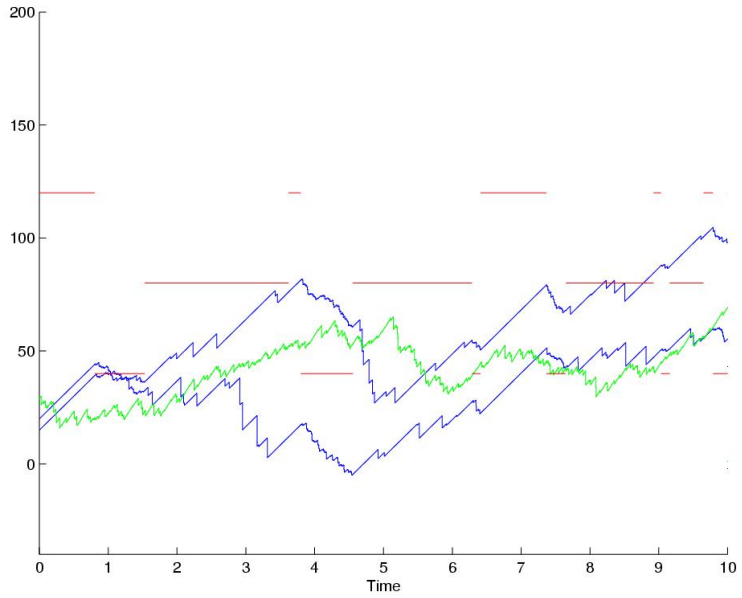

FIG. 1.1 – Sample path for three lines of business : The green one does not depend on the state of the environment. The two blue lines of business have identical parameters, and are independent conditionally on the environmental state. Occupation periods for environment states in red.

Another example is the influence of police controls. Recently, in France, the development of speed controls downed the number of accidents, and the number of severely injured people on the road. The frequency of controls may vary over time and create time-correlation. This is another aspect of Markov-modulation, which may generate over-dispersion for some lines of business.

Common Poisson shock models are quite easy to understand. To fix ideas, figure 1.1 shows a sample path of the surpluses of the 3 lines of business of the insurance company, under a Markovian environment, but without common shock. The set of states of the environment has cardinality three. State 3 is the most favorable for the company, almost no claim occurs for lines 1 and 2 (in blue) in this state. State 1 is the least favorable state for the company, claim frequencies and severities are higher for lines 1 and 2. The state of the environment is represented in red. Events for the third line of business (in green) are independent from the state of the environment. One can see the strong positive dependence between lines 1 and 2 (in blue), but also their independence conditionally to the environment state. At some moment, the two blue curves separate each other because of this conditional independence.

Let us define more precisely the model we propose, which takes into account these two different sources of dependence : the Markov-modulated, Common Shock, Multivariate Compound Poisson Process model ((MM,CS)-MCPP),

Conditionally on the state of the environment, the multivariate claim process is modelled by a Compound Poisson Process with Common Shocks. The intensity, the claim size distribution and even the common shock parameters may vary in function of the state of the environment, which is modelled by a Markov process.

The environment state process, denoted by $J(t)$, is a Markov process with state space $\mathcal{S} = \{1, \ldots, N\}$, initial distribution μ and intensity matrix A.

For example, state 1 might correspond to periods of frequent heavy rains, or very hot weather, to hurricane seasons. It might also indicate frequent speed controls, law modifications,...

There are $m \geq 1$ different types of shock. If $J(t) = i$, then shocks of type e $(1 \leq e \leq m)$ occur according to a Poisson process with intensity $\lambda_{e,i}$. These shock counting processes are independent conditionally on $J(.)$.
For example, a shock may be a big car accident, a particular hurricane, an explosion, a medical mistake with consequences in a hospital,...

If $J(t) = i$, at the r^{th} occurrence of type e $(1 \leq e \leq m)$, the Bernouilli vector $I_i^{e,r} = (I_{1,i}^{e,r}, \ldots, I_{Ki}^{e,r})$ indicates whether a loss occurs for branch $k \in [1, K]$, and the potential losses are represented by $W_i^{e,r} = (W_{1,i}^{e,r}, \ldots, W_{K,i}^{e,r})$.
For example, a car accident may cause claims in driving insurance, liability, incapacity, death.

For a fixed state i and fixed shock type e, the successive $I_i^{e,r}$ are i.i.d.,
the successive $W_i^{e,r}$ are i.i.d.

Besides, the $I_i^{e,r}$ are independent from the $W_i^{e,r}$. However, for a fixed event i, e, r, the loss triggers $(I_{1,i}^{e,r}, \ldots, I_{K,i}^{e,r})$ and the potential losses generated by this event $W_i^{e,r} = (W_{1,i}^{e,r}, \ldots, W_{K,,i}^{e,r})$ may be dependent. In most real-world cases, the by-claims amounts seem to be positively correlated. This is the reason why we allow this kind of dependence. Between time 0 and time t, denote by $N_i^e(t)$ the number of shocks of type e that occurred while J was in state i. Then the aggregate claim amount vector up to time t is $S(t) = (S_1(t), \ldots, S_K(t))$ where for a branch $k \in [1, K]$,

$$S_k(t) = \sum_{i=1}^{N} \sum_{e=1}^{m} \sum_{r=1}^{N_i^e(t)} I_{k,i}^{e,r}.W_{k,i}^{e,r}$$

In case of no common shock, $S_k(t)$ (the aggregate claim amount vector up to time t for branch k) is a compound Cox process with intensity $\lambda_{k,J(t)}$ and claim size distribution $F_{k,J(t)}$.

$(S(t), J(t))$ is a Markov process. (and of course $S(t)$ is not !) This has an impact on computation times, because we have to keep track of the environment state during the computations.

The advantage of the multi-dimensional approach is that probabilities of various unfavorable events may be computed, Considering the total wealth of the company would not be enough to compute them, and some of these events have a strong impact on the evaluation of the holding company by the market. However, recursive computations become too long for more than 3 lines of business and 3 states of the environment. For higher dimensions, one has to use simulation techniques, which seem to work quite well. The additional information provided by the multidimensional model is counterbalanced by the high computation times, which are still acceptable for low dimensions. In the case of large state space cardinality, the problem of calibrating the model, and choosing the right state space becomes even more difficult. Future work is needed to be able to calibrate the model correctly.

1.3 Multi-dimensional, finite-time ruin probabilities

To compute finite-time ruin probabilities, we approximate the model with a discrete time risk model, in which the change of environment state can only happen at some fixed dates. For integer-valued claim amounts, the continuous-time ruin probability would be equal to the discrete time ruin probability for well-chosen inventory dates. So, the approximation only concerns the Markov-modulation part. It seems reasonable as changes in the environment are supposed to happen more rarely than claims.

1.3.1 Generalization of the Picard-Lefèvre formula

In the sequel, boldface math letter represent multi-indices. References : Picard et al. (2003) and Collamore (1998) (large deviations). The presentation of this section is inspired from Picard et al. (2003).
Ruin concept : for all $t \in \mathbb{N}^*$, $D(t) \subset \mathbb{R}^n$ (resp. $H(t)$) denotes the insolvency (resp. solvency) region at time t, and ruin within time t corresponds to the aggregate claim amount process $S(t)$ entering $D(t)$ for some time $t \leq T$.

For discrete claim size distribution, ruin is equivalent to ruin at inventory for well-chosen dates.

Discretization of the claim amount distributions. Time-discretization of J(t) into a Markov chain. Particular ruin concept : at least one branch gets ruined. Define the marginal time to ruin : $T_k = \inf\{t > 0, u_k + c_k(t) - S_k(t) < 0\}$. $T = \min\{T_1, \ldots, T_K\}$. The probability of ruin within finite time t is $\psi(\mathbf{u}, t) = \mathbb{P}(T > t)$, where $\mathbf{u} = (u_1, \ldots, u_K)$. For $k \leq K$, $c_k(t)$ and $S_k(t)$ respectively denote the cumulated premium and the aggregated claim amount up to time t for line of business k. For $t \in \mathbb{N}^*$ and $k \leq K$, denote the cumulated claim amount of period $]t-1, t]$ for line of business k by

$$X_k(t) = S_k(t) - S_k(t-1),$$

and define $X(t) = (X_1(t), \ldots, X_K(t))$.

In Picard et al. (2003), the $X(t), t \in \mathbb{N}^*$ are assumed to be i.i.d. vectors with distribution described by $\mathbb{P}(X(t) = \mathbf{j}) = a_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^K$. The choice of space correlation (between lines of business) is counterbalanced by the time-independence. In the present model, the increments are no longer independent, and one must take $J(t)$ into account to get a Markov process.

Set $v_{\mathbf{n}} = \max\{t, \mathbf{n} \in D(t)\}$ for $\mathbf{n} \in D(0)$, $v_{\mathbf{n}} = 0$ otherwise. Let

$$e_{\mathbf{n}}^{i_0,i}(t) = \mathbb{P}(S(t) = \mathbf{n} \cap J(t) = i \mid J(0) = i_0) \quad \text{and}$$

$$A_{\mathbf{n}}^{i_0,i}(t) = \mathbb{P}(S(t) = \mathbf{n} \cap T > t \cap J(t) = i \mid J(0) = i_0).$$

Here, $e_{\mathbf{n}}(t)$ and $A_{\mathbf{n}}(t)$ are not polynomials in t.

Then the probability of non-ruin is :

$$P(T > t) = \sum_{i_0 \in \mathcal{S}} \sum_{j \in \mathcal{S}} \sum_{\mathbf{n} \in H(t)} \mu(i_0) A_{\mathbf{n}}(t)^{i_0,j}$$

From Chapman-Kolmogorov equations, the previous probabilities satisfy the following properties : for $0 \leq t' \leq t$, $\mathbf{n} \in \mathbb{N}^K$, and $i_0, i \in \S$,

$$e_{\mathbf{n}}^{i_0,i}(t) = \sum_{\mathbf{k}=0}^{\mathbf{n}} \sum_{j \in \mathcal{S}} e_{\mathbf{k}}^{i_0,j}(t') e_{\mathbf{n}-\mathbf{k}}^{j,i}(t - t'), \quad \text{and} \tag{1.1}$$

$$A_{\mathbf{n}}^{i_0,i}(t) = \sum_{\mathbf{k}=0}^{\mathbf{n}} \sum_{j \in \mathcal{S}} A_{\mathbf{k}}^{i_0,j}(v_{\mathbf{n}}) e_{\mathbf{n}-\mathbf{k}}^{j,i}(t - v_{\mathbf{n}}). \tag{1.2}$$

Besides, if we denote $|\mathbf{n}| = n_1 + \cdots + n_K$,

$$A_{\mathbf{n}}^{i_0,i}(v_{\mathbf{n}}) = \delta_{|\mathbf{n}|,0} \delta_{i_0,i}. \tag{1.3}$$

In some cases, some generalized Appell type properties save computation time. Numerical results are given in section 1.5.

1.3.2 Generalization of other results

Comparison with independence and average Poisson model

With no common shocks, in case $K = 1$ (1 branch only)

Theorem II.1 *Rolski et al. (1999) : Assume that $\lambda_1 \leq \cdots \leq \lambda_N$, $F_1 \geq \cdots \geq F_N$, $J(t)$ has initial stationary distribution π , and that A is stochastically monotone. Then*

$$\psi_\pi(u) \geq \psi^*(u),$$

where $\psi^*(u)$ is the ruin probability in the compound Poisson risk model with intensity $\lambda = \sum_{i=1}^N \pi_i \lambda_i$ and claim size distribution $F = \sum_{i=1}^N \pi_i F_i$.

Interpretation of these optimal conditions : States are classified from the most favorable state (1) to the least favorable state (N), both for claim frequency and size. The future evolution of the environment should be in average more favorable in state i than in state $j \geq i$:
If $\beta \leq_{\text{st}} \beta'$ then $\beta A \leq_{\text{st}} \beta' A$.

This result generalizes into the following theorem.

Theorem II.2 *Assume that for $k \in I = \{1, \ldots, K\}$, $\lambda_{k,1} \leq \cdots \leq \lambda_{k,N}$, and $F_{k,1} \geq \cdots \geq F_{k,N}$, that $J(t)$has initial stationary distribution π and that Q is stochastically monotone.*
Then for $t > 0$, $(S_i(t))_{i\in I}$ is an associated vector, and so is $(S_i'(t))_{i\in I}$. Similarly, for all $(u_i)_{i\in I}$, $(T_i(u_i))_{i\in I}$ is associated, where $T_i(u_i)$ is the time to ruin for branch i with initial reserve u_i.
Besides, for all $t \in (\mathbb{R}^+)^K$, for all $u \in (\mathbb{R}^+)^K$,

$$\psi(u,t) = P[\exists i \in I, T_i(u_i) \leq t_i] \geq \psi_\pi^*(u,t),$$

where $\psi_\pi^*(u,t)$ is the probability of ruin of at least one line of business i before t_i with initial reserve u_i in the average independent Poisson model (without modulation).

The proof is omitted here, and is a simple generalization of the proof one may found in Rolski et al. (1999). It uses the following lemma.

Lemma II.1 *With the previous hypotheses, for $i \leq j$,*

$$\psi_i(u,t) \leq \psi_j(u,t).$$

Similar results may be obtained for the independent case.

Lundberg bounds, heavy tails and asymptotical optimal allocation

Hojgaard and many others derived asymptotical properties of univariate ruin probabilities with Markov modulation and heavy tails. Many of these results may be adapted to our model. As an example, for large $u \geq 0$, define

$$\psi(u) = \min_{u_1+\cdots+u_n=u} \psi(u_1, \ldots, u_n).$$

Theorem II.3 *Assume that the lines of business are independent. Assume that the heavier tail is for line of business k, in environment state $j \in \S$. Assume for example that in the classical model with parameters of line k and state j, $\psi_{k,j}(x) \sim C_j x^{-\alpha}$ as $x \to +\infty$. Then as $u \to +\infty$, the optimal allocation tends to be $u_i = u\delta_{i,j}$ for $1 \leq i \leq K$, and there exists a constant C such that*

$$\psi(u) \sim Cu^{-\alpha}.$$

The proof is based on the fact that the univariate probability of ruin is convex and decreasing for each branch.

Many other results, that may be found for example in Rolski et al. (1999), may be adapted to our framework.

1.4 Impact of dependence on multi-dimensional risk measures

1.4.1 Impact of dependence

Cossette et al. (2003) show that positive dependence between risks increases the probability of ruin in the model where the total wealth of the company is considered. This is consistent with coherence of risk measures in the sense of Artzner et al. (1999).
Recall the definition of the lower and upper concordance ordering.

Definition II.1 *Let $(Z_1, ..., Z_K)$ and $(Y_1, ..., Y_K)$ be two random vectors with the same marginal distributions. Y is said to be smaller than Z for the lower concordance order (denoted by $\leq_{lc}$) if*

$$\forall y_1, ..., y_K \in \mathbb{R},\ P[Y_1 \leq y_1, ..., Y_K \leq y_K] \leq P[Z_1 \leq y_1, ..., Z_K \leq y_K],$$

and smaller than Z for the upper concordance order (denoted by $\leq_{uc}$) if

$$\forall y_1, ..., y_K \in \mathbb{R},\ P[Y_1 > y_1, ..., Y_K > y_K] \leq P[Z_1 > y_1, ..., Z_K > y_K].$$

In our multidimensional model, as noted previously by Picard et al. (2003) in the case of independent increments, positive lower orthant dependence between risks decreases the multidimensional probability of ruin. Besides, if for all t,

$$(X_1(t), \ldots, X_K(t)) \leq_{lc} (X'_1(t), \ldots, X'_K(t)),$$

then for all $u_1, \ldots, u_K \geq 0$,

$$\psi(u_1, \ldots, u_K) \leq \psi'(u_1, \ldots, u_K).$$

In particular, for $e \in \mathcal{S}$, $1 \leq k, l \leq K$, let $p^e_{k,l} = \mathbb{P}(I^e_k = I^e_l = 1)$. Then for all $u_1, \ldots, u_K \geq 0$, $\psi(u_1, \ldots, u_K)$ is decreasing in each $p^e_{k,l}$, for $e \in \mathcal{S}$, $1 \leq k, l \leq K$.
This fundamental difference between these two kinds of risk measures may be illustrated by the geometry of the acceptance regions at time t, or solvency regions. These regions, denoted by $H(t)$ as in Picard et al. (2003), correspond to the multidimensional values of the aggregate claim amounts up to time t that may be compensated by the initial reserves and the premium received up to time t. Figure 1.2 shows these regions, which are triangles, or kinds of simplexes, if ruin corresponds only to a negative value of the total wealth process, and rectangles, or boxes in the case where ruin occurs as soon as at least one line of business is in the red. An interesting compromise is to define ruin as the event "at least one line of business gets ruined with a certain severity, or the total company gets ruined". This is illustrated by figure 1.3. Many ruin concepts may be defined like this. All computations described in section 1.3.1 remain valid. One just has to

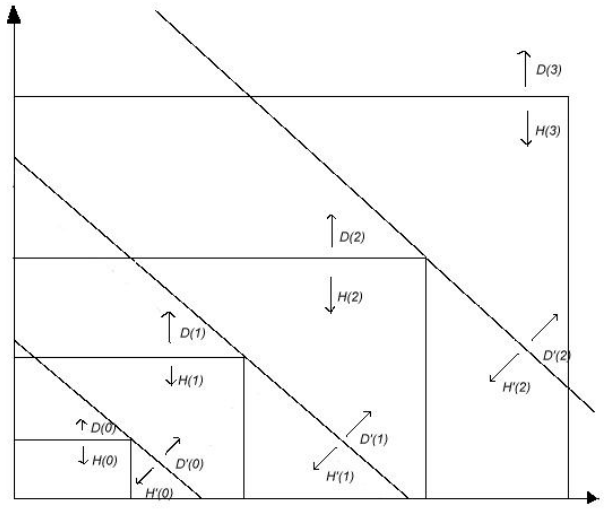

FIG. 1.2 – Regions.

replace regions $H(t)$ and $D(t)$ by the adapted ones. Numerical results (see section 1.5) compare the obtained ruin probabilities in the three cases, showing that these concepts should be used together to get a good idea of what might happen. Figure 1.4 compares the $\mathbb{P}(T > t)$, $1 \leq t \leq 10$, for these three kinds of regions, with two negatively correlated lines of business. The state space has cardinality 2. State 1 is favorable for line 1, unfavorable for line 2, and this is the opposite for state 2. Define $c_i(t)$ as the total risk premium received by the company up to time t. We consider the probability that at least one line gets ruined before t

$$\mathbb{P}_1(t) = 1 - \mathbb{P}(T_1 > t, T_2 > t) = \mathbb{P}(\exists s \leq t, \quad S_1(s) > u_1 + c_1(s) \text{ or } S_2(s) > u_2 + c_2(s)),$$

the probability that the total wealth becomes negative at some time less than t

$$\mathbb{P}_2(t) = \mathbb{P}(\exists s \leq t, \quad S_1(s) + S_2(s) > u_1 + c_1(s) + u_2 + c_2(s)),$$

and the probability that either the total wealth becomes negative at some time less than t, or some line is ruined with severity greater than a fixed limit $v > 0$

$$\mathbb{P}_2(t) = \mathbb{P}(\exists s \leq t, \quad S_1(s)+S_2(s) > u_1+c_1(s)+u_2+c_2(s) \text{ or } S_2(s) > u_2+c_2(s)+v \text{ or } S_2(s) > u_2+c_2(s)+v).$$

$\mathbb{P}_1(t)$ corresponds to regions $H(t)$ and $D(t)$, and $\mathbb{P}_2(t)$ corresponds to regions $H'(t)$ and $D'(t)$ in figure 1.2. $\mathbb{P}_3(t)$ corresponds to regions $H(t)$ and $D(t)$ in figure 1.3.

A neutral possibility is to consider the sum over the number of lines of business of the time-integrated negative parts of each surplus process, as in Loisel (2005). This multi-component risk indicator does not depend on the dependence structure between the lines of business, the marginal distributions being given. This provides a reference, which may be useful to investigate the impact of dependence on other risk measures.

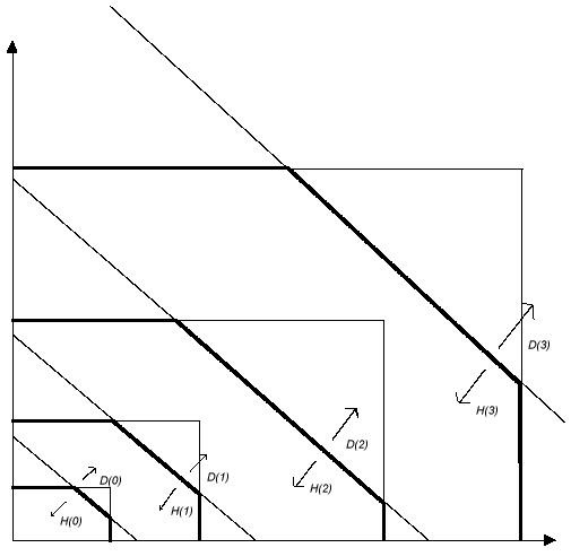

FIG. 1.3 – Compromise regions.

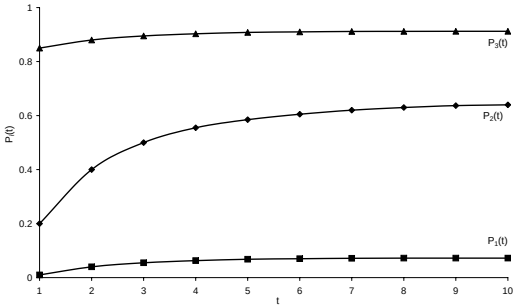

FIG. 1.4 – Probabilities of ruin with different ruin concepts :
values of $\mathbb{P}_1(t), \mathbb{P}_2(t), \mathbb{P}_3(t)$ for $1 \leq t \leq 10$.

1.4.2 The pros and the cons of some risk measures

Some authors, like Cossette and Marceau (2000), and many others, considered multi-risk models. However, in most cases, they focus on the unidimensional risk process representing the total wealth of the company.
From now on, we consider a fixed accounting time horizon T, and study the evolution of the wealths of the lines of business of the company between times 0 and T. With three lines of business (three kinds of activities), for example health, invalidity and driving insurance, it is not the same situation to have $(1M, 2M, -2.8M)$ (*id est* 1 million euros for the first branch, 2 million euros for the second one and to be short of 2.8 million euros for the last branch), or to have $(0, 0.1M, 0.1M)$. Considering only the total wealth (0.2 million euros) does not reflect the situation of the company very well. A few years ago, a holding company mainly had two large airlines companies. The first one was doing well, say its wealth was 10M dollars. The second one was undergoing a bad period, with a debt of, say, 2M dollars. Even if the subcompanies were collateralized, the holding company was estimated 4M dollars, instead of 8M dollars, by the market. The reason was that analysts expected the healthy line of business to be penalized by the other one, which was in the red. To be able to detect such penalties, and to compute probabilities of such unfavorable events, the multi-dimensional process has to be studied as an additional indicator. There is the same phenomenon for a single line of business during different time periods : managers tend to smooth results over time artificially. Of course, it is possible to smooth results between the lines of business of activities and over time up to a certain point, but it is interesting to try to take this into account for the choice of risk measures. Some of them should so reward good, smooth results better than a mix of excellent and bad results corresponding to equivalent global results.

A good risk measure for multidimensional risk processes must satisfy some common-sense properties, similar to those of unidimensional risk measures. However, they may express some preferences for multi-branch companies, which could not be taken into account by classical risk measures. Indeed, if the new risk measures had the same properties of the existing ones, it would not be interesting to introduce them, one had better use the simpler, existing ones. For some strategy reasons, a company could prescribe that one particular line of business, or a finite number of its lines of business do not get ruined before T. It could also be required that no more than two lines of business get in the red simultaneously, or that the global severity is less than a fix amount. One may combine all these requirements to build a risk measure which corresponds to the global strategy of the company.
To be short, a good multidimensional risk measure should in general "favor" good smooth results, and be flexible enough to take possibly into account special management objectives.

The probability to hit a certain region : for example the probability that no more than k lines of business from K get ruined before T, or the probability that no more than k lines of business from K get ruined simultaneously.
The sum of the severities at ruin : if τ_k denotes the time to ruin for line of business k, it corresponds to

$$f(\omega) = \sum_{k=1}^{K} |X_k(\tau_k, \omega)| \mathbb{1}_{\{\tau_k(\omega) \leq T\}}$$

The severity at first ruin time, with

$$f(\omega) = |X_{k_0}(\tau_{k_0}, \omega)| \mathbb{1}_{\{\inf_k \tau^k(\omega) \leq T\}},$$

where $\tau_{k_0} = \inf\{\tau_k, \quad 1 \leq k \leq K\}$.
The sum of the time-aggregated severities of ruin (see Loisel (2005) for definition and properties) with

$$f(\omega) = \sum_{k=1}^{K} I_T^k(\omega),$$

where for $1 \leq k \leq K$,

$$I_T^k(\omega) = \int_0^T |X_k(t,\omega)| \mathbb{1}_{\{X_k(t,\omega)<0\}} dt.$$

The sum of the times in the red.
If one wants to focus on the allocation of the initial reserve, one wants to minimize the drawbacks of having lines of business in the red while the global company has global positive wealth, one can define as in Loisel (2005) the risk measure defined by

$$f(\omega) = \sum_{k=1}^{K} \tau_k'(\omega),$$

where

$$\tau_k'(\omega) = \int_0^T \mathbb{1}_{\{X_k(t,\omega)<0\}} \mathbb{1}_{\{\sum_{k=1}^K X_k(t,\omega)>0\}} dt.$$

τ_k' represents the time spent in the red by line of business k while the global wealth of the company is positive. One can imagine to replace $\mathbb{1}_{\{X_k(t,\omega)\}}$ with $-g(X_k(t,\omega))\mathbb{1}_{\{X_k(t,\omega)\}}$, where g is any convex, increasing function in $\mathcal{F}(\mathbb{R}^+, \mathbb{R}^+)$.
More generally, all functions

$$f(\omega) = \int_0^T g\left(X_1(t,\omega), \ldots, X_K(t,\omega)\right) dt$$

where g is any nondecreasing function from $\mathbb{R}^K$ to $\mathbb{R}$, define a risk measure. This is a very important class of risk measures, because Fubini's theorem provides an easier way to compute them :

$$Ef(\omega) = \int_0^T E\left[g\left(X_1(t,\omega), \ldots, X_K(t,\omega)\right)\right] dt$$

For these risk measures, one can split the problem into different problems for different times and then make integration over time, which is not possible for the probability of ruin for example. Computing the probability of ruin requires to keep track of the history of the process. Besides, differentiation theorems may be useful to make computations. (see Loisel (2005)).
It is of course possible to mix different risk measures to obtain a new one.

1.5 Numerical results

The following figures illustrate the impact of positive dependence, and reinsurance. The example taken here is the one illustrated by figure 1.1. Recall that we consider a model with 3 states, 3 lines of business, and no common shock.
Lines of business 1 and 2 (blue) may depend on $J(t)$ (red), whereas line of business 3 (green) is always independent from $J(t)$, and thus from the two other lines of business.
These graphs illustrate the fact that, if the reinsurance premium is small enough, reinsurance

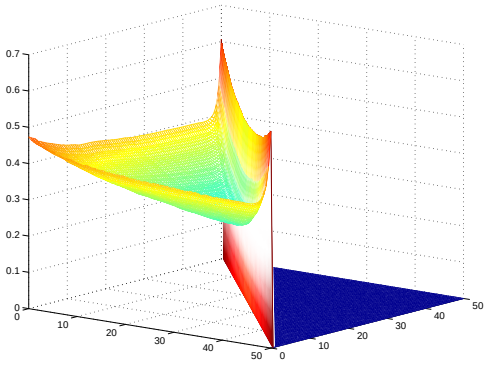

FIG. 1.5 – Dependence (lines of business 1-2), no reinsurance.
$\psi_{\min} = \psi(22.75, 22.75, 4.5) = 0.214$

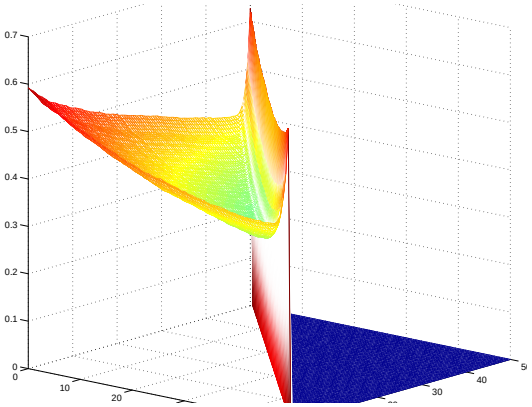

FIG. 1.6 – Independence (lines of business 1-2), no reinsurance.

$\psi_{\min} = \psi(23.5, 23.5, 3) = 0.325$

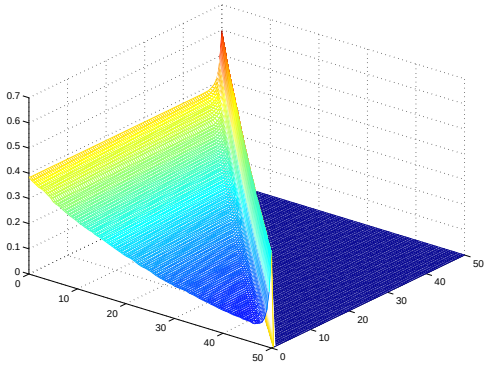

FIG. 1.7 – Dependence and reinsurance for branch 1.
$\psi_{\min} = \psi(0.5, 46, 3.5) = 0.074$

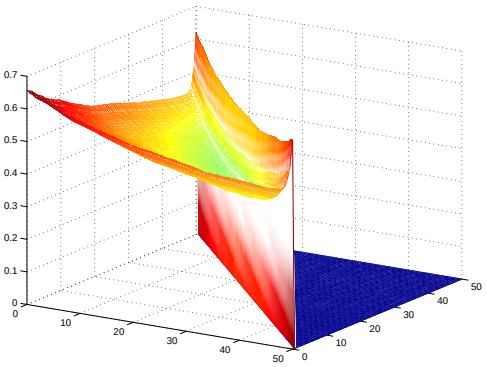

FIG. 1.8 – Independence and reinsurance for branch 1.
$\psi_{\min} = \psi(28, 19, 3) = 0.317$

decreases the ruin probability. Besides, positive dependence (see figures 1.5 and 1.7) also decreases the ruin probabilities (see figures 1.6 and 1.8). This is consistent with theorem II.2. The probability that at least one line of business gets ruined favors massive reinsurance compared to the classical ruin probability for the total reserve process. For these four graphs, the initial state is 3, the most favorable. It is possible to get similar graphs with predefined reinsurance strategy adapted to the environment, in about the same amount of time. For example, assuming that the state of the environment is observable, one might choose to sign excess of loss treaties when the state of environment is 1 (the least favorable for the insurance company).
The impact of the initial state of the environment is clearly illustrated by the comparison between figures 1.5 and 1.7 (initial state 3), and figures 1.9 and 1.10 (initial state 1). This illustrates lemma II.1.

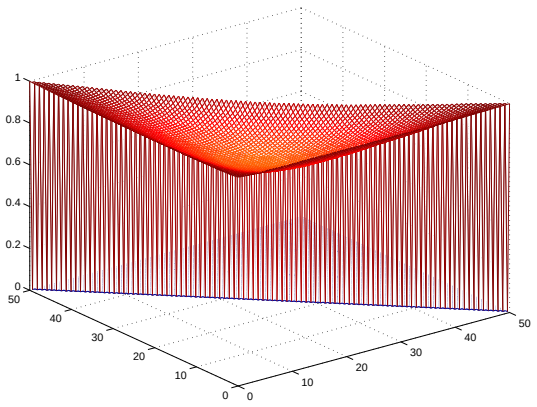

FIG. 1.9 – Dependence and no reinsurance, initial state 1.

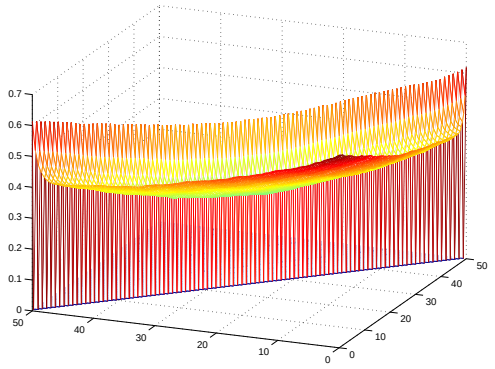

FIG. 1.10 – Dependence and reinsurance for line of business 1, initial state 1.

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2

Aggios et allocation optimale de réserve initiale
Differentiation of some functionals of stochastic processes and optimal reserve allocation

For general risk processes, the expected time-integrated negative part of the process on a fixed time interval is introduced and studied. Differentiation theorems are stated and proved. They make it possible to derive the expected value of this risk measure, and to link it with the average total time below zero studied by Dos Reis (1993), and the probability of ruin. Differentiation of other functionals of unidimendional and multidimensional risk processes with respect to the initial reserve level are carried out. Applications to ruin theory, and to the determination of the optimal allocation of the global initial reserve which minimizes one of these risk measures, illustrate the variety of application fields and the benefits deriving from an efficient and effective use of such tools.

Introduction

For unidimensional risk processes $R_t = u + X_t$ (representing the surplus of an insurance company at time t, with initial reserve u and with $X_t = ct - S_t$, where $c > 0$ is the premium by unit time, and S_t is in the most classical case a compound Poisson process (here we do not limit ourselves to the Poisson case)), many risk measures have been considered (see for example Gerber (1988), Dufresne et Gerber (1988) and Picard (1994)) : the time to ruin $T_u = \inf\{t > 0, u + X_t < 0\}$, the severity of ruin $u + X_{T_u}$, the couple $(T_u, u + X_{T_u})$, the time in the red (below 0) from the first ruin to the first time of recovery $T'_u - T_u$ where $T'_u = \inf\{t > T_u, u + X_t = 0\}$, the maximal ruin severity ($\inf_{t>0} u + X_t$), the aggregate severity of ruin until recovery $J(u) = \int_{T_u}^{T'_u} |u+X_t|dt$,... Dos Reis (1993) studied the total time in the red $\tau(u) = \int_0^{+\infty} 1_{\{u+X_t<0\}} dt$ using results of Gerber (1988).
All these random variables are drawn from the infinite time ruin theory, or involve the behavior of the risk process between ruin times and recovery times. It seems interesting to consider risk measures based on some fixed time interval $[0, T]$ (T may be infinite).
One of the simplest penalty functions may be the expected value of the time-aggregated negative part of the risk process :

$$E(I_T) = E\left(\int_0^T 1_{\{R_t<0\}}|R_t|dt\right).$$

Note that the probability $P(I_T = 0)$ is the probability of non ruin within finite time T. I_T may be seen as the penalty the company will have to pay due to its insolvency until the time horizon T. From an economic point of view, it seems more consistent to consider

$$I_{g,h}(u) = \left(\int_0^T \left(1_{\{u+X_t\geq 0\}} g(|u + X_t|) - 1_{\{u+X_t\leq 0\}} h(|u + X_t|)\right) dt\right)$$

with $0 \leq g \leq h$, where g corresponds to a reward function for positive reserves, and h is a penalty function in case of insolvency. To be consistent with the theory of utility functions, g should be increasing and concave, and h should be increasing and convex in the classical case. Besides, $g \leq h$ because usually the cost of ruin is higher than the reward of the opposite wealth level.
These risk measures may be differentiated with respect to the initial reserve u, which makes it possible to compute them quite easily as integrals of other functions of u such as the probability of ruin or the total time in the red. Moreover, they have the advantage that the integral over t and the mathematical expectation may be permuted thanks to Fubini's theorem.
Statements and proofs of differentiation theorems can be found in Sections 1 and 2.

Section 3 presents examples of applications to unidimensional risk measures, in particular a closed-form formula is derived for $E(I_\infty(u))$ in the Poisson-exponential case.
One can also use these concepts to construct risk measures for multidimensional risk processes, modelling different lines of business of an insurance company (car insurance, health insurance, ...). In this framework, determining the global initial reserve needed for the global expected penalty to be small enough requires finding the optimal allocation of this reserve. Differentiation of unidimensional risk measures are useful for this purpose. All this is illustrated in Section 4.

2.1 Differentiation theorems

In the sequel, we will denote for $T \in [0, +\infty]$ the time in the red until time T by

$$\tau(u, T) = \int_0^T 1_{\{u+X_t<0\}} dt.$$

In most cases, T will be fixed, and we will use the notation $\tau(u)$ instead of $\tau(u, T)$. In Section 1, we assume that $T < +\infty$.

Theorem II.4 *Assume $T \in \mathbb{R}^+$. Let $(X_t)_{t\in[0,T)}$ be a stochastic process with almost surely time-integrable sample paths. For $u \in \mathbb{R}$, denote by $\tau(u)$ the random variable corresponding to the time spent under zero by the process $u + X_t$ between the fixed times 0 and T :*

$$\tau(u) = \int_0^T 1_{\{u+X_t<0\}} dt,$$

Let $\tau_0(u)$ correspond to the time spent in zero by the process $u + X_t$:

$$\tau_0(u) = \int_0^T 1_{\{u+X_t=0\}} dt.$$

Let $I_T(u)$ represent the time-integrated negative part of the process $u + X_t$ between 0 and T :

$$I_T(u) = \left(\int_0^T 1_{\{u+X_t<0\}} |u + X_t| dt \right)$$

and $f(u) = E(I_T(u))$.
For $u \in \mathbb{R}$, if $E\tau_0(u) = 0$, then f is differentiable at u, and $f'(u) = -E\tau(u)$.

$I_T(u)$ is illustrated by Figure 2.1.

Proof. Fix $u \in \mathbb{R}$. For $\epsilon \geq 0$, set

$$\tau_\epsilon(u) = \int_0^T 1_{\{|u+X_t|<\epsilon\}} dt.$$

Here, $\tau_\epsilon(u)$ represents the time spent by the process $u + X_t$ in the interval $]-\epsilon, \epsilon[$ between dates 0 and T.
For each sample path (considered as a function of time t),

$$t \to 1_{\{|u+X_t|<\epsilon\}}$$

pointwise converges, decreasingly to

$$t \to 1_{\{u+X_t=0\}}.$$

Besides, each of the integrals of the indicator functions is bounded by T. From the monotone convergence theorem, τ_ϵ is decreasing with respect to ϵ and converges surely to τ_0.
From the monotone convergence theorem (this time for mathematical expectation), $E\tau_\epsilon \downarrow E\tau_0$ as $\epsilon \downarrow 0$, because for all $\epsilon \geq 0$, $E\tau_\epsilon \leq T$.

Lemma II.2 *For $\epsilon \in \mathbb{R}$,*

$$|I_T(u+\epsilon) - I_T(u) + \epsilon\tau(u)| \leq |\epsilon|\tau_\epsilon(u).$$

Proof of the lemma. For $\epsilon > 0$, $\{u+\epsilon+X_t < 0\} \subset \{u+X_t < 0\}$, whence $I_T(u+\epsilon) - I_T(u) =$

$$\int_0^T \left(|u+\epsilon+X_t| - |u+X_t|\right) 1_{\{u+X_t<0\}} dt - \int_0^T |u+\epsilon+X_t| 1_{\{-\epsilon<u+X_t<0\}} dt.$$

$$I_T(u+\epsilon) - I_T(u) = -\epsilon \int_0^T 1_{\{u+X_t<0\}} dt - \int_0^T |u+\epsilon+X_t| 1_{\{-\epsilon<u+X_t<0\}} dt. \tag{2.1}$$

On the right side of (2.1), the left term corresponds to $-\epsilon\tau(u)$. The absolute value under the integral of the second term is less than ϵ on the support of the indicator function. Hence

$$|I_T(u+\epsilon) - I_T(u) + \epsilon\tau(u)| < \int_0^T \epsilon 1_{\{-\epsilon<u+X_t<0\}} dt,$$

which proves the lemma for $\epsilon > 0$. A symmetrical procedure solves the case $\epsilon \leq 0$.

From Lemma II.2,

$$|EI_T(u+\epsilon) - EI_T(u) + \epsilon E\tau(u)| \leq |\epsilon| E\tau_\epsilon(u)$$

and

$$EI_T(u+\epsilon) = EI_T(u) - \epsilon E\tau(u) + \epsilon v(u,\epsilon)$$

where

$$|v(u,\epsilon)| \leq E\tau_\epsilon(u) \to E\tau_0(u) = 0$$

as $\epsilon \to 0$, which proves that f is differentiable with respect to u and that for $u \in \mathbb{R}$, $f'(u) = -E\tau(u)$.

Corollary II.1 *Using the notation of Theorem II.4, let $X_t = ct - S_t$, where S_t is a jump process such that, almost surely, S_t has a finite number of nonnegative jumps in every finite interval, and that X_t has a positive drift ($X_t \to +\infty$ a.s.). Then f defined by $f(u) = E(I_T(u))$ for $u \in \mathbb{R}$ is differentiable on $\mathbb{R}$, and for $u \in \mathbb{R}$, $f'(u) = -E\tau(u)$.*

Proof. Only

$$E\tau_0(u) = \int_0^T 1_{\{u+ct-S_t=0\}} dt = 0$$

has to be shown. Note that $R_t = u + ct - S_t$ is a process whose sample paths are almost surely increasing between two consecutive jump instants. The number of jumps is almost surely finite on the time interval $[0, T]$. Between two times when the process is 0, there must be at least one

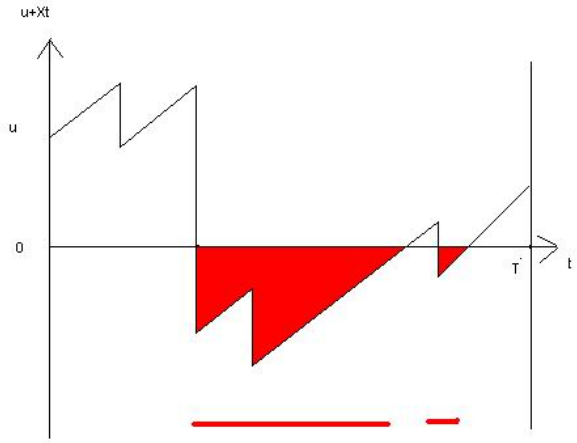

FIG. 2.1 – The area in red represents $I_T(u) = \int_0^T 1_{\{u+X_t<0\}}|u+X_t|dt$

jump instant.
This implies that the number of visits of 0 is almost surely finite as it is less than $N_T + 1$, where N_T is the number of jumps between 0 and T.
So $E\tau_0 = 0$ and the result comes from Theorem II.4.

Proposition II.1 *More generally, all processes for which the distribution of R_t is diffuse for all $t \in \mathbb{R}^+ - N$ satisfy the condition $E\tau_0 = 0$, if N is a null subset of $\mathbb{R}^+$ for the Lebesgue measure. Theorem II.4 is also satisfied for this wide class of processes.*

Proof. For $T \in \bar{\mathbb{R}}$, from Fubini's theorem,

$$E\tau_0(T) \leq E\left(\int_0^{+\infty} 1_{\{R_t=0\}}dt\right) = \int_0^{+\infty} P(R_t = 0)dt$$

which provides the required result.

Theorem II.5 *Let $g \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R}^+)$ be a convex or concave function, such that $g(0) = 0$. Let X_t be a stochastic process such that, for $u \in \mathbb{R}$, $t \to g(-(u + X_t))1_{\{u+X_t<0\}}$ is almost surely integrable with respect to t. Let I_g be the function from $\mathbb{R}$ into the space of nonnegative random variables, and defined by*

$$I_g(u) = \left(\int_0^T 1_{\{u+X_t<0\}} g(-(u + X_t))dt\right)$$

for $u \in \mathbb{R}$ and let $f(.) = EI_g(.)$.
For $u \in \mathbb{R}$, if $f(u) < +\infty$, $EI_{g'}(u) < +\infty$ and $E\tau_0(u) = 0$, then f is differentiable at point u, and

$$f'(u) = -E\left(\int_0^T 1_{\{u+X_t<0\}} g'(|u + X_t|)dt\right).$$

Proof. Fix $u \in \mathbb{R}$. For $\epsilon > 0$, $\{u + \epsilon + X_t < 0\} \subset \{u + X_t < 0\}$, whence

$$\frac{I_g(u+\epsilon) - I_g(u)}{\epsilon} = \int_0^T \frac{g(|u+\epsilon+X_t|) - g(|u+X_t|)}{\epsilon} 1_{\{u+X_t<0\}} dt$$

$$- \int_0^T \frac{g(|u+\epsilon+X_t|)}{\epsilon} 1_{\{-\epsilon<u+X_t<0\}} dt.$$

For $t \in [0,T]$,

$$\frac{g(-(u+\epsilon+X_t)) - g(-(u+X_t))}{-\epsilon} 1_{\{u+X_t<0\}} \uparrow \text{ (resp. } \downarrow \text{) } g'(-(u+X_t)) 1_{\{u+X_t<0\}}$$

almost surely as $\epsilon \downarrow 0$, from the increase (resp. decrease) of the rates of increase of convex (resp. concave) functions.
From the monotone convergence theorem, for $t \in [0,T]$,

$$E\left(\frac{g(-(u+\epsilon+X_t)) - g(-(u+X_t))}{\epsilon} 1_{\{u+X_t<0\}}\right) \to -E\left(g'(-(u+X_t)) 1_{\{u+X_t<0\}}\right).$$

From Fubini's theorem,

$$E\left(\int_0^T \frac{g(-(u+\epsilon+X_t)) - g(-(u+X_t))}{\epsilon} 1_{\{u+X_t<0\}} dt\right) \to -EI_{g'}(u)$$

as $\epsilon \downarrow 0$, where

$$I_{g'}(u) = \int_0^T g'(-(u+X_t)) 1_{\{u+X_t<0\}} dt.$$

Hence

$$|f(u+\epsilon) - f(u) + \epsilon EI_{g'}(u) + \epsilon w(u,\epsilon)| \leq E\left(\int_0^T g(-(u+\epsilon+X_t)) 1_{\{-\epsilon<u+X_t<0\}} dt\right)$$

with $w(u,\epsilon) \to 0$ as $\epsilon \downarrow 0$, and

$$|f(u+\epsilon) - f(u) + \epsilon EI_{g'}(u) + \epsilon w(u,\epsilon)| \leq \epsilon E\tau_\epsilon(u) E\left(\sup_{t\in[0,\epsilon]} g'(t)\right).$$

$$EI_g(u+\epsilon) = EI_g(u) - \epsilon EI_{g'}(u) + \epsilon(v(u,\epsilon) - w(u,\epsilon))$$

where

$$|v(u,\epsilon)| \leq KE\tau_\epsilon(u) \to KE\tau_0(u) = 0$$

as $\epsilon \downarrow 0$, which proves that f is right-differentiable at point u and that

$$f'_r(u) = -E\left(\int_0^T g'(-(u+X_t)) 1_{\{u+X_t<0\}} dt\right).$$

With similar reasoning, f is left-differentiable and $f'_l = f'_r$, which ends the proof.

2.2 Differentiation of the average time in the red and other generalizations

Recall that the time in the red is the time spent by the wealth process below 0, between time 0 and some fixed time horizon T :

$$\tau(u) = \int_0^T 1_{\{u+X_t<0\}} dt.$$

T is first supposed to be finite.

Theorem II.6 *Let $X_t = ct - S_t$, where S_t is a jump process satisfying hypothesis (H1) : S_t has a finite expected number of nonnegative jumps in every finite interval, and for each t, the distribution of S_t is absolutely continuous.*
For example, S_t might be a compound Poisson process with a continuous jump size distribution. Consider $T < +\infty$ and define h by $h(u) = E(\tau(u))$ for $u \in \mathbb{R}$. h is differentiable on $\mathbb{R}_*^+$, and for $u > 0$,

$$h'(u) = -\frac{1}{c} E N^0(u, T),$$

where $N^0(u,T) = Card(\{t \in [0,T], \quad u + ct - S_t = 0\})$.

Proof. Almost surely in ω, the number of jumps $N(T)$, and so $N^0(u,T)$, is finite. Consider a sample path $(X_t(\omega))_{0\le t\le T}$. Let $R_t = u + X_t$ and denote by T_i the i^{th} jump instant. Define

$$\epsilon_0(\omega) = \inf_{n\le N(T), R_{T_n}>0} R_{T_n}.$$

If $N^0(u,T) = 0$, then define

$$\epsilon^+ = \inf\left(\{u + X_t, \quad 0 \le t \le T\} \cap \mathbb{R}^+\right)$$

and

$$\epsilon^- = -\sup\left(\{u + X_t, \quad 0 \le t \le T\} \cap \mathbb{R}^-\right).$$

Here, ϵ^- and ϵ^+ are almost surely positive. If $|\epsilon| < \inf(\epsilon^+, \quad \epsilon^-)$, then $\tau(u-\epsilon) - \tau(u) = 0$, and the following reasoning remains valid.

Otherwise, for $1 \le i \le N^0(u,T)$, denote by t_i the instant of the i^{th} visit of R_t in 0, and by t_i' the instant of the first jump of R_t after t_i. The sample paths of the process R_t are almost surely right-continuous, and the probability that $R_T = 0$ is zero. So one may consider

$$\epsilon_1(\omega) = \min\left(\min_{1\le i\le N^0(u,T)} c(t_i' - t_i), c(T - t_{N^0(u,T)})\right).$$

Then, for $0 < \epsilon < \min(\epsilon_0(\omega), \epsilon_1(\omega))$,

$$\{0 < u + ct - S_t < \epsilon\} = \bigcup_{i=1}^{N^0(u,T)}]t_i, t_i + \epsilon/c[$$

and so

$$\tau(u-\epsilon) - \tau(u) = \int_0^T \left(1_{\{u-\epsilon+ct-S_t<0\}} - 1_{\{u+ct-S_t<0\}}\right) dt$$

$$= \int_0^T 1_{\{0 \leq u+ct-S_t<\epsilon\}} dt = \sum_{k=1}^{N^0(u,T)} \frac{\epsilon}{c}.$$

Hence

$$\frac{\tau(u-\epsilon)-\tau(u)}{\epsilon} \to \frac{1}{c} N^0(u,T)$$

almost surely as $\epsilon \to 0$. Moreover, between two consecutive jumps of R_t, the difference between the two integrals is less than $\frac{\epsilon}{c}$ in absolute value, whence

$$\int_{T_i}^{T_{i+1}} 1_{\{0 \leq u+ct-S_t<\epsilon\}} dt \leq \frac{\epsilon}{c}.$$

So for $\epsilon > 0$ small enough, with notations $T_{N(T)+1} = T$ and $T_0 = 0$,

$$\left(\frac{\tau(u-\epsilon)-\tau(u)}{\epsilon}\right) = \left(\sum_{i=0}^{N(T)} \frac{1}{\epsilon} \int_{T_i}^{T_{i+1}} 1_{\{0 \leq u+ct-S_t<\epsilon\}} dt\right).$$

$$\left(\frac{\tau(u-\epsilon)-\tau(u)}{\epsilon}\right) \leq \left(\sum_{i=0}^{N(T)} \frac{1}{\epsilon}\frac{\epsilon}{c}\right) \leq \frac{1}{c}(N(T)+1).$$

Hence, from the dominated convergence theorem,

$$E\left(\frac{\tau(u-\epsilon)-\tau(u)}{\epsilon}\right) \to \frac{1}{c} E N^0(u,T)$$

as $\epsilon \to 0$. This proves that h is left-differentiable on $\mathbb{R}_*^+$, and that for $u > 0$,

$$h'_l(u) = -\frac{1}{c} E N^0(u,T).$$

With similar reasoning, h is right-differentiable on $\mathbb{R}_*^+$, and $h'_l = h'_r$. Hence h is differentiable on $\mathbb{R}_*^+$, and for $u > 0$, $h'(u) = -\frac{1}{c} E N^0(u,T)$.

Remark II.1 *This provides the second-order derivative of $EI_T(.)$, which appears to be positive. Thus, $EI_T(.)$ is strictly convex, which will be very important for the minimization in Section 2.4.*

Remark II.2 *This second-order differentiate corresponds in the general case to the expectation of the local time $L_T(0)$ in 0 of the process $u + X_t$ up to time T :*

$$L_T(0) = \lim_{\epsilon \downarrow 0} \left(\frac{1}{2\epsilon} \int_0^T 1_{\{|u+X_t|<\epsilon\}} dt\right).$$

Theorem II.7 *Let g, h be two convex or concave functions in $\mathcal{C}^1(\mathbb{R}^+, \mathbb{R}^+)$, such that for $x \geq 0$, $g(x) \geq g(0)$ and $h(x) \geq h(0)$. Let X_t be a stochastic process such that $t \to g(u + X_t)$ and $t \to h(u + X_t)$ are almost surely integrable on $[0, T]$. Let I_g^+ be the function from $\mathbb{R}$ into the space of nonnegative random variables, and defined by*

$$I_g^+(u) = \int_0^T 1_{\{u+X_t \geq 0\}} g(u + X_t) dt$$

for $u \geq 0$ and let $f(.) = EI_g^+(.) - EI_h(.)$.
If, for $u \in \mathbb{R}$,

$$EI_g^+(u), \quad EI_h(u), \quad EI_{g'}^+(u), \quad EI_{h'}(u) \quad < +\infty,$$

and if $E\tau_0(u) = 0$, then f is differentiable on $\mathbb{R}_*^+$, and for $u > 0$,

$$f'(u) = EI_{g'}^+(u) - EI_{h'}(u) - (g(0) + h(0))EL_T(0).$$

Corollary II.2 *With the hypotheses of Theorem II.7, if besides $X_t = ct - S_t$, where S_t satisfies hypothesis (H1) of Theorem II.6, then the differentiate may be rewritten for $u > 0$ as :*

$$f'(u) = EI_{g'}^+(u) - EI_{h'}(u) + \frac{(g(0) + h(0))EN^0(u,T)}{c} \tag{2.2}$$

where $N^0(u,T) = Card(\{t \in [0,T], \quad u + ct - S_t = 0\})$.

Proof of Corollary II.2. Immediate from Theorem II.7, after replacing the last term in (2.2) following the proof of Theorem II.6.

Proof of Theorem II.7. Decompose

$$I_g^+(u) - I_h(u) = -\tilde{I}_{(g-g(0))}(-u) - I_{(h-h(0))}(u) - h(0)\tau(u) + g(0)(T - \tau(u)),$$

where $\tilde{I}_g$ is obtained from I_g by changing X_t into $-X_t$. From linearity of expectation and of differentiation, applying theorem II.5 to $g - g(0)$ with $-X_t$ and to $h - h(0)$ with X_t, and using Theorem II.6 end the proof of Theorem II.7.

Theorem II.8 *If besides the process X_t converges almost surely to $+\infty$ as $t \to +\infty$, and if for $u \geq 0$, $EI_\infty < +\infty$ and $E\tau(u,\infty) < +\infty$, then Theorem II.4 remains valid with $T = +\infty$.*

Proof. Same kind of reasoning as previously.

Remark II.3 *These conditions of integrability are fulfilled if the time spent below 0 for a single ruin is integrable.*

Denote by $\psi(u)$ the probability of ruin in infinite time with initial reserve u.

Theorem II.9 *Theorem II.6 remains valid with $T = +\infty$ if besides X_t has a positive drift and if $\tau(u)$ is integrable for all $u > 0$. Besides, in the compound Poisson case, for $u > 0$,*

$$h'(u) = -\frac{1}{c}\frac{1}{1 - \psi(0)}\psi(u).$$

Proof. For $T \in [0, +\infty]$, recall the notation

$$\tau(u,T) = \int_0^T 1_{\{u+X_t<0\}} dt.$$

Note that $(N^0(u,n))_{n\geq 0}$ is a nondecreasing sequence of random variables which converges surely to $N^0(u,+\infty)$, possibly infinite.

We shall show that $EN^0(u,+\infty) < +\infty$.
Almost surely, $u + X_t \to +\infty$ as $t \to +\infty$. Hence, almost surely, $N^0(u,+\infty) < +\infty$ and is equal to the number of ruins :

$$N^0(u,\infty) = \text{Card}\left(\{t > 0, \quad u + ct - S_t < 0 \text{ and } u + ct^- - S_{t^-} > 0\}\right).$$

Indeed, after each ruin, there is a recovery because X_t converges almost surely to $+\infty$ as t goes to $+\infty$, and the number of jumps which lead exactly to the value 0 is finite almost surely. Besides, in the compound Poisson case, the number of ruins has the following distribution :

$$P(N^0(u,\infty) = n) = \psi(u)\psi(0)^{n-1}(1-\psi(0))$$

for $n \geq 1$ and $P(N^0(u,\infty) = 0) = 1 - \psi(u)$. So $N^0(u,\infty)$ follows a zero-modified geometric distribution : $P(N^0(u,\infty) = 0) = 1 - \psi(u)$ and for $n > 0$,

$$P(N^0(u,\infty) = n | N^0(u,\infty) > 0) = \psi(0)^{n-1}(1-\psi(0)).$$

Hence $N^0(u,\infty)$ is integrable and

$$EN^0(u,\infty) = \psi(u)\frac{1}{1-\psi(0)}.$$

For all ω and for $\epsilon > 0$, the function

$$(T,\omega) \to \frac{\tau(u+\epsilon,T) - \tau(u,T)}{\epsilon}(\omega)$$

is increasing with respect to T, and its limit expectation is equal to $-\frac{1}{c}EN^0(u,T)$ as $\epsilon \downarrow 0$. From the monotone convergence theorem,

$$E\left[\lim_{\epsilon\downarrow 0}\left(\frac{\tau(u+\epsilon,\infty) - \tau(u,\infty)}{\epsilon}\right)\right] = -\frac{1}{c}EN^0(u,\infty).$$

Remark II.4 *In infinite time, the probability of ruin may be regarded as the expectation of the local time in 0 of the process (up to multiplication by a constant number).*

2.3 Applications to the unidimensional case

Theorem II.10 *In the Poisson(λ)-Exponential($1/\mu$) case, with positive safety loading $\rho = \frac{c-\lambda\mu}{\lambda\mu}$,*

$$\psi(u) = (1-\mu R)e^{-Ru},$$

with $R = \frac{1}{\mu}\left(1 - \frac{\lambda\mu}{c}\right)$. Hence, for $T = +\infty$,

$$E\tau(u) = \frac{(1-\mu R)}{c\mu R^2}e^{-Ru}$$

and

$$EI_\infty(u) = \frac{(1-\mu R)}{c\mu R^3}e^{-Ru}.$$

Proof. This comes simply from integration of the well-known formula for $\psi(u)$, as the functions considered tend to 0 as $u \to +\infty$.

This method provides a way to get back the average total time in the red from the integration of the probability of ruin. Dos Reis (1993) derived this result for $E\tau(u,\infty)$ by considering the number of ruins, and using the distributions of the length of the first period in the red (until recovery), and of those of the following periods in the red, which had been derived by Gerber (1988).

Remark II.5 *It is possible to derive $EI_\infty(u)$ for Gamma-distributed or phase-type-distributed claim amounts, as we know the probability of ruin in these cases. The results are not reported here in the interest of conciseness.*

The parallel with the Brownian case is also interesting. The local time of a standard Brownian motion W_t in x is defined by

$$L_t(x) = \lim_{\epsilon \downarrow 0} \frac{1}{4\epsilon} \int_0^t 1_{\{|W_s - x| < \epsilon\}} ds.$$

This provides a density for the occupation time $\Gamma_t(B)$ of a Borelian set B between 0 and t :

$$\Gamma_t(B) = \int_B 2L_t(x)dx.$$

Paul Lévy's Brownian local time representation theorem with downcrossings states that

Theorem II.11 *(Paul Lévy)*

$$2L_t(0) = \lim_{\epsilon \downarrow 0} \epsilon D_t(\epsilon)$$

where $D_t(\epsilon)$ is the number of downcrossings of the interval $[0,\epsilon]$ by the process W_s between 0 and t .

This well-known theorem might be viewed as a limit case of Theorem II.6.

2.4 Multidimensional risk measures and optimal allocation

For a unidimensional risk process, one classical goal is to determine the minimal initial reserve u_ϵ needed for the probability of ruin to be less than ϵ.
In a multidimensional framework, modelling the evolution of the different lines of business of an insurance company by a multirisk process $(u_1 + X_t^1, \ldots, u_K + X_t^K)$ ($u_k + X_t^k$ corresponds to the wealth of the k^{th} line of business at time t), one could look for the global initial reserve u which ensures that the probability of ruin ψ satisfies

$$\psi(u_1, \ldots, u_K) \leq \epsilon$$

for the optimal allocation $(u_1, \ldots, u_K)$ such that

$$\psi(u_1, \ldots, u_K) = \inf_{v_1 + \cdots + v_K = u} \psi(v_1, \ldots, v_K)$$

with

$$\psi(u_1, \ldots, u_K) = P(\exists k \in [1, K], \exists t > 0, u_k + X_t^k < 0).$$

Instead of the probability of crossing some barriers, it may be more interesting to minimize the sum of the expected cost of the ruin for each line of business until time T, which may be

represented by the expectation of the sum of integrals over time of the negative part of the process. In both cases, finding the global reserve needed requires determination of the optimal allocation. It has just been shown in the previous sections how to compute $E(I_T)$ for one line of business, and the linearity of the expectation makes it possible to compute the sum for K dependent lines of business just as in the independent case. The structure of dependence between lines of business has no impact on this risk measure. This may be considered as a problem of optimal allocation of resource under budget constraints as in economics, the goal being to maximize the utility function given by the opposite of the sum of the $E(I_T^i)$.

2.4.1 Minimizing the penalty function

Recall that what has to be minimized is

$$A(u_1, \dots, u_K) = \sum_{i=k}^{K} EI_T^k$$

where

$$EI_T^k = E\left[\int_0^T |R_t^k| 1_{\{R_t^k<0\}} dt\right]$$

with $R_t^k = u_k + X_t^k$ under the constraint $u_1 + \cdots + u_K = u$. This does not depend on the dependence structure between the lines of business because of the linearity of the expectation. Denote $v_k(u_k)$ the differentiate of EI_T^k with respect to u_k. Using Lagrange multipliers implies that if $(u_1, \dots, u_K)$ minimizes A, then $v_k(u_k) = v_1(u_1)$ for all $1 \leq k \leq K$. Compute $v_k(u_k)$:

$$v_k(u_k) = \left(E\left[\int_0^T |R_t^k| 1_{\{R_t^k<0\}} dt\right]\right)' = -E\tau^k = -\int_0^T P\left[\{R_t^k < 0\}\right] dt$$

where τ^k represents the time spent in the red between 0 and T for line of business k.
The differentiation theorem of the previous section justifies the previous derivation. The sum of the average times spent under 0 is a decreasing function of the u_k. So A is strictly convex. On the compact space

$$\mathcal{S} = \{(u_1, \dots, u_K) \in (\mathbb{R}^+)^K, \quad u_1 + \cdots + u_K = u\},$$

A admits a unique minimum. The optimal allocation is thus the following : there is a subset $J \subset [1, K]$ such that for $k \notin J$, $u_k = 0$, and for $k, j \in J$, $E\tau_k = E\tau_j$. The interpretation is quite intuitive : the safest lines of business do not require any reserve, and the other ones share the global reserve in order to get equal average times in the red for those lines of business.

Relaxing nonnegativity, on $\{u_1 + \cdots + u_K = u\}$, if $(u_1, \dots, u_K)$ is an extremum point for A, then for the K lines of business, the average times spent under 0 are equal to one another. If it is a minimum for the sum of the times spent below 0 for each line of business, then the average number of visits is proportional to the c_k, and in infinite time the ruin probabilities are in fixed proportions. However the existence of a minimum is not guaranteed, because $(u_1, \dots, u_K)$ is no longer compact. The problem would be more tractable with the average time in the red or with minimization on the c_k, because some factors penalize very negative u_k in these cases.

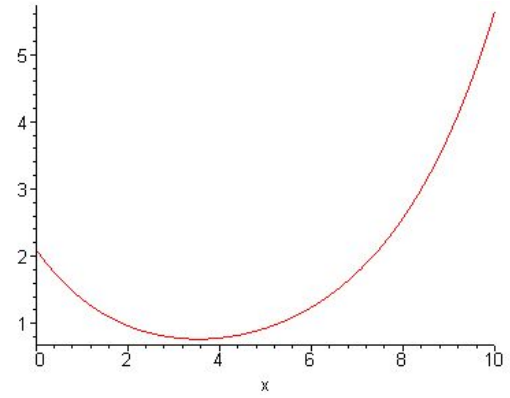

FIG. 2.2 – Graph of $A(x, 10-x)$ with $R_1 = 0.5$. : line of business 2 should have greater initial reserve than line 1.

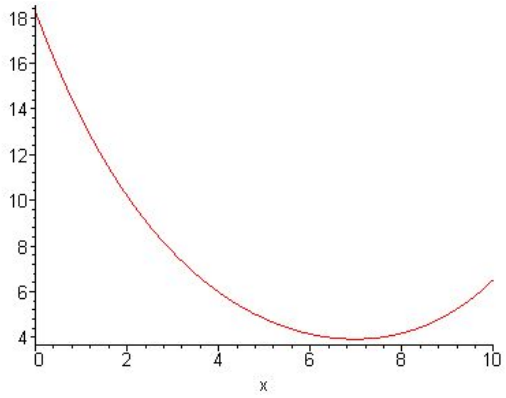

FIG. 2.3 – Graph of $A(x, 10-x)$ with $R_1 = 0.3$. : line of business 1 should have greater initial reserve than line 2.

2.4.2 Example

In the Poisson(λ)-Exponential($\frac{1}{\mu}$) case, recall that

$$EI_\infty(u) = \frac{(1-\mu R)}{c\mu R^3} e^{-Ru}.$$

Consider a two-line-of-business model, with the following parameters :
$\mu_1 = \mu_2 = 1$, $c_1 = c_2 = 1$, $R_2 = 0.4$ and $u = 10$. We want to minimize $A(u_1, u_2)$ for $0 \leq u_1, u_2 \leq 10$ such that $u_1 + u_2 = 10$. A mere modification of the adjustment coefficient R_1 makes the optimal allocation vary strongly. When $R_1 = 0.5 > R_2$, (Figure 2.2), line of business 1 is safer than line 2 from the comparison between the adjustment coefficients, and line 2 should receive a greater initial reserve than line 1. The optimal allocation is about ($u_1 = 3.543410871$, $u_2 = 6.456589129$). When $R_1 = 0.3 < R_2$ (Figure 2.3), line of business 1 is riskier, and this is the opposite. The optimal allocation is in this case ($u_1 = 6.976665007$, $u_2 = 3.023334993$). When $R_1 = 0.08$ (Figure 2.4), the optimal allocation is ($u_1 = 10$, $u_2 = 0$). In that case, line of business 1 is much more risky than line of business 2, which justifies the transfer of the global initial reserve u to line of business 1. For more properties or examples about optimal reserve allocation, the interested reader may consult Loisel (2004).

2.4.3 Further applications

The multidimensional risk measure A, which does not depend on the structure of dependence between lines of business, is one example of what can be considered. Another possibility would be to minimize the sum

$$B = \sum_{k=1}^{K} E\tau_k'(u)$$

where

$$E\tau_k'(u) = E\left(\int_0^T 1_{\{R_t^k<0\}} 1_{\{\sum_{j=1}^K R_t^j>0\}} dt\right).$$

Here B takes dependence into account, and the following proposition shows what can be done :

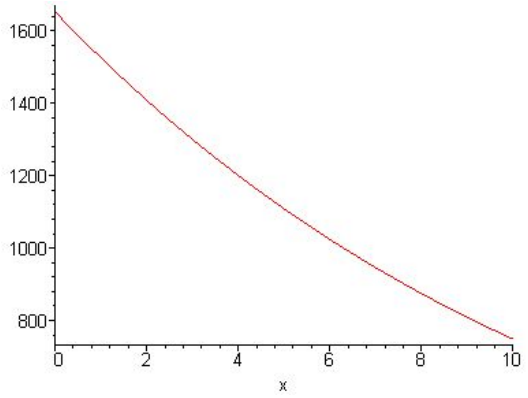

FIG. 2.4 – Graph of $A(x, 10-x)$ with $R_1 = 0.08$. : line of business 1 should receive the whole initial reserve.

Proposition II.2 *Let $X_t = ct - S_t$, where S_t satisfies hypothesis (H1) of Theorem II.6. Define B by $B(u_1, \dots, u_K) = \sum_{k=1}^{K} E(\tau'_k(u))$ for $u \in \mathbb{R}^K$. B is differentiable on $(\mathbb{R}_*^+)^K$, and for $u_1, \dots, u_K > 0$,*

$$\frac{\partial B}{\partial u_k} = -\frac{1}{c_k} E N_k^0(u, T),$$

where $N_k^0(u,T) = Card\Big(\{t \in [0,T], \quad \big(R_t^k = 0\big) \cap \Big(\sum_{j=1}^{K} R_t^j > 0\Big)\}\Big)$.

It is also possible to differentiate with respect to c instead of u.

Theorem II.12 *With the notation of Theorem II.4, consider the case $X_t = ct - S_t$, where S_t satisfies hypothesis (H1) of Theorem II.6, and define $\tilde{f}(c) = E(I_T(c))$.*
If for all c, $E\tau_0(c) = 0$, then $\tilde{f}$ is differentiable on $\mathbb{R}$ and for $c \in \mathbb{R}$,

$$\tilde{f}'(u) = -\int_0^T tP(R_t < 0)dt.$$

It is interesting to look for the optimal allocation of the global premium $c = c_1 + \cdots + c_K$ because if c_k is small enough to make the safety loading negative, the process R_t^k tends to $-\infty$. Quite often, optimizing with the c_k will be easier than with the u_k for this reason. These examples illustrate how these differentiation results may be used.
The differentiation developed here is quite general and may be useful to solve many problems involving multirisk models. For a discussion about multidimensional risk measures, optimal allocation procedures, and impact of dependence between lines of business, the interested reader may consult Loisel (2004).

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3

Modèle multidimensionnel et dividendes
Time to ruin, insolvency penalties and dividends in a Markov-modulated multi-risks model with common shocks

We consider a main insurance company with K subcompanies. The joint evolution of the surpluses of these lines of business is modelled by a Markov-modulated multivariate compound Poisson model with Poisson common shocks, modified by interactions between the lines of business and paiement of dividends. We assume that the financial situation of the subcompanies has an impact on the other companies, for example because they have part of their surplus invested in each other. If a line of business is in the red, the others have to pay a penalty, which is traduced by a decrease of the premium received by unit of time, or by a lost of dividends for the shareholders if the other line of business is "doing well". Conversely, a line of business with a high surplus level may increase the premium by unit of time of the others as they receive part of the dividends. In this paper, we focus on a particular line of business, and provide an approximation for expected time to ruin, and the expected amounts of dividends paid to the shareholders, and used to pay penalty due to insolvency of some subcompany. The method is to discretize claim amounts and to approximate the multidimensional surplus process of the subcompanies with a continuous time Markov process with finite state space. A technique of Frostig (2004) and Kella et Whitt (1992) enables us to get approximates, which are shown to converge at a certain asymptotical rate to the desired values. It is possible to compare the behavior of the main company with and without the other subcompanies, which could provide a tool to help making consortium building decision.

Introduction

This article defines and studies risk and dividend-related problems for a K-dimensional process $R(t)$ modelling the evolution of the surpluses of the $K \geq 1$ lines of business of an insurance company. The model we propose takes into account dependence between lines of business both for the multivariate claim process, and for the premium incomes and dividends.
We consider that line of business 1 behaves slightly differently from the other ones (it might correspond for example to a main company with $K-1$ subcompanies). To take dependence between claim arrivals and amounts of the K lines of business, $R(t)$ is assumed to be a K-dimensional process based on a compound Poisson process with Poisson common shocks in Markovian environment. More precisely, conditionally on the state of the environment, the multivariate claim process is supposed to be compound Poisson with common shocks. As for the environment, it is modelled by a Markov process with finite state space. Claim amounts, taking values in $(\mathbb{R}^+)^K$, are supposed to have exponentially distributed marginals.
Besides, the interaction between lines of business is also traduced by penalties and dividends which modify the vector of premium income rates, which becomes a function of the position of the K-dimensional surplus process. Informally,

- If one or several lines of business become insolvent, this may trigger a penalty for the other lines of business, modelled by a decrease of their premium income rates.
- The surplus level of each line of business is limited to an upper barrier. The excess is transformed into dividends for the shareholders if no line of business is in the red. Otherwise, the excess is used to pay a penalty due to the insolvency of some other lines of business, for example because of financial consequences, or as a way to help the other lines borrow some money in order to recover.
- Line of business 1 may benefit from part or totality of the excesses of other lines of business if they are at their maximum wealth level. This is modelled by an increase of the premium income rate of line of business 1 when another line of business reaches its maximum level.

Typical questions that arise are :

- what proportion of the excess of the surplus of line of business 1 is in average lost for the shareholders due to the insolvency of some other lines of business ? This represents for the shareholders a lost of dividends that they would not have undergone if the subcompanies were completely separate.
- Does the expected time to ruin of a line of business increase or decrease due to the possible financial support or penalty coming from the impact of the surpluses of other lines of business ?
- What is the probability for a line of business to get recovered after its ruin ?

To provide a clue for the last question, we shall derive the distribution of the joint surplus process for the K lines of business at the time to ruin of one line of business.
All these questions seem very tricky, because they involve the *simultaneous* behavior of the different lines of business. The first approach is to try to generalize results of Asmussen et Kella (2000) for multidimensional martingales to multidimensional processes. In section 3.1, we precise the motivations for the model, which we formally present. We show that it is possible to obtain the Laplace transform and a martingale for the multidimensional process.

In section 3.2, we try to answer the three questions mentioned earlier. We show how to approximate the model with a Markov-modulated risk model in which the evolution of $K-1$ lines of business are now contained in the Markovian environment process. We apply the methods of Asmussen et Kella (2000) and Frostig (2004), and compute the expected amounts of dividends paid by line of business 1 to the shareholders until its ruin, and the expected amount paid to the other lines of business. We also derive the distribution of the K-dimensional surplus process at the time to ruin of a line of business.

In section 3.3, we give other ideas of applications, including reinsurance reliability, and adding quasi-default states.

3.1 The multi-dimensional model

3.1.1 Why this model

Consider first the process modelling the wealth of the K lines of business of an insurance company, without notions of dividends and impact of insolvency of one line of business on the other one.
Typical lines of business are driving insurance, house insurance, health, incapacity, death, liability,... They may also be subcompanies of a main insurance company, which may be a line of business or not. They may also face risks of the same nature, but in different countries or regions of the world.
Two main kinds of phenomena may generate dependence between the aggregated claim amounts of these lines.

- Firstly, in some cases, claims for different lines of business may come from a common event : for example, a car accident may cause a claim for driving insurance, liability and disablement insurance. Hurricanes might cause losses in different countries. This should correspond to simultaneous jumps for the multivariate process. The most common tool to take this into account is the Poisson common shock model.
- Secondly, there exist other sources of dependence, for example the influence of the weather on health insurance and on agriculture insurance. In this case, claims seem to outcome independently for each branch, depending on the weather. This seems to correspond rather to models with modulation by a Markov process which describes the evolution of the state of the environment.

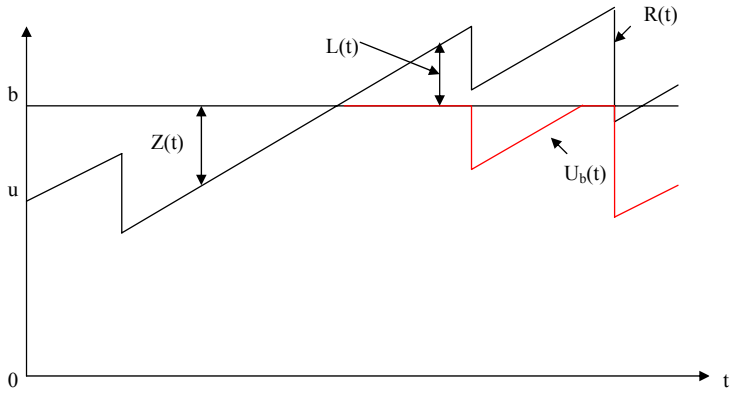

FIG. 3.1 – Explanation for $U_b(t)$, $Z(t)$ and $L(t)$.

Another example is the influence of police controls. Recently, in France, the development of speed controls downed the number of accidents, and the number of severely injured people on the road. The frequency of controls may vary over time and create time-correlation. This is another aspect of Markov-modulation, which may generate over-dispersion for some lines of business.

Each jump may be specific to a line of business, or occur at the same instant as a jump of the other line. Here, amounts are positively correlated, which can be seen on the graph.
To illustrate the effect of a Markovian environment on a multidimensional claim process, figure 3.2 shows a sample path of the surpluses of 3 lines of business of an insurance company, in a Markovian environment, but without common shock. The set of states of the environment has cardinality three. State 3 is the most favorable for the company, almost no claim occurs for lines 1 and 2 (in blue) in this state. State 1 is the least favorable state for the company, claim frequencies and severities are higher for lines 1 and 2. The state of the environment is represented in red. Events for the third line of business (in green) are independent from the state of the environment. One can see the strong positive dependence between lines 1 and 2 (in blue), but also their independence conditionally to the environment state. At some moment, the two blue curves separate each other because of this conditional independence.

We also assume that the surplus process of each line of business is limited by an upper horizontal barrier. The excess of surplus is instantaneously transformed into dividends paid to the shareholders, or used to pay penalty due to insolvency of another line. Besides, when a line of business is in the red (its surplus process is below zero), the premium income rate is supposed to be decreased for the other lines of business. If line 1 is the main company, and the other ones its subcompanies, this may be understood because of the surplus of the main company which may be partly invested in some subcompanies.

3.1.2 The model

We first define the multidimensional claim process $X(t)$. Then we define the multidimensional surplus process $R(t)$.

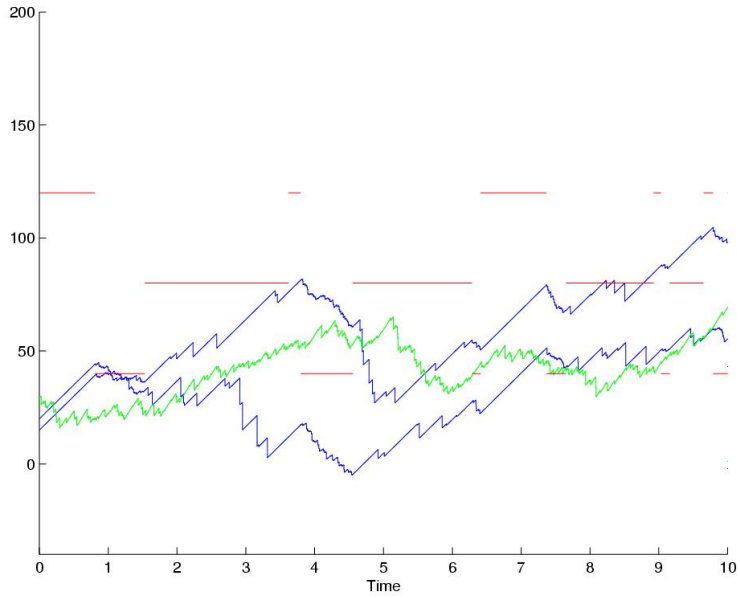

FIG. 3.2 – Sample path for three lines of business : The green one does not depend on the state of the environment. The two blue lines of business have identical parameters, and are independent conditionally on the environmental state. Occupation periods for environment states in red.

Claim process

Let n be the number of states of the environment and K the number of lines of business. Assume that the evolution of the state of the environment is modelled by the Markov process $J(t)$ with initial distribution π_0 and rate transition matrix Q. To define the risk process with K lines of business, consider for $1 \leq i \leq n$ a sequence of i.i.d. random vectors $(W_m^i)_{m\geq 1}$ taking values in $(\mathbb{R}^+)^K$, with distribution function F_{W^i} and exponentially distributed marginals, and independent from a Poisson process $N^i(t)$ with parameter λ^i, and define the n independent K-dimensional Lévy processes

$$X^i(t) = \sum_{l=1}^{N^i(t)} W_m^i - c^i t$$

whose Lévy exponents are denoted by $\varphi^i(\alpha_1, \ldots, \alpha_K)$ (the N^i and the W_m^i are independent from one another), where $c^i = (c_1^i, \ldots, c_K^i)$ is the vector of the premium income rates for all lines of business when the environment state is i. Then, define $X(t) = (X_1(t), \ldots, X_K(t))$ as follows : let T_p be the instant of the p^{th} jump of the process J_t, and

$$\forall k \leq K, \quad X(t) - X(0) = \sum_{p\geq 1} \sum_{1\leq i\leq n} (X^i(T_p) - X^i(T_{p-1}) \mathbf{1}_{\{J_{T_{p-1}}=i, T_p \leq t\}}$$

$$+ \sum_{p\geq 1} \sum_{1\leq i\leq n} (X^i(t) - X^i(T_{p-1}) \mathbf{1}_{\{J_{T_{p-1}}=i, T_{p-1} \leq t < T_p\}}.$$

Define then

$$F(\alpha_1, \ldots, \alpha_K) = Q + \text{diag}(\varphi^1(\alpha_1, \ldots, \alpha_K), \ldots, \varphi^n(\alpha_1, \ldots, \alpha_K)).$$

Note that in case of no common shock, when the environment state is i, each claim only attains one single line of business k, with probability λ_k^i/λ^i, and severity exponentially distributed with parameter $1/\mu_k^i$. In this case, each $\varphi^i(\alpha_1,\ldots,\alpha_k)$ simplifies into :

$$\varphi^i(\alpha_1,\ldots,\alpha_K)=\sum_{k=1}^{K}\varphi_k^i(\alpha_k),$$

where for $1\leq i\leq n$ and $1\leq k\leq K$,

$$\varphi_k^i(\alpha_k)=-c_k^i\alpha_k+\lambda_k^i\frac{1}{1-\alpha_k\mu_k^i}-\lambda_k^i.$$

Theorem II.13

$$M'(t,\alpha)=e^{<\alpha,X(t)>}\widetilde{\mathbf{1}}_{J_t}e^{-F(\alpha_1,\ldots,\alpha_K)t}$$

is a n-dimensional martingale (n is the environment state space size) for all $\alpha\in\mathbb{C}^K$ such that the $\varphi_k^i(\alpha_k)$ all exist, and for all distribution of $(X(0),J_0)$.
If $h(\alpha_1,\ldots,\alpha_K)$ is a right eigenvector of $F(\alpha_1,\ldots,\alpha_K)$ with eigenvalue $\lambda(\alpha_1,\ldots,\alpha_K)$, then

$$N'(t,\alpha)=e^{<\alpha,X(t)>-\lambda(\alpha_1,\ldots,\alpha_K)t}h_{J_t}(\alpha_1,\ldots,\alpha_K)$$

is a martingale.

This martingale cannot really be used in this form. Looking forward to apply Doob's optimal stopping theorem, we would not be able to get the position of the multidimensional process at the considered stopping time, as often in a multidimensional setting. Nevertheless, the Laplace transform of $X(t)$ may be used to make recursive computations after discretizing time and space. The corresponding algorithm involves computation of generalized Appell functionals. However, if the state space is not small, or if the number of lines of business is large, it may take too long to run the algorithm. For more information on this, see Loisel (2004), and Picard et al. (2003) for the case without Markovian environment.

3.1.3 Surplus process modified by the barriers and interactions

Let

$$Y(t)=(Y_1(t),\ldots,Y_K(t))$$

be the n dimensional surplus process defined by :

$$\text{for } 1\leq k\leq K,\quad Y_k(t)=u_k-X_k(t),$$

where u_k is the initial reserve level for line of business k. Let

$$U(t)=(U_1(t),\ldots,U_K(t))$$

correspond to the process $Y(t)$ modified with the barrier strategy $b=(b_1,\ldots,b_K)$, as in Frostig (2004). For $1\leq k\leq K$, define first

$$L_k(t)=-\inf{}_{0\leq s\leq t}\{b_k-u_k+X_k(s)\}^-,$$

where $x^-=\min(x,0)$, and

$$Z_k(t)=b_k-u_k+X_k(t)+L_k(t).$$

Then,

$$U_k(t) = b_k - Z_k(t).$$

This defines the surplus process modified by the barrier strategy $(b_1, \ldots, b_K)$: when a line of business k reaches the level b_k, all the premium income is paid as dividends until the next claim for line of business k.
Define now the new process $R(t) = (R_1(t), \ldots, R_K(t))$ as the modification of $U(t)$ induced by the fact that the c_k^i in the $\varphi^i(\alpha_1, \ldots, \alpha_K)$ now vary in time, and are actually random processes

$$c_k^i(t) = g_k^i(s_1(t), \ldots, s_{k-1}(t), s_{k+1}(t), \ldots, s_K(t)),$$

where the g_k^i are nondecreasing functions from $(\{-1, 0, 1\})^{K-1}$ to $]0, +\infty[$, and for $1 \leq k \leq K$,

$$\begin{array}{rcll} s_k(t) & = & -1 & \text{if } \mathrm{R}_k(t) < 0, \\ & = & +1 & \text{if } \mathrm{R}_k(t) = b_k, \\ & = & 0 & \text{otherwise.} \end{array}$$

In this example, line 2 becomes insolvent at t_1, which decreases the premium income rate for line 1. Then, line 2 recovers at t_2 and reaches its maximum b_2 at t_3, which increases the premium income rate for line 1.

3.2 Expected values of time to ruin, dividends and insolvency penalty

Let τ be the time to ruin of one of the lines of business (called line of business zero). We are interested in knowing the expected value of τ, the distribution $\mathbb{P}_{X_1,\ldots,X_K}(\tau)$ of the surpluses of all lines of business at time τ (to have an idea on possible recovery), and the expected value of the amount of dividends $L_0(\tau)$ paid to the shareholders until τ, and those of penalties paid due to insolvency of at least line of business k, denoted by $L_k(\tau)$.

3.2.1 Outline of the method

We shall now focus on one line of business, which we can choose as line of business 1, and the idea is the following. First, we will approximate the process modelling the evolution of lines of business from 2 to K with a finite state space continuous time Markov process. Then, we will incorporate the position of this $(K-1)$-dimensional process into the environment space. Definition and the way to obtain parameters of this new Markovian environment process are explained in subsection 3.2.2. We have to make the assumption that all common shocks involving line of business 1 have an exponential marginal distribution for line of business 1, independent from the other severities, but with a probability that may depend on the other severities. This is necessary to use a method of Frostig (2004), slightly modified to add common shocks into the model. We state and prove our main result in subsection 3.2.3 : we answer the three questions mentioned in the introduction in the approximate model, and we prove the convergence of the expected time to ruin of line of business 1, and the expected values of the dividends paid to the shareholders and of the penalty paid because of insolvency of some other line(s) of business, as well as the joint distribution of the surplus of the K lines of business at the time to ruin of line of business 1.

3.2.2 Construction of the approximating process

We have to incorporate the position of a $K-1$ dimensional process into a finite state space. We first try to explain the approximation procedure for the simple case of a compound Poisson process with drift on a finite time interval.
To approximate a compound Poisson process with drift by a continuous Markov process with finite state space, the idea is to consider a Poisson process of parameter a, and a discretization of the space with a step d, and to define the paths of the approximated process from the ones of the real process. We will let the approximated process jump upwards if no claim occurred since its previous jump, and jump downwards with the same jump size as the real process with a d approximation. We let $ad = c$, d tends to zero and $a \to +\infty$.
To ensure a finite number of states at the end, we can make the following approximation : consider a common state for all positions less than some lower level.

Let us now build the approximated process from a generalization of the previous idea. The Markov modulation is reproduced from the impact of the environment component of the whole modulating process on the transition probabilities of the surpluses of the subcompanies. To model common shocks, we may allow the process to jump at jump instants of the environment process as in Asmussen et Kella (2000). However, we must remain able to stop the process at the time of ruin of the main company. We must thus know the severity at ruin conditionally on the state of the environment at ruin. Therefore, we assume that the following hypothesis is satisfied :
(H1) : the jump distribution $G_{i,j}$ should be equal to $p_{i,j}F_j$, where $p_{i,j}$ represents the probability that the jump occurs, and F_j is the exponential distribution function with same parameter as for the jumps of $X(t)$ in state j.
Choosing the $p_{i,j}$ enables us to introduce dependence between claim amounts for the different lines of business for common shocks. (H1) may be relaxed to hypothesis (H2), under which the claim distribution for line of business 1 depends only on the state of the original Markovian environment, and not of the position of the other lines of business. As announced in the introduction, we consider here the case where hypothesis (H2) is satisfied, but the results obtained hereafter may be generalized to the case where only (H1) is satisfied, as explained in section 3.3.

Following the previous statements, we want now to define a new environment state space

$$\mathcal{J} = \mathcal{J}^1 \times \cdots \times \mathcal{J}^K$$

with

$$n = n_1 . \prod_{2 \le k \le K} (n_k + 1)$$

states, where $\mathcal{J}^1$ is the original environment state space(n becomes n_1, and $\mathcal{J}$ becomes $\mathcal{J}^1$).
Let $y_2, \ldots, y_K < 0$ be the lower bounds for each line of business, and $n_2, \ldots, n_K \ge 0$ the numbers of subdivisions, and $d_k = \frac{b_k + y_k}{n_k}$ the corresponding discretization steps. Let a_k^i be the Poisson parameter of the process which determines the upwards jumps of the approximating process for line k when the original environment state is i. The d_k should satisfy $c_k^i / a_k^i = d_k$ to have only common upwards jumps. These parameters and the corresponding approximation are illustrated by figure 3.3.

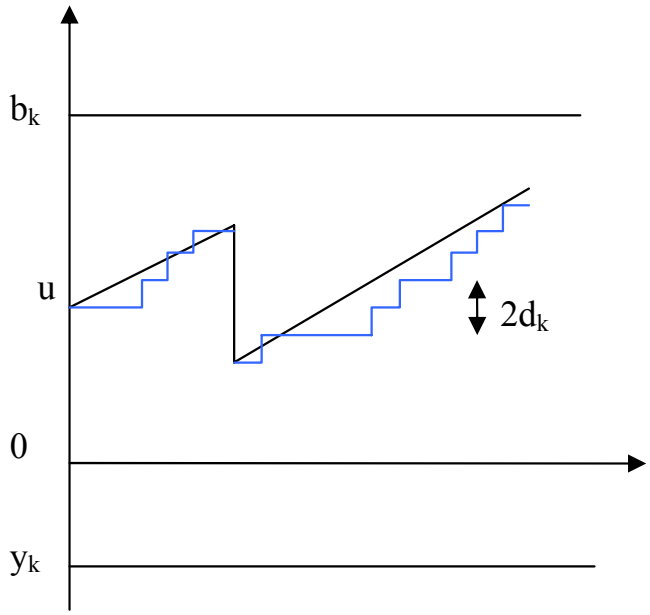

FIG. 3.3 – Sample path of the approximated process, and interpretation of the d_k, a_k^i and n_k for one line of business.

Let Q_1 be the rate transition matrix of the original environment (with state space $\mathcal{J}_1$). We shall use the notation $\underline{n} = (n_2, \ldots, n_K)$, $\underline{a} = (a_2^1, \ldots, a_2^{n_1}, \ldots, a_K^1, \ldots, a_K^{n_1})$ and $\underline{y} = (y_2, \ldots, y_K)$. Denote also $\underline{j} = (j_2, \ldots, j_K)$ and $\underline{l} = (l_2, \ldots, l_K)$ for any $j_2, \ldots, j_K, l_2, \ldots, l_K$. Let

$$Q^{\underline{n},\underline{a},\underline{d}}$$

be the transition rate matrix of the new environment for these parameters.

For an instant, let us fix $\underline{n}$, $\underline{a}$ and $\underline{d}$ and abbreviate objects like $Q^{\underline{n},\underline{a},\underline{d}}$ as Q.
We choose to number the states as follows :
for each line $k \in [2, K]$, $j_k = 0$ corresponds to the minimum level y_k,
$j_k = n_k$ corresponds to the top level b_k,
and more generally the state

$$(j_1, j_2, \ldots, j_K)$$

corresponds to the case where the original environment state is j_1, and the position of the $(K-1)$-dimensional surplus level of the $K-1$ last lines of business is

$$(y_2 + j_2 d_2, \ldots, y_K + j_K d_K).$$

For $\underline{d}$, and $1 \leq j_1 \leq n_1$, let $f_{j_1}(\underline{l})$ (resp. $F_{j_1}(\underline{l})$) be the probability mass (resp. distribution) function of the discretized distribution of $F_{W^{j_1}}$ restricted to the $K-1$ last lines of business. The $f_{j_1}(\underline{l})$ and the $F_{j_1}(\underline{l})$ may be obtained by a procedure generalizing the one described in De Vylder (1999), to preserve mathematical expectations for each line of business.
We can now describe the transition rate matrix Q. Denote $\widetilde{\underline{1}} = (1, \ldots, 1)$ and $\widetilde{\underline{0}} = (0, \ldots, 0) \in \mathbb{R}^{K-1}$.

For $j_1 \neq j_2$, $Q_{(j_1,\underline{j}),(j_2,\underline{l})} = Q_{1(j_1,j_2)}$ if $\underline{j} = \underline{l}$ and 0 otherwise.
This corresponds to jumps of the original environment process $J_1(t)$, and comes from the fact that, in a small interval of length dt, the probability to have a change for $J_1(t)$ and a claim for a line of business is of order $(dt)^2$.
For $\underline{j} = (j_2, \ldots, j_K)$, define

$$A(\underline{j}) = \{k \in [2, K], \quad j_k \notin \{0, n_k\}\}.$$

For $\underline{j}$ such that $A(\underline{j}) \neq \emptyset$, for $k \in A(\underline{j})$ and $1 \leq j_1 \leq n_1$,

$$Q_{(j_1,\underline{j}),(j_1,\underline{j}+\widetilde{\underline{1}}_k)} = a_k^{j_1}.$$

This corresponds to the case where lines of business which are neither at their top level, nor at their minimum level increase because they receive some premium income. Lines of business which are at their top level than remain at their top level. Lines of business at the minimum level must wait longer to jump upwards, because of the severity of the last claim, which may have sent the wealth of this line further down. This is why we treat this case separately.
Denote by $\underline{j} \geq \underline{l}$ the fact that

$$\forall k \in [2, K], j_k \geq l_k,$$

and by $\underline{j} > \underline{l}$ the fact that

$$\underline{j} \geq \underline{l} \quad \text{and that} \quad \exists k \in [2, K], j_k > l_k.$$

For each $\underline{l}$, define the subset $\mathcal{K}(\underline{l})$ of $[2, K]$ of the indices of the lines of business which are at their minimum level when the state of the environment is $(l_1, \underline{l})$ for any state of the original environment $l_1 \in [1, n_1]$:

$$\mathcal{K}(\underline{l}) = \{k \in [2, K], l_k = 0\}.$$

For each subset $\mathcal{K}$ of $[2, K]$, denote by

$$\widetilde{\underline{1}}_{\mathcal{K}}(+\infty)$$

the vector whose k^{th} entry is $+\infty$ if $k \in \mathcal{K}$, and 0 otherwise.

For $\underline{j} > \underline{l}$, and $1 \leq j_1 \leq n_1$,

$$Q_{(j_1,\underline{j}),(j_1,\underline{l})} = \lambda^{j_1} f_{j_1,\underline{l}}(\underline{j} - \underline{l}),$$

where

$$f_{j_1,\underline{l}}(\underline{j} - \underline{l}) = f_{j_1}(\underline{j} - \underline{l}) + F_{j_1}(\underline{j} - \underline{l} + \widetilde{\underline{1}}_{\mathcal{K}(\underline{l})}(+\infty)) - F_{j_1}(\underline{j} - \underline{l}). \tag{3.1}$$

This corresponds to the case where a multivariate claim, concerning some lines of business (between 2 and K) makes the position of the joint surplus of the $K-1$ last lines of business change, provoking a change in the environment from state $(j_1, \underline{j})$ to state $(j_1, \underline{l})$. We already explained that there cannot be a change of j_1 and a claim at the same time, whence j_1 cannot change here. Besides, the real surplus of some lines of business k may fall in fact to a lower level than the minimum level y_k in this approximated model. Thus, we have to incorporate the probability of these events into the transition rates from any state to states $(j_1, \underline{l})$ for which at least one line of business is at its minimum level, *id est* for which $\mathcal{K}(\underline{l}) \neq \emptyset$. To do this, we need to take differences of distribution functions and limits of distribution functions in (3.1).
Now, we have to compensate this by adapting the transition rates for which one line of business

exits its minimum level.
For all $k \in [2, K]$, and for all $j_1, j_2, j_{k-1}, 0, j_{k+1}, \ldots, j_K$, define

$$Q_{(j_1,j_2,j_{k-1},0,j_{k+1},\ldots,j_K),(j_1,j_2,j_{k-1},1,j_{k+1},\ldots,j_K)} = \frac{c_k^{j_1} - \lambda_k^i \mu_k^i}{\mu_k^i + d_k}$$

in order to respect the average time the process of line k would asymptotically take to reach $y_k + d_k$ from $y_k - W_k^{j_1}$ at an average increase rate $c_k^{j_1} - \lambda_k^i \mu_k^i$ if the case of frequent, small claims.
If

$$\text{Card } (\{k \in [2, K], (\, j_k = 0 \text{ and } l_k = 1\,)\}) \geq 2,$$

then for any $1 \leq j_1 \leq n_1$,

$$Q_{(j_1,\underline{j}),(l_1,\underline{l})} = 0.$$

This is because we allow only one line to exit its minimum level state at a time.

For all $\underline{l}$ and $\underline{j}$ such that

$$\exists k \in [2, K], l_k \geq j_k + 2,$$

as we only allow small upwards jumps,

$$Q_{(j_1,\underline{j}),(l_1,\underline{l})} = 0.$$

As usual, define the diagonal terms of the transition rate matrix for all $j_1, \underline{j}$ as

$$Q_{(j_1,\underline{j}),(j_1,\underline{j})} = - \sum_{(l_1,\underline{l}) \neq (j_1,\underline{j})} Q_{(j_1,\underline{j}),(l_1,\underline{l})}.$$

Consider now the parameter $\lambda_{j_1,\underline{j}} = \lambda_{j_1}$ of the Poisson process modelling the jumps which only concern line of business 1, and the parameter $1/\mu_{j_1,\underline{j}} = 1/\mu_{j_1}$ of the exponential distribution of these claim amounts. These parameters only depend on the state j_1 of the original environment, and may be easily obtained from the common shock distribution F_W and the common shock intensity λ. Usually, the common shock distribution will be built from these parameters first, as they can most often be estimated with the data of insurance companies.
Now, we have to incorporate common shocks which involve line of business 1 and some other lines of business. We thus consider also jumps of the surplus process of line of business 1 at instants of change of the environment from state i to state j. For $i = (j_1, \underline{j})$ and $j = (j_1, \underline{l})$, under (H2), these jumps (with size $U_{i,j}$) have a probability $p_{i,j}$ to occur at each transition from i to j. If they occur, the conditional jump size distribution is exponentially distributed with parameter $1/\mu_{j_1}$. Let $G(\alpha)$ be the corresponding $n \times n$ Laplace transform matrix defined by

$$G_{(j_1,\underline{j}),(j_2,\underline{l})}(\alpha) = p_{(j_1,\underline{j}),(j_2,\underline{l})} E\left[e^{\alpha U_{(j_1,\underline{j}),(j_2,\underline{l})}}\right] = p_{(j_1,\underline{j}),(j_2,\underline{l})} \frac{1}{1 - \alpha\mu^{j_2}}.$$

Assumption (H2) may be relaxed to $(H1)$ as explained in section 3.3. Now we are ready to apply the formalism of Frostig (2004) with the adding of common shocks. Define

$$c_{j_1,\underline{j}} = g_1^{j_1}(s_2(j_2), \ldots, s_K(j_k))$$

the premium income rate effectively received by line of business 1 when the state of the new environment is $(j_1, \underline{j})$.

3.2.3 Main result

The idea is now to consider line of business 1, in the new, global environment

$$\mathcal{J} = J^1 \times \cdots \times J^K.$$

First, consider some fixed $\underline{n}$, $\underline{d}$ and $\underline{\overline{a}}$, and omit superscripts dedicated to these quantities. Define for $\alpha \in \mathbb{C}$

$$K(\alpha) = Q \circ G(\alpha) - \alpha \text{ diag } (c) + \text{ diag } (\lambda \frac{1}{1-\alpha\mu}) - \text{ diag } (\lambda_{j_1,\underline{j}}),$$

where for two $n \times n$ matrices, $M \circ N$ denotes the $n \times n$ matrix defined by

$$(M \circ N)_{ij} = M_{ij} N_{ij},$$

and for a vector

$$x = (x_{j_1,\underline{j}})_{1 \leq j_1 \leq n_1, 0 \leq j_k \leq n_k \text{ for } k \in [2,K]},$$

diag (x) is $n \times n$ diagonal matrix whose entries are

$$\text{diag } (x)_{j_1,\underline{j} l_1,\underline{l}} = x_{j_1,\underline{j}} \mathbf{1}_{(j_1,\underline{j})=(l_1,\underline{l})}.$$

From lemma 2.1 of Asmussen et Kella (2000),

$$M^W(t,\alpha) = e^{\alpha X(t)} \widetilde{\underline{\mathbf{1}}}_{J_t} e^{-K(\alpha)t}$$

is a n-dimensional martingale for all $\alpha \in \mathbb{C}$ such that all the $\varphi^i(\alpha)$ exist and for all distribution of $(X(0), J_0)$. If $h(\alpha)$ is a right eigenvector of $K(\alpha)$ for eigenvalue $\kappa(\alpha)$, then

$$N(t,\alpha) = e^{\alpha X(t) - \kappa(\alpha)t} h_{J_t}(\alpha)$$

is a martingale.
Consider the dividend processes

$$L_j^{\underline{n},\underline{d},\underline{\overline{a}}}(t) = \int_0^t \mathbf{1}_{\{J(s)=j\}} dL_1(s),$$

with $L_1(t)$ as in the previous section.

For $j = (j_1, \underline{j})$, it is possible to obtain the $l_j = \mathbb{E}L_j^{\underline{n},\underline{d},\underline{\overline{a}}}(\tau)$, the $p_j = p_{j_1,\underline{j}}^{\underline{n},\underline{d},\underline{\overline{a}}}(\tau) = \mathbb{P}\left[J(\tau) = (j_1, \underline{j})\right]$ and the expected time to ruin $\mathbb{E}\tau^{\underline{n},\underline{d},\underline{\overline{a}}}$ from a slight modification of the method of Frostig (2004). Denote by π the stationary distribution of Q. Note that $K(\alpha)$ has a real eigenvalue $\kappa(\alpha)$ with maximum real part. Define $h(\alpha)$ the corresponding right eigenvector with positive components satisfying $\pi h(\alpha) = 1$. Let k be the derivative of $h(\alpha)$ at $\alpha = 0$, and $m = \kappa'(0)$. As in page 12 of Frostig (2004),

$$\mathbb{E}X(\tau) = m\mathbb{E}\tau + \mathbb{E}\left[k_{J(0)}\right] - \mathbb{E}\left[k_{J(\tau)}\right],$$

but

$$m = \kappa'(0) = \pi.\left[-\text{ diag } (c - \lambda\mu) + Q \circ G'(0)\right] \widetilde{\underline{\mathbf{1}}}$$

has here a different expression. The adapted end of the proof shows that

$$k = (Q - \widetilde{\underline{\mathbf{1}}}\pi)^{-1} \left(m\widetilde{\underline{\mathbf{1}}} + \left[\text{ diag } (c - \lambda\mu) + Q \circ G'(0)\right] \widetilde{\underline{\mathbf{1}}} \right).$$

Theorem II.14 *(Theorem 2.1, Asmussen et Kella (2000)) The multidimensional process*

$$M(t,\alpha) = \int_0^t e^{\alpha Z(t)} \widetilde{\underline{\mathbf{1}}}_{J(s)} ds K(\alpha) + e^{\alpha Z(0)} \widetilde{\underline{\mathbf{1}}}_{J(0)} - e^{\alpha Z(t)} \widetilde{\underline{\mathbf{1}}}_{J(t)} + \alpha \int_0^t \widetilde{\underline{\mathbf{1}}}_{J(s)} dL(s) \tag{3.2}$$

is a n-dimensional martingale for all $\alpha \in \mathbb{C}$ such that the $\varphi_k^i(\alpha)$ exist and for all distributions of $(X(0), J_0)$.

Note that $\det(K(\alpha))$ may be written as a quotient of two polynomials where the numerator is of degree $2n$. Assume that the numerator has $2n$ distincts roots $\alpha_1, \ldots, \alpha_{2n}$. Let $h^j(\alpha)$ be a column vector such that $K(\alpha_j)h^j(\alpha) = 0$. By multiplying (3.2) by $h^j(\alpha)$, we get the following system of $2n$ equations for the $p_j =$ and l_j : for $1 \leq j \leq 2n$,

$$\mathbb{E}\left[e^{\alpha_j Z(0)} h^j_{J(0)}(\alpha_j)\right] - \sum_{i=1}^n p_i e^{\alpha_j b} \frac{1}{1-\alpha_j \mu_i} h_i^j(\alpha_j) + \alpha_j \sum_{i=1}^n l_i h_i^j(\alpha_j) = 0. \tag{3.3}$$

Then, using $\mathbb{E}X(\tau) = \mathbb{E}Z(\tau) - b + u - \mathbb{E}L(\tau)$, and $\mathbb{E}Z(\tau) = \sum_{i=1}^n p_i(b+\mu_i)$,

$$\mathbb{E}\tau^{\underline{n},\underline{d},\overline{\underline{a}}} = \frac{1}{m}\left(\sum_{i=1}^n \left(\mathbb{P}[J(0)=i] - p_i\right) k_i\right) + \sum_{i=1}^n p_i(b+\mu_i) - \sum_{i=1}^n l_i + u - b. \tag{3.4}$$

Theorem II.15 *As all components of $\overline{\underline{a}}$and $\underline{d}$ tend to zero (satisfying $c_k^i = d_k a_k^i$ for all k and i), and all components of $\underline{n}$ tend to $+\infty$, for all $i \in \mathcal{J}$ and $\mathcal{K} \subset [2, K]$,*

$$\tau^{\underline{n},\underline{d},\overline{\underline{a}}} \to \tau$$

and

$$\sum_{j_k > n_k - [b_k/d_k], k\in\mathcal{K}, 0\leq j_l \leq n_l, l\notin\mathcal{K}} \mathbb{E}L_{j_1,\underline{j}}^{\underline{n},\underline{d},\overline{\underline{a}}}(\tau^{\underline{n},\underline{d},\overline{\underline{a}}}) \to \mathbb{E}L_1^{j_1\mathcal{K}}(\tau),$$

where τ is the time to ruin in the original model, $L_1^{j_1\mathcal{K}}(t)$ is the part of dividends used by line 1 to pay some penalty because of insolvency of at least all lines $k \in \mathcal{K}$, and where $L_{j_1,\underline{j}}^{\underline{n},\underline{d},\overline{\underline{a}}}(t)$ and $\tau^{\underline{n},\underline{d},\overline{\underline{a}}}$ are respectively defined by (3.3) and (3.4).
Let $F^{\underline{n},\underline{d},\overline{\underline{a}}}(\tau^{\underline{n},\underline{d},\overline{\underline{a}}})$ be the joint distribution of the K-dimensional surplus process at time $(\tau^{\underline{n},\underline{d},\overline{\underline{a}}})$ (it is directly obtained by (3.3) and the memoryless property satisfied by exponentials distributions). As all components of $\overline{\underline{a}}$and $\underline{d}$ tend to zero, and all components of $\underline{n}$ tend to $+\infty$,

$$F^{\underline{n},\underline{d},\overline{\underline{a}}}(\tau^{\underline{n},\underline{d},\overline{\underline{a}}})$$

converges pointwisely to $F(\tau)$, the joint distribution of the multidimensional surplus in the original model.

The discretized process converges almost surely as all components of $\overline{\underline{a}}$and $\underline{d}$ tend to zero (satisfying $c_k^i = d_k a_k^i$ for all k and i), and all components of $\underline{n}$ tend to $+\infty$. To show this, let us start with classical convergence results in $L^\infty([0,1])$. The following theorem is the analogue of Schilder's theorem for Poisson processes.

Theorem II.16 *(See Dembo et Zeitouni (1998))*
Let μ_ϵ be the probability measures induced on $L^\infty([0,1])$ by $\epsilon\hat{N}(t/\epsilon)$, where $\hat{N}(.)$ is a Poisson process on $\mathbb{R}^+$ of intensity one. The $\{\mu_\epsilon\}$ satisfy the large deviation principle with good rate function :

$$I_{\hat{N}}(\varphi) = \int_0^1 [\dot{\varphi}(t)\ln\dot{\varphi}(t) - \dot{\varphi}(t) + 1]\,dt$$

if φ is absolutely continuous and increasing with $\varphi(0) = 0$, and

$$I_{\hat{N}}(\varphi) = \infty \quad \textit{otherwise.}$$

The previous theorem gives the almost sure convergence of the modified Poisson process to the deterministic linear process in the Skorohod space $D([0,1])$ equipped with the topology of uniform convergence.

Consider first a classical risk process

$$R(t) = u + ct - \sum_{k=1}^{N(t)} U_k,$$

where $N(t)$ is a Poisson process with parameter λ. Define then the approximating process

$$R^{\mu,d}(t)$$

sample path by sample path, from the paths of $R(t)$: the downwards jumps of $R^{\mu,d}(t)$ occur at the same instants as those of $R(t)$, and have severity $d\,[x/d]$, where x is the severity of the corresponding jump for $R(t)$. Upwards jumps are described by a Poisson process with parameter μ, and their size is deterministic, and equal to ηd (which prescribes $\eta d = c/\mu$). Let $[0,T]$ be a fixed time interval. Sample path by sample path, downwards jumps are the same up to the discretization step d, and we superimpose a compound Poisson process with parameter c/d and deterministic jump size $+d$. From Theorem II.16, we can restrict our attention to the downwards jumps part, as for all $p \geq 1$,

$$\mathbb{P}\left(\forall\epsilon > 0, \exists m \geq 1, \forall 0 \leq k \leq p, \quad \| R(kT/p) - R^{m,1/m}(kT/p) \| \leq \epsilon\right) = 1.$$

So for all $p \geq 1$,

$$\mathbb{P}\left(\forall\epsilon > 0, \exists m \geq 1, \forall 0 \leq k \leq p, \quad \| R(t) - R^{m,1/m}(t) \| \leq 2\epsilon + cT/p\right) = 1.$$

By choosing ϵ' and p such that $2\epsilon + cT/p \leq \epsilon$, we obtain the almost sure convergence of $R^{m,1/m}(t)$ to $R(t)$ in the space of càdlàg functions on $[0,T]$ endowed with the norm of the uniform convergence. The sojourn and hitting times until T thus converge almost surely. From the dominated convergence theorem, the expected sojourn times and hitting times until τ converge to those of the continuous model.

For the multidimensional Markov modulated process with common shocks, it is possible to do a similar reasoning. Path by path, the approximating strategy consists in decomposing the path into a finite number (almost surely) of smaller paths on a partition of $[0,T]$ given by the instants of change of the environment, and then doing the same as previously. We must just be careful and respect the relations

$$c_k^i = a_k^i d_k.$$

We thus obtain the almost sure uniform convergence (see also Dembo et al. (2004) for large deviations in a Markovian environment, and Dembo et Zajic (1995)).

As all hitting times and sojourn times before T in the approximating model converge almost surely to the corresponding random variables in the continuous model, the approximated multi-dimensional process taking into account impact of the position of the other lines of business on the premium income rate converges also almost surely to the continuous corresponding process. The sojourn times and hitting times until the time to ruin of line of business 1 thus converge almost surely. Besides, we know that in the most favorable case (other lines of business always at their top level), $\mathbb{E}\tau < +\infty$ from the standard model. Thus, by the dominated convergence theorem, the mathematical expectations of the sojourn times and of hitting times until time to ruin of line of business 1 in the approximate model converge to the corresponding times in the continuous model, which ends the proof..

3.3 Extensions, other ideas of applications

We may also introduce a quasi-default state for each secondary line of business, modelled by a state with a very high exit time (different from $+\infty$ to ensure existence and uniqueness of the stationary distribution), and with a severe jump distribution for the main process at the arrival time to this state.
The $c_k^i(t)$ might also be general increasing functions of the $X_k(t)$.
The jump marginals for lines of business $k \geq 2$ do not have to be exponentially distributed.
Hypothesis (H2) may be relaxed to (H1).

Interest of the problem with the ratings model in itself

Consider a company which has the opportunity to buy another one, it may be easier to obtain data about that company to model its surplus evolution with a rating-like model than with a compound Poisson process. Besides, it could be easier to model the impact of the subcompanies on the risk premium by unit of time received by the company only through their signatures (from AAA to D).

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Conclusion

Nous avons proposé des méthodes fondées notamment sur des techniques de martingales et sur un théorème de différentiation de fonctionnelles que nous avons établi pour calculer des probabilités de ruine et pour évaluer d'autres mesures de risque. La prise en compte du comportement conjoint des branches d'activité d'une compagnie d'assurance nous a amené à définir de nouveaux concepts dans le modèle multidimensionnel. Ce cadre multidimensionnel fait apparaître de nouveaux problèmes, comme l'allocation de la réserve initiale ou la modélisation des interactions entre les branches, qui peuvent fournir de nombreuses pistes de recherche pour un travail futur. Il serait en particulier intéressant d'obtenir des résultats sur l'évolution du coupleur entre les montants agrégés de sinistres au temps t des différentes branches.

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Contribution à l'étude de processus univariés et multivariés de la théorie de la ruine

Résumé

La théorie de la ruine définit et étudie des processus stochastiques introduits dans la modélisation de l'évolution de la richesse d'une compagnie d'assurances. L'objet de cette thèse est d'approfondir certains aspects mathématiques récemment développés dans ce domaine et de proposer quelques nouveaux concepts. Des méthodes de calcul explicite ou numérique des probabilités de franchissement d'une ou de plusieurs barrières sont proposées. Grâce à un résultat sur la différentiation de fonctionnelles de processus de risque, certains problèmes d'allocation optimale de réserve initiale sont résolus en modélisant l'évolution conjointe des richesses de plusieurs branches dactivités par un processus de risque multidimensionnel. Dans ce cadre multivarié, des concepts et modèles tenant compte de l'évolution conjointe des processus de risque sont étudiés. Dans un environnement markovien commun à toutes les branches, des méthodes de martingales sont utilisées pour évaluer des mesures de risque.

Mots-clés: processus multidimensionnel, temps d'atteinte, théorie de la ruine, environnement markovien.

Contribution to the study of unidimensional and multidimensional processes in ruin theory

Abstract

Ruin theory defines and studies stochastic processes involved in the modelling of the wealth evolution of an insurance company. The thesis aims at deepening some mathematical aspects recently developed in this field, and to propose some new concepts. Methods for explicit or numerical computation of hitting probabilities are proposed. A differentiation theorem for functionals of risk processes allows us to solve some optimal allocation problems for the initial reserve level, by modelling the joint evolution of the surpluses of several lines of business with a multidimensional risk process. In this framework, concepts and models taking into account the joint evolution of risk processes are studied. In a Markovian environment, common to all lines of business, some martingale methods are used to evaluate risk measures.

Keywords: multidimensional risk process, hitting times, ruin theory, Markovian environment.

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