Ambiguous Life Expectancy and the Demand for Annuities *

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Abstract: In this paper, ambiguity aversion to uncertain survival probabilities is introduced in a life-cycle model with a bequest motive to study the optimal demand for annuities. Provided that annuities return is sufficiently large, and notably when it is fair, positive annuitization is optimal in the ambiguity neutrality limit case. Conversely, the optimal strategy is to sell annuities in case of infinite ambiguity aversion. Then, in a model with smooth ambiguity preferences, there exists a finite degree of ambiguity aversion above which the demand for annuities is non positive. To conclude, ambiguity aversion appears as a relevant candidate for explaining the annuity puzzle.

Keywords: Demand for Annuities; Uncertain Survival Probabilities; Ambiguity Aversion.

JEL codes: D11, D81, G11, G22.

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1 Introduction

According to the life-cycle model of consumption with uncertain lifetime proposed by Yaari [41], full annuitization should be the optimal strategy followed by a rational individual without altruistic motives provided that annuities are available. Since this theoretical prediction does not meet the facts, the initial framework has hence been extended and many explanations of the so-called annuity market participation puzzle have been proposed (see Brown [4]). However, as shown by Davidoff et al. [7], positive annuitization still remains optimal under very general specifications and assumptions, including intergenerational altruism. According to the authors, the literature to date has failed to identify a sufficiently general explanation for consumers' aversion to annuities, suggesting that psychological or behavioral biases might be rather important in decisions involving uncertain longevity[1].

In this paper, we consider the possibility that a fully rational individual displaying some aversion toward ambiguity survival probabilities may be likely to exhibit a low demand for annuities. Indeed, in a life-cycle model with a bequest motive, we show that it may be optimal to sell annuities short if maxmin utility preferences are assumed. We note that the selling of annuities is, in our simple framework, equivalent to the purchase of pure life insurance policies (see Yaari [41] and Bernheim [1]). We shall therefore not make any distinction between the two financial products and, for the sake of simplicity, only deal with the demand for annuities. Before detailing our results, let us first discuss the two main assumptions of the paper: the uncertainty on survival probabilities and the aversion toward this ambiguity.

Despite all the available information displayed in Life Tables, we think that survival probabilities are nevertheless ambiguous for individuals. First, there is a rather strong heterogeneity in the age at death. According to Edwards and Tuljapurkar [8], past

[1] Most notably, Brown et al. [6] use the framing hypothesis and show that in an investment frame, individuals prefer non-annuitized products.

age 10, its standard deviation is around 15 years in the US. After having controlled for sex and race differentials and for various socioeconomic statuses, they found a residual heterogeneity that remains significant. Heterogeneity is notably explained by biological differences that are not necessarily known to the individual. Second, the large increase in life expectancy experienced by populations over the last two centuries was characterized by changes in the distribution of survival probabilities at each age. This is referred to as the epidemiological transition and features an increase in the dispersion of heterogeneity computed in the later years. Moreover, opposite factors such as medical progress versus the emergence of new epidemic diseases cause some uncertainty in the dynamics of the distribution per age. Based on Life Tables, an individual belonging to a given cohort may, at best, only know the true distributions of past cohorts but remains uncertain about his/her own. Finally, due to the small number of observations, data concerning the much later years are not reliable and there is not even a consensus among demographers about the mean survival rate (see especially Oeppen and Vaupel [33]).

The assumption of an aversion toward the ambiguity of survival probabilities is also supported by a great deal of evidence. Initial intuitions concern health risks and can be found in the study by Keynes [23], in which he considers patients who must decide between two medical treatments. Keynes argues that most individuals would choose a treatment that has been extensively used in the past and has a well-known probability of success, rather than a new one, for which there is no information about its probability of success[2]. More recently, the celebrated Ellsberg [11] experiment has also been applied to health and longevity risks in many studies using hypothetical scenarios. Among them, Viscusi et al. [39] show that individuals have a significant aversion to ambiguous information about the risk of lymphatic cancer. More recently, real case studies have confirmed that individuals are ambiguity averse. Riddel and Show [36] have found a negative relationship between the perceived uncertainty about

[2] This idea was formalized and developed by Manski [29].

the risks associated with nuclear waste transportation and the willingness to be exposed to such risks. Similarly, scientific disagreement about the efficiency of vaccination (see Meszaros et al. [31]) or screening mammography recommendations (see Han et al. [19], [18]) has been found to be negatively correlated with the perception of disease preventability and the decisions of preventive behaviors. Concerning portfolio and life-cycle decisions, Post and Hanewald [35] have shown that individuals are aware of longevity risk and that this awareness affects their savings decisions, while Huang et al. [22] study a life-cycle model with stochastic survival probability.

We consider a life-cycle model with consumption and bequest similar to Davidoff et al. [7]. We do not focus on market imperfections but instead assume, as in Yaari [41], some warm-glow altruism. Remark that a bequest motive is necessary to obtain some partial annuitization but it does not eliminate the advantage of annuities since they return more, in case of survival, than regular bonds. Our main departure from Yaari [41] and Davidoff et al. [7]'s works hinges on the assumption of ambiguity aversion toward uncertain survival probabilities. We apply the recent model for ambiguity aversion proposed by Klibanoff et al. [24] to a decision problem in a state-dependent utility framework yield by uncertain lifetimes[3]. This allows us to study the optimal demand for annuities in the expected utility case (hereafter EU case), the maxmin expected utility case (hereafter Mm case) of Wald [40] or Gilboa and Schmeidler [14] and in a continuum of intermediary cases for which the ambiguity aversion is finite. Klibanoff et al. [24]'s representation functional is an expectation of an expectation. The inner expectation evaluate the expected utilities corresponding to possible first order probabilities while the outer expectation aggregates a transform of these expected utilities with respect to a second order prior[4]. Applications of non-expected utility models to health and longevity uncertainties have been scarce[5]. Interestingly, Ponzetto

[3]See Nau [32] for an axiomatic model of ambiguity aversion and state dependent utilities.

[4]Interestingly, Gollier [15] has applied this framework to a standard portfolio problem and shown that ambiguity aversion does not necessarily reinforce risk aversion.

[5]Among exceptions, Eeckhoudt and Jevela [10] study medical decisions and Treich [37] the value

[34] and Horneff et al. [21] apply Epstein and Zin [12]'s recursive utility framework to uncertain longevity. They show that the utility value of annuitization is decreasing in both risk aversion and elasticity of intertemporal substitution. Positive annuitization nevertheless remains optimal. Moreover, Groneck et al [17] apply the Bayesian learning model under uncertain survival developed by Ludwig and Zimper [28] to a life-cycle model without annuity markets. The dynamics of savings over the life-cycle is closer to the empirical observations. Similarly, Holler et al. [20] study a life-cycle model extended to incorporate an aversion to uncertain lifespans, as developed by Bommier [2].

We obtain the following results. Provided that annuities return is sufficiently larger than bonds return, and notably when it is fair, the optimal share of annuities in the portfolio is positive if the individual is ambiguity neutral. The individual indeed maximizes a standard expected utility computed with a subjective mean survival probability. Conversely, if ambiguity aversion is infinite, the optimal strategy is to sell annuities. In that case, the maxmin expected utility is not the expected utility computed with the lowest survival probability considered by the individual since some utility is derived from the bequest. The maxmin optimal behavior aims at equalizing the utilities in the two states of nature, namely being alive or not, which is obtained for a negative demand for annuities. Then, as the index of ambiguity aversion of Klibanoff et al. [24] increases, the optimal demand for annuities decreases and there exists a finite threshold above which the demand is non positive. Theses results are obtained under general specifications of both utility functions and survival probability distributions[6]. The aversion to ambiguous survival probabilities hence appears as a good candidate to explain the observed aversion to annuities. A numerical application of our model suggests that the impact of ambiguity aversion is likely to be quantitatively large.

Section 2 proposes a model of consumption and bequest with uncertain lifetimes

of a statistical life.

[6]Moreover, our results could also be derived within the framework proposed by Gajdos et al. [13].

and ambiguity aversion. It studies two benchmark cases: ambiguity neutrality and maxmin expected utility. Section 3 introduces the Klibanoff et al. [24] framework to analyze the impact of ambiguity aversion on optimal choices. Proofs are gathered in the Appendix.

2 The benchmark cases

This section presents the model and studies the optimal demand for annuities in two polar cases: expected utility and maxmin expected utility.

2.1 The basic framework

We consider a static model of consumption and bequest under uncertain lifetime similar to Yaari [41] and Davidoff et al. [7]. The length of life is at most two periods with the second one being uncertain. The Decision Maker (DM) derives utility from a bequest that might happen at the end of periods 1 and 2 and, if alive, from consumption in period 2. At the first period, the DM is endowed with an initial positive income w that can be shared between bonds and annuities. Bonds return $R > 0$ units of consumption in period 2, whether the DM is alive or not, in exchange for each unit of the initial endowment. Conversely, annuities return $R_a \geq R$ in period 2 if the DM is alive and nothing if she is not alive. Due to the possibility of dying, the demand for bonds should be non negative and therefore annuities are the only way to borrow. The selling of annuities is here equivalent to the purchasing of pure life insurance policies (see Yaari [41] and Bernheim [1]). In the remaining of the paper, we will consider the selling of annuities as a zero annuitization strategy. If alive during the second period, the DM may allocate her financial wealth between consumption and bequest. Since death is certain at the end of period 2, the latter is exclusively a demand for bonds.

Denote by a, the demand for annuities and by $w-a$, the demand for bonds, decided in period 1. Moreover let c and x denote the consumption and the bequest decided in period 2. The budget constraint writes:

$$c = c(a, x) = R_a a + R(w - a) - x, \tag{1}$$

and the non negativity constraints are:

$$c \geq 0,\ x \geq 0,\ a \leq w. \tag{2}$$

Following Davidoff et al. [7], we assume that whatever the length of the DM's life, bequests are received in period 3, involving additional interest: the bequest is therefore xR, if the DM is alive in period 2, while it is $(w - a)R^2$, if she is not.

We assume that the DM's utility is $u[c, xR]$ if alive in period 2 and $v[(w - a)R^2]$ if she is not alive. Functions u and v are supposed to satisfy the following assumptions:

Assumption 1. *The (twice continuously differentiable) function* $u : \mathbb{R}_+^2 \to \mathbb{R}_+$ *is strictly concave*[7] *and satisfies* $u_1'[c, \varrho] > 0$, $u_1'[c, \varrho] > 0$, $\lim\limits_{c \to 0} u_2'[c, \varrho] = +\infty$ *and* $\lim\limits_{\varrho \to 0} u_2'[c, \varrho] = +\infty$. *The (twice continuously differentiable) function* $v : \mathbb{R}_+ \to \mathbb{R}_+$ *is increasing, strictly concave and satisfies* $\lim\limits_{\varrho \to 0} v'[\varrho] = +\infty$. *Goods c and x are normal*[8].

Assuming infinite marginal utilities when consumption or bequest go to zero ensures that inequalities lying in (2) are strict at the optimum. This, of course, has no impact on the sign of the optimal demand for annuities.

Assumption 2. $u[0, 0] = 0$ *and, for all non negative c and ϱ,* $u[c, \varrho] \geq v[\varrho]$ *and* $u_2'[c, \varrho] \geq v'[\varrho]$.

Both the utility and the marginal utility of a given bequest is larger if the DM is alive than if she is not. These assumptions are commonly used in the literature on the economic valuation of risks to health and life (see e.g. Viscusi and Aldy [38]). They are obviously satisfied when the utility if alive is separable, i.e. for $u[c, xR] = h[c] + v[xR]$, as in Yaari [41] and Davidoff et al. [7].

[7] The function u is strictly concave if and only if the Hessian of u is negative definite, or equivalently, if and only if $u''_{11} < 0$ and $u''_{11}u''_{22} - {u''_{12}}^2 > 0$ (see Theorems M.C.2 and M.D.2 in Mas-Colell et al [30]).

[8] Assuming the normality of c and x means assuming $u''_{12} - Ru''_{22} > 0$ and $Ru''_{12} - u''_{11} > 0$ (see Appendix A).

2.2 The expected utility (EU) case

The first benchmark case we consider is the traditional expected utility (EU) model used intensively in the literature since Yaari [41]'s seminal work. Let us suppose that the survival probability, denoted $p \in (0,1)$, is known by the DM[9]. Given p, the DM maximizes his expected utility i.e. solves the EU problem, denoted $(\mathcal{P}_{EU_p})$:

$$\max_{a,x} \mathcal{U}(a,x,p)$$
$$\text{s.t.} \begin{cases} \mathcal{U}(a,x,p) = pu[c(a,x), xR] + (1-p)v[(w-a)R^2], \\ c = c(a,x) = R_a a + R(w-a) - x. \end{cases} \tag{$\mathcal{P}_{EU_p}$}$$

The solution of $(\mathcal{P}_{EU_p})$, denoted $(\bar{a}, \bar{x})$, satisfies the FOCs:

$$p(R_a - R)u_1'[c(\bar{a},\bar{x}), xR] - (1-p)R^2 v'[(w-\bar{a})R^2] = 0, \tag{3}$$

$$-u_1'[c(\bar{a},\bar{x}), \bar{x}R] + Ru_2'[c(\bar{a},\bar{x}), \bar{x}R] = 0. \tag{4}$$

We remark that the survival probability affects the optimal demand for annuities but not, as shown by equation (4), the optimal allocation of the financial wealth between consumption and bequest. It will be useful to use the condition (4) to define the application $\bar{x} = f(\bar{a})$, which satisfies $0 \leq f'(a) \leq R_a - R$. Hence, at the optimum, if the DM survives, her consumption $\widehat{c}(a) = c(a, f(a))$ and her bequest $f(a)$ increase with the demand for annuities. Our first result is then given in the following proposition:

Proposition 1 .

The optimal demand for annuities, denoted $\bar{a}$, that solves the problem $(\mathcal{P}_{EU_p})$:

(i) is lower than w but larger than $-Rw/(R_a - R)$.

(ii) is positive if and only if R_a is larger than a threshold $\widehat{R}_a \in (R, R/p)$.

(iii) is positive if and only if p is larger than a threshold $\widehat{p} \in (0,1)$.

Proof – See Appendix A. □

[9] Equivalently, we may think p as a survival probability subjectively evaluated by the DM and thus the problem is the one of a subjective expected utility maximizer.

The optimal demand for annuities can be positive or negative and the role of the annuities return is crucial in that matter. This can be shown by merging equations (3) and (4) to have:

$$(R_a - R)pu_2'[\widehat{c}(\bar{a}), \bar{x}R] - R(1-p)v'[(w-\bar{a})R^2] = 0. \tag{5}$$

Suppose first that annuities and bonds returns are equal i.e. $R_a = R$. Consumption is then independent of a and it is immediate from equation (5) to conclude that the optimal behavior is to sell an infinite quantity of annuities to finance the purchase of an infinite quantity of bonds ($-\bar{a} = \bar{x} \to +\infty$). Conversely, for an actuarially fair return of annuity, such that $R_a = R/p$, equation (5) rewrites:

$$u_2'[\widehat{c}(\bar{a}), \bar{x}R] - v'[(w-\bar{a})R^2] = 0.$$

Using the fact that the marginal utility of a given bequest is larger in case of survival (Assumption 2), one concludes that $\bar{x} \geq (w - \bar{a})\,R$ and thus, with the budget constraint, that $\bar{c} = \widehat{c}(\bar{a}) \leq R_a\bar{a}$. As a consequence, and as previously pointed out by Davidoff et al. [7], the optimal demand for annuities is significantly positive.

Low return of annuities, justified by some market imperfections, can thus serve as an argument to explain the low observed annuitization. If the return is small enough, we find that the demand can be negative. This argument has notably been investigated quantitatively by Lockwood [27]. In what follow, we will consider the possibility of having a non positive demand for annuities for large R_a, and notably in the case of an actuarially fair pricing.

2.3 The maxmin (Mm) expected utility case

The second benchmark case we consider is the maxmin (Mm) expected utility framework, notably developed by Gilboa and Schmeidler [14]. We suppose that the DM has no idea about her survival probability and that she maximizes her utility in the worst possible state of nature. There are two possible states: being alive or not during the second period. As the bequest yields some utility, the worst state is not a priori given. Hence, the DM solves the Mm problem, denoted $(\mathcal{P}_{Mm})$:

$$\max_{a,x} \min\{u[c(a,x), xR], v[(w-a)R^2]\}$$
$$\text{s.t.} \quad c = c(a,x) = R_a a + R(w-a) - x. \tag{$\mathcal{P}_{Mm}$}$$

The solution $(\underline{a}, \underline{x})$ of $(\mathcal{P}_{Mm})$ is such that $\underline{x} = f(\underline{a})$ where $f(a)$ is derived from (4). Since the utility if alive increases with the holding of annuities while utility if not decreases with it (see Appendix B), the optimal behavior is to equalize the utilities in the two states of nature. Formally, the optimal demand $\underline{a}$ solves:

$$\xi(\underline{a}) = u[\widehat{c}(\underline{a}), f(\underline{a})R] - v[(w-\underline{a})R^2] = 0. \tag{6}$$

Then, we establish that:

Proposition 2 .

The optimal demand for annuities $\underline{a}$ that solves the problem $(\mathcal{P}_{Mm})$ is negative.

Proof – See Appendix B. □

Equalizing the utility if alive with the utility if not alive requires the purchase of pure life insurance contracts. Whatever the returns, the endowment and the utility functions, the optimal portfolio hence features a zero annuitization strategy. To understand this result, let us suppose that $a = 0$. The utility if alive is then $\max_x u(Rw - x, R^2 x)$ while the utility if not alive is $v(R^2 w)$. By the definition of a maximum and by the fact that, for a given bequest, the utility is always larger if the individual survives (see Assumption 2) we obtain that $\max_x u(Rw - x, R^2 x) > v(R^2 w)$ which is not optimal. To equalize the utilities in the two states of nature, one should then reduce the utility if alive and increase the utility if not, which is done by selling annuities short.

Behaving following a maxmin rule is sometime referred as extreme pessimism. Pessimism on survival chances would indeed be a natural candidate for a low demand for annuities. This is theoretically (see Eeckhoudt and Gollier [9]) and empirically (see Brown et al. [5]) relevant. It is however not clear in our problem that surviving is the best state of nature. Assumption 2 only says that surviving yields more utility provided that the bequest is the same in the two states, which is not the result of an optimal decision. Rather than focusing on pessimism, we are going to argue in the next section that ambiguity aversion explains a significant part of the low annuitization. Before that, let us compare the optimal behaviors characterized in Propositions 1 and 2.

Proposition 3 .

The optimal demand for annuities $\underline{a}$*, that solves the problem* $(\mathcal{P}_{Mm})$*, is lower than (resp: larger)* $\bar{a}$*, the one which solves the problem* $(\mathcal{P}_{EU_p})$*, if* R_a *is sufficiently large (resp: low).*

At the optimum, the difference between the utility if alive and the utility if not:

(i) equals zero in the maxmin expected utility case.

(ii) is positive if and only if p *is larger than a threshold* $\check{p} \in (0, \widehat{p})$*.*

(iii) is positive (resp: negative) if R_a *is sufficiently large (resp: low) in the expected utility case.*

Proof – See Appendix C. □

The first part of Proposition 3 compares the optimal demand for annuities in our EU and Mm benchmark cases. Obviously, if the annuities return is sufficiently large, we have seen that the demand is positive in the EU case and therefore larger than in the Mm case. However, for a low enough annuities return, the demand in the EU case is lower. This is not so surprising because, in the Mm case, the infinite selling of annuities is never an optimal strategy as it implies a utility if not alive larger than if alive. When the annuities return is close to bonds return, the demand for annuities is hence lower in the EU case than in the Mm case.

The second part of Proposition 3 compares the utility derived at the optimum in the two states of nature. It has been shown that they are equal in the Mm case. In the EU case, comparing the utilities is equivalent to comparing the optimal demand for annuities with the one derived in the Mm case. Indeed, the utility if alive increases with the demand for annuities while the utility if not decreases with it.

Let us now illustrate these results with a numerical application. The calibration of the model has been done to reproduce a proportion of the initial endowment invested in annuities of about 70% in the EU case with fair annuities return. We have thus chosen the following functional forms: $u[c, xR] = c^{0.7} + (xR)^{0.3}$ and $v[(w-a)R^2] = [(w-a)R^2]^{0.3}$ and the following parameters: $w = 1$, $p = 0.68$, $R = 2$. The fair annuities return is thus 2.9412.

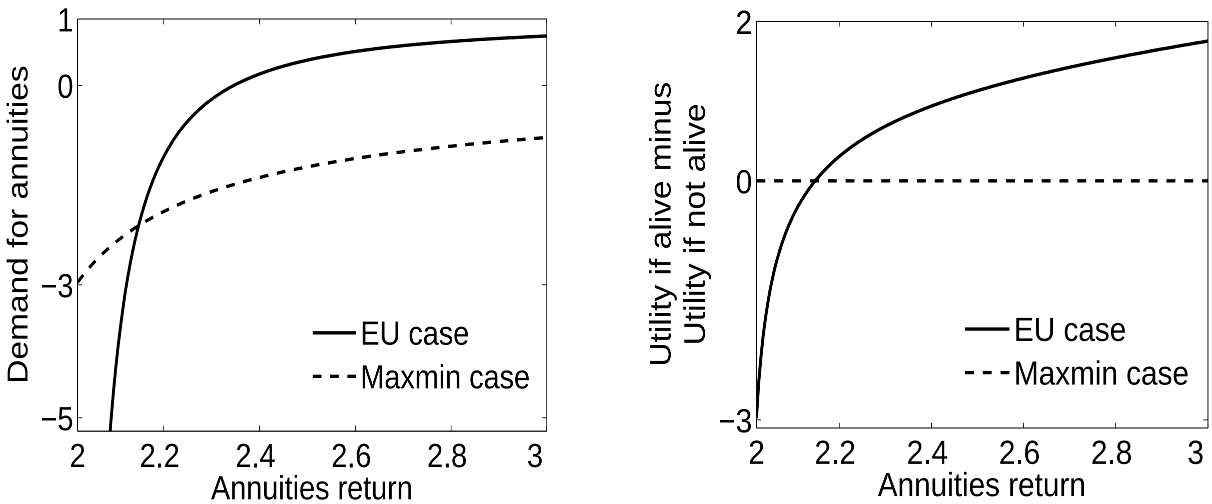

Figure 1: EU case versus Mm case according to annuities return R_a

The LHS of Figure 1 plots the optimal demand for annuities as a function of its return in the empirically relevant interval $[R, R/p]$ in the EU case (solid line) and in the Mm case (dashed line). We observe a threshold for the annuities return, which is here equal to about 2.148, such that $\bar{a} \geq \underline{a}$ if and only if the annuities return is larger than the threshold. Moreover, if the return is larger than about 2.347, the demand for annuities is positive in the EU case while it remains negative in the Mm case. The RHS of Figure 1 plots the difference between the utility if alive and the utility if not alive computed at the optimum as a function of the annuities return. As previously shown, the difference is always equal to zero in the Mm case while it monotonically increases with the demand for annuities in the EU case.

This maxmin framework can be considered as an extreme case when dealing with the demand for annuities. There is indeed some information on the survival probabilities which is notably included in the annuity premium. As discussed in the introduction, this information is however not perfect and the survival probabilities are uncertain. In the next section, we propose a general model with ambiguous survival probabilities.

3 Annuitization and ambiguity aversion

This section applies the recent rationale for ambiguity aversion proposed by Klibanoff et al. [24] to our life-cycle problem with uncertain lifetimes. This framework embodies the two benchmark cases studied in the previous section as special cases.

We assume that survival probabilities are uncertain. Moreover, the DM does not know her own probability distribution but only knows the set of possible distributions. There is a given number of states of nature corresponding to different survival probabilities, and that may be interpreted as health types, to which the DM subjectively associates a probability to be in. Ambiguity is hence modeled as a second order probability over first order probability distributions. Let us denote the random (continuous or discrete) survival probability by $\widetilde{p}$ whose support is denoted $\text{Supp}(\widetilde{p})$, and the survival expectancy as it is evaluated by the DM by $E(\widetilde{p}) = p$. It is supposed, since it does not affect the main results, that p corresponds to the mean survival probability.

The expected utility, denoted $\mathcal{U}(a, x, \widetilde{p})$, is thus also a random variable that writes:

$$\mathcal{U}(a, x, \widetilde{p}) = \widetilde{p}u[c(a, x), xR] + (1 - \widetilde{p})v[(w - a)R^2], \tag{7}$$

Following Klibanoff et al. [24], the DM has smooth ambiguity preferences. The aversion to ambiguity is introduced using a function ϕ whose concavity represents an index of this aversion. The utility function of the DM is then given by an expectation of an expectation. The inner expectations evaluate the expected utilities corresponding to possible first order probabilities while the outer expectation aggregates a transform of these expected utilities with respect to the second order prior. The utility function writes:

$$\phi^{-1}(E(\phi(\mathcal{U}(a,x,\widetilde{p})))), \tag{8}$$

where ϕ satisfies the following restrictions:

Assumption 3. *The (twice continuously differentiable) function* $\phi : \mathbb{R}_+ \to \mathbb{R}$ *is increasing and concave.*

Limit cases of (8) are interesting. First, when ϕ is linear, the DM is ambiguity neutral, and using (7), the utility function (8) rewrites:

$$\phi^{-1}(E(\phi(\mathcal{U}(a,x,\widetilde{p}))))|_{\phi''=0} = \mathcal{U}(a,x,p), \tag{9}$$

which is linear with respect to the mean survival probability, p. Equation (9) corresponds to the EU case studied subsection 2.2. Alternatively, under some conditions[10] that will be assumed to be satisfied, the maxmin expected utility is obtained when the absolute ambiguity aversion, given by $-\phi''/\phi'$, is infinite. The DM then maximizes the expected utility for the worst possible realization of $\widetilde{p}$. Formally:

$$\lim_{-\frac{\phi''(.)}{\phi'(.)}\to+\infty} \phi^{-1}(E(\phi(\mathcal{U}(a,x,\widetilde{p})))) = \max_{a,x} \min_{p_\varepsilon \in \mathrm{Supp}(\widetilde{p})} \mathcal{U}(a,x,p_\varepsilon). \tag{10}$$

The comparison of this limit case with the benchmark case studied in 2.3 will be analyzed latter.

The ambiguity averse DM faces the following problem, denoted $(\mathcal{P}_\phi)$:

$$\max_{a,x} \phi^{-1}(E(\phi(\mathcal{U}(a,x,\widetilde{p}))))$$
$$\text{s.t.} \begin{cases} \mathcal{U}(a,x,\widetilde{p}) = \widetilde{p}u[c(a,x),xR] + (1-\widetilde{p})v[(w-a)R^2], \\ c(a,x) = R_a a + R(w-a) - x. \end{cases} \tag{$\mathcal{P}_\phi$}$$

Then, the solution $(a^\star, x^\star)$ of $(\mathcal{P}_\phi)$ satisfies the FOCs:

$$E(\phi'(\mathcal{U}(a^\star,x^\star,\widetilde{p}))[\widetilde{p}(R_a-R)u_1'[c(a^\star,x^\star),x^\star R] - (1-\widetilde{p})R^2 v'[(w-a^\star)R^2]]) = 0, \tag{11}$$

[10] See Proposition 3 page 1867 in Klibanoff et al. [24].

$$-u_1'[c(a^\star, x^\star), x^\star R] + R u_2'[c(a^\star, x^\star), x^\star R] = 0. \quad (12)$$

Equation (12) is equivalent to (4). Since the trade-off between consumption and bequest if alive is not affected by the survival probability, it is also independent from ambiguity aversion. As previously, we define $x^\star = f(a^\star)$. From (11), it is clear that ambiguity aversion nevertheless affects the demand for annuities. We are going to consider two distinct DM distinguished only by their ambiguity attitude. Ambiguity aversion of the first one is characterized by the function ϕ, while the one of the second by an increasing and concave transformation of ϕ; the second DM being thus said more averse to ambiguity[11].

Denoting by p_m the lower bound of Supp$(\widetilde{p})$ and $E(\widetilde{p}) = p$, we consider some restrictions on the probability distribution of the survival probability $\widetilde{p}$.

Assumption 4. *$p_m < \widehat{p} < p$, where $\widehat{p}$ is defined Proposition 1.*

Assumption 4 implicitly implies that the annuity return is sufficiently strong. In particular, a fair price for annuities is included. Our results can be summarized as follows:

Proposition 4.

The optimal demand for annuities $a^\star$ that solves the problem $(\mathcal{P}_\phi)$:

(i) is lower than $\bar{a}$ but larger than $\underline{a}$.

(ii) decreases with ambiguity aversion.

(iii) is always negative when ambiguity aversion is infinite.

Proof – See Appendix D. □

Ambiguity aversion determines the optimal degree of exposure to the uncertainty. A more ambiguity adverse DM will hence choose to smooth the expected utilities computes to each realization of $\widetilde{p}$. According to Proposition 4, this is done by reducing the demand for annuities. Using $\check{p}$, the threshold defined Proposition 3, we then have:

[11]See Theorem 2 page 1865 in Klibanoff et al. [24].

Corollary 1 .

When ambiguity aversion is infinite the optimal demand for annuities is:

(i) $\underline{a}$ *if and only if* $0 \leq p_m < \check{p}$ *where* $\underline{a}$ *solves the problem* $(\mathcal{P}_{Mm})$.

(ii) $\check{a}$ *if and only if* $\check{p} < p_m$ *where* $\check{a}$ *solves the problem* $(\mathcal{P}_{EU_{pm}})$.

Let us consider the impact of a marginal increase of the random variable $\widetilde{p}$ on the expected utility evaluated at the optimal point $(a^\star, x^\star)$. Using (7), we have:

$$\frac{\partial \mathcal{U}(a^\star, x^\star, \widetilde{p})}{\partial \widetilde{p}} = u[c(a^\star, x^\star), x^\star R] - v[(w - a^\star)R^2]. \tag{13}$$

Since the expected utility is linear with respect to $\widetilde{p}$, the marginal increase is equal to the difference between the utility if alive in the second period, i.e. $u[c(a^\star, x^\star), x^\star R]$, and the utility if not alive, i.e. $v[(w - a^\star)R^2]$. As discussed previously, the bequest yielding some utility, the sign of this difference is not immediate. We first show that Assumption 4 ensures its positiveness. Indeed, for sufficiently large values of R_a, the difference is positive in the EU case (see Proposition 3). Conversely when ambiguity aversion is infinite, the optimal behavior is to eliminate the exposure to the uncertainty. Given equation (13), this is done by equalizing the utility if alive and the utility if not. The limit case of problem $(\mathcal{P}_\phi)$ is thus equivalent to the benchmark case studied in 2.3. For any finite degree of ambiguity aversion, the difference between the utility if alive and the utility if not is non negative. Consequently, the demand for annuities lies in $[\underline{a}, \widehat{a}]$, where $\underline{a}$ is negative (see Proposition 2). We finally showed that increasing ambiguity aversion drives to reduce the difference in utilities. This is obtained by purchasing fewer annuities and more bonds. As a consequence, consumption is also reduced. By a continuity argument, we derive the following corollary:

Corollary 2 .

There exists a finite degree of ambiguity aversion such that the zero annuitization strategy is optimal if and only if the DM's ambiguity aversion is larger than this threshold.

To our knowledge, previous articles in the rich literature on annuities have not been able to show that a zero annuitization strategy can be optimal when annuities are fairly priced. A large bequest motive or a large risk aversion can reduce the demand for annuities by not yield to a selling. Of course, allowing for a low annuities return, precisely for $R_a < \widehat{R}_a$, reinforces our results: the demand for annuities is then negative whatever the degree of ambiguity aversion.

To illustrated numerically our results we consider a simple case where there are only two survival distributions which represent two types of respectively "good" and "bad" health. With subjective probability $q \in [0,1]$, the DM thinks she is of the "good" type for which the survival probability is $p_1 \in (0,1]$, while she is, with probability $1-q$, of the "bad" type for which the survival probability is $p_2 \in [0, p_1)$. We assume moreover that the absolute ambiguity aversion is constant and is denoted $\alpha \geq 0$. The functional $\phi(.)$ is then an exponential function which satisfies:

$$\phi(x) = \begin{cases} \dfrac{1-e^{-\alpha x}}{\alpha} & \text{if } \alpha > 0, \\ x & \text{if } \alpha = 0. \end{cases}$$

In this case, the objective function is:

$$\phi^{-1}(E(\phi(\mathcal{U}(a,x,\widetilde{p})))) = \\ -\ln\left\{(qe^{-\alpha(p_1-p_2)(u[c(a,x),xR]-v[(w-a)R^2])}+1-q\right\}/\alpha+\mathcal{U}(a,x,p_2).$$

Limit cases are derived using the Hôpital's rule. We obtain:

$$\lim_{\alpha \to 0} \phi^{-1}(E(\phi(\mathcal{U}(a,x,\widetilde{p})))) = \mathcal{U}(a,x,p_1 q + (1-q)p_2),$$

which is the EU case and where $p_1 q + (1-q)p_2$ is the subjective survival probability. Moreover[12]:

$$\lim_{\alpha \to +\infty} \phi^{-1}(E(\phi(\mathcal{U}(a,x,\widetilde{p})))) = \min\{\mathcal{U}(a,x,p_1), \mathcal{U}(a,x,p_2)\}. \tag{14}$$

The calibration of the model is similar to the one done in the previous section. We have thus chosen the following functional forms: $u[c,xR] = c^{0.7} + (xR)^{0.3}$ and $v[(w-a)R^2] = [(w-a)R^2]^{0.3}$, and the following parameters: $w = 1$, $p_1 = q = 0.8$,

[12] See Appendix E for details.

$p_2 = 0.2$, $R = 2$ and $R_a = 2.9412$ which correspond to an actuarially fair annuity system. Since there is no consensus in the literature about the value of the ambiguity aversion coefficient but that it is likely to be heterogeneous among individual (see Borghans et al. [3]), we plotted in Figure 2 the optimal demand for annuities and the difference between the utility if alive and the utility if not as functions of the absolute ambiguity aversion.

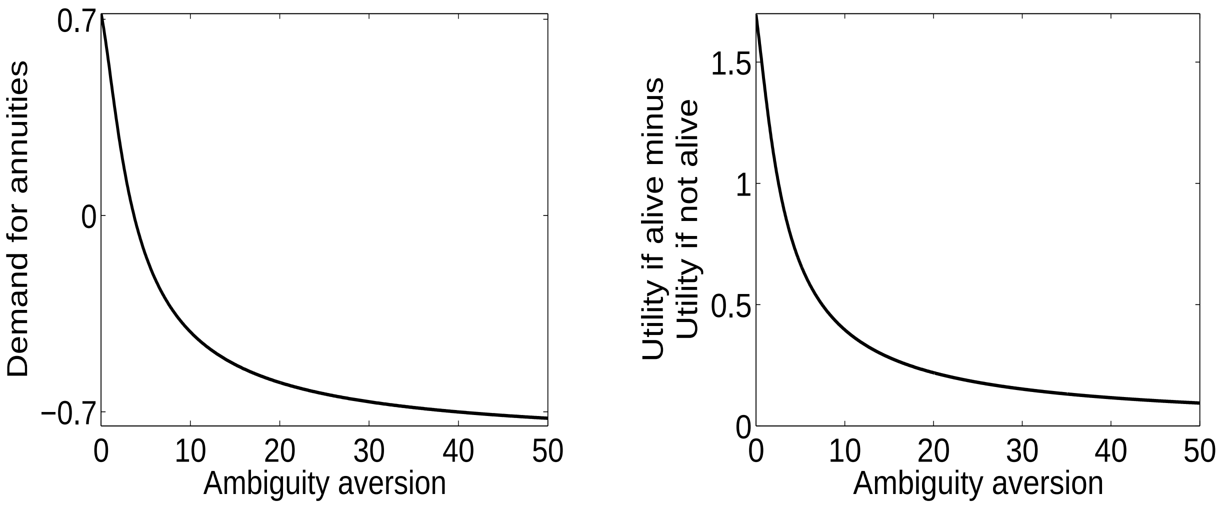

Figure 2: The impact of ambiguity aversion

The LHS shows that ambiguity aversion strongly affects the demand for annuities: the proportion of the initial endowment being invested in annuities goes from about 70% in the EU case to 0% in the case such that absolute ambiguity aversion is about 6. The RHS shows that the difference in utilities monotonically reduces with the aversion.

4 Conclusion

In this paper, we have studied the optimal demand for annuities in a life-cycle model with a bequest motive. We claimed that ambiguity aversion to uncertain survival probabilities can explain the observed low demand for annuities. Indeed, purchasing annuities means participating to a lottery, which is very specific since it is a bet on longevity. A more ambiguity averse individual will be more reluctant to participate to the lottery and thus will held less annuities in her portfolio. Moreover, we proved that for sufficiently large ambiguity aversion, the demand for annuities is non positive.

Importantly, this result notably holds for arbitrarily low bequest motives and for fair annuity return. We therefore think that aversion to uncertain survival probabilities can contribute to explain, with other factors such as bequest motives or unfair pricing of annuities, the observed low demand for annuities. A natural extension of our work would be considering a multi-period life-cycle model. The dynamic framework proposed by Kreps and Porteus [26] and Klibanoff et al. [25] would be useful for that purpose.

Appendix

Appendix A – Proof of Proposition 1.

As a preliminary, let us establish that $u''_{11} + R^2 u''_{22} - 2Ru''_{12} < 0$, $u''_{12} - Ru''_{22} > 0$ and $Ru''_{12} - u''_{11} > 0$.

First, as the Hessian of u is negative definiteness, it is straightforward that $u''_{11} + R^2 u''_{22} - 2Ru''_{12} < 0$. Second, consider the static program $\max_{c,x} u[c, xR]$ subject to the constraint $\Omega = c + x$ where Ω is the life-cycle income of an agent. This program is equivalent to $\max_c u[c, (\Omega - c)R]$ or $\max_x u[\Omega - x, xR]$. Then, using the FOC of these programs and the implicit function theorem, there exists two functions A and B such that $c = A(\Omega)$ and $x = B(\Omega)$. It is straightforward that $A'(\Omega)$ has the sign of $u''_{12} - Ru''_{22}$ while $B'(\Omega)$ has the one of $Ru''_{12} - u''_{11}$. By definition, c and x are normal goods if and only if $\partial c/\partial\Omega$ and $\partial x/\partial\Omega$ are positive. Then, Assumption 1 implies $u''_{12} - Ru''_{22} > 0$ and $Ru''_{12} - u''_{11} > 0$.

A.1 – Proof of statement (i).

Using (4), let us define $F(a, x) = -u'_1[c(a, x), xR] + Ru'_2[c(a, x), xR] = 0$. Under Assumption 1, $F'_1(a, x) = (R_a - R)(Ru''_{12} - u''_{11})$ is non negative whereas $F'_2(a, x) = u''_{11} + R^2 u''_{22} - 2Ru''_{12}$ is negative. Then, there exists a continuous and differentiable function f

such that $x = f(a)$ and, under Assumption 1, $0 \leq f'(a) = -F'_1(a,x)/F'_2(a,x) \leq R_a - R$. Replacing $x = f(a)$ and (4) in (3) allows us to define the solution $(\bar{a}, f(\bar{a}))$ as a pair satisfying:

$$G(a, R_a, p) = p(R_a - R)u'_2[\widehat{c}(a), f(a)R] - (1-p)Rv'[(w-a)R^2] = 0.$$

Then, an optimal demand for annuities $\bar{a}$ is a real root of $G(a, R_a, p) = (R_a - R)p\varphi(a) = R(1-p)\psi(a)$ where $\varphi(a) = u'_2[\widehat{c}(a), f(a)R]$ and $\psi(a) = v'[(w-a)R^2]$. After computations, $\varphi'(a) = -R(R_a - R)(u''^2_{12} - u''_{11}u''_{22})/F'_2(a,x)$ and $\psi'(a) = -R^2v''[(w-a)R^2]$. Then, under Assumption 1, φ is a decreasing function whereas ψ is an increasing one, implying $G'_1(a, R_a, p) < 0$. Moreover $\varphi(-Rw/(R_a - R)) = +\infty$, $0 < \varphi(w) < +\infty$, $0 < \psi(-Rw/(R_a - R)) < +\infty$ and $\psi(w) = +\infty$. Then, as $G'_1(a, R_a, p) < 0$, $G(-Rw/(R_a - R), R_a, p) > 0$ and $G(w, R_a, p) < 0$, there exists a unique optimal pair $(\bar{a}, \bar{x})$. This pair is such that $\bar{a} \in (-Rw/(R_a - R), w)$ and $0 < \bar{x} = f(\bar{a}) < w$.

A.2 – Proof of statement (ii).

Since $G(\bar{a}, R_a, p) = 0$ and $G'_1(a, R_a, p) < 0$, $\bar{a}$ is positive if and only if $G(0, R_a, p)$ is positive. Importantly $f(0)$ is independent of R_a because (4) is independent of R_a when $a = 0$. Consequently, $\varphi(0) = u'_2[Rw - f(0), f(0)R]$ and $\psi(0) = v'[wR^2]$ are independent of R_a and we have $G'_2(0, R_a, p) = p\varphi(0) > 0$. Note that $G(0, R/p, p) = R(1-p)(u'_2[Rw - f(0), f(0)R] - v'[wR^2])$. As $f(0) < wR$ and $u'_2[c, y] \geq v'[y]$, we have $G(0, R/p, p) > 0$. Since $G(0, R, p) < 0$ and $G'_2(0, R_a, p) > 0$, there exists a unique $\widehat{R}_a \in (R, R/p)$ such that $G(0, \widehat{R}_a, p) = 0$. Consequently, $\bar{a}$ is positive if and only if R_a is larger than $\widehat{R}_a$. $\square$

Appendix B – Proof of Proposition 2.

Let $\tau(a) = u[\widehat{c}(a), f(a)R]$ be the utility if alive. As $0 \leq f'(a) \leq R_a - R$ and $\tau'(a) = [(R_a - R) - f'(a)]u'_1[\widehat{c}(a), f(a)R] + Rf'(a)u'_2[\widehat{c}(a), f(a)R]$, the utility if alive increases with the demand for annuity while the utility if not, namely $v[(w-a)R^2]$, decreases with a. Hence, the optimal behavior in the Mm case is to equalize the utilities in the two states of nature. Formally, the optimal demand for annuities $\underline{a}$ solves $\xi(\underline{a}) = 0$ where:

$$\xi(a) = u[\widehat{c}(a), f(a)R] - v[(w - a)R^2]$$

increases with respect to a. Moreover one has under Assumption 2 $\xi(-wR/(R_a - R)) = u[0, 0] - v[R^2 R_a w/(R_a - R)] < 0$ and $\xi(0) = u[\widehat{c}(0), f(0)R] - v[wR^2] > 0$. As $\xi'(a) > 0$, there exists an unique optimal demand for annuities $\underline{a}$ which is negative. $\square$

Appendix C – Proof of Proposition 3.

Using Appendix A.2 it is straightforward to show that $\bar{a}$ is larger than $\underline{a}$ if and only if $G(\underline{a}, R_a, p) > 0$. Since for R_a sufficiently close to R, $G(\underline{a}, R_a, p)$ is negative, there exists a threshold $R_m > R$ such that, for all $R_a \in (R, R_m)$ we have $\bar{a} < \underline{a} < 0$. Conversely, using Proposition 1, when $R_a > \widehat{R}_a$ we have $\bar{a} > 0 > \underline{a}$. By continuity, there exists a threshold R_M satisfying $R/p > \widehat{R}_a > R_M \geq R_m > R$ such that for all $R_a > R_M$ we have $\bar{a} > \underline{a}$. According to Appendix B, $\xi(a)$ is an increasing function of a satisfying $\xi(\underline{a}) = 0$. Consequently $\xi(\bar{a}) > \xi(\underline{a}) = 0$ for all $R_a > R_M$ and $\xi(\bar{a}) < \xi(\underline{a}) = 0$ for all $R \leq R_a < R_m$. $\square$

Appendix D – Proof of Proposition 4.

We consider the problem $(\mathcal{P}_\phi)$ and denote by $(a^\star, x^\star)$ its solution which is the solution of (11) and (12).

D.1 – Proof of statement (i).

Let us define the random variable $\widehat{\mathcal{U}}(a, \widetilde{p}) = \mathcal{U}(a, f(a), \widetilde{p})$ where the application $x = f(a)$ is derived from (12). Then, $\widehat{\mathcal{U}}(a, \widetilde{p})$ represents the expected utility when the budget constraint (1) and the consumption-bequest optimal allocation (12) are satisfied. It is a function of the random variable $\widetilde{p}$. Moreover, let $\widehat{\mathcal{U}}(a, p_\varepsilon)$ denotes the expected utility associated to p_ε, a realization of $\widetilde{p}$. Then, the system of equations (11) and (12) rewrites as a single equation in a:

$$\eta_{\phi}(a) = E(\phi'(\widehat{\mathcal{U}}(a,\widetilde{p}))\{\widetilde{p}(R_a - R)u_1'[\widehat{c}(a), f(a)R] - (1-\widetilde{p})R^2 v'[(w-a)R^2]\}) = 0.$$

We are going to prove that $\eta_{\phi}(\underline{a}) > 0$, $\eta_{\phi}(\bar{a}) < 0$ and $\eta_{\phi}'(a) < 0$ for $a \in [\underline{a}, \bar{a}]$.

As $\widehat{\mathcal{U}}(\underline{a},\widetilde{p})$ is independent of $\widetilde{p}$, $\eta_{\phi}(\underline{a})$ has the sign of:

$$E(\widetilde{p}(R_a - R)u_1'[\widehat{c}(\underline{a}), f(\underline{a})R] - (1-\widetilde{p})R^2 v'[(w-\underline{a})R^2])$$

i.e., since $E\widetilde{p} = p$ and (12), the one of $G(\underline{a}, R_a, p)$. Under Assumption 4 and according to Propositions 1, 2 and 3, we have $G(\underline{a}, R_a, p) > 0$ and, consequently, $\eta_{\phi}(\underline{a}) > 0$. Since $G(\bar{a}, R_a, p) = 0$, using (12) we have $\eta_{\phi}(\bar{a}) = E(\phi'(\widehat{\mathcal{U}}(\bar{a},\widetilde{p}))RG(\bar{a}, R_a, \widetilde{p})) = Cov(\phi'(\widehat{\mathcal{U}}(\bar{a},\widetilde{p})), RG(\bar{a}, R_a, \widetilde{p})) < 0$. Since ϕ is concave and $G_1'(a, R_a, \widetilde{p}) < 0$, $\eta_{\phi}'(a) < 0$. Consequently, η_{ϕ} has a unique real root that shall be denoted $a^{\star}$ and that belongs to $(\underline{a}, \bar{a})$.

Importantly the derivative of function $\widehat{\mathcal{U}}(a, p_{\varepsilon})$ with respect to a realization p_{ε} is:

$$\widehat{\mathcal{U}}_2'(a, p_{\varepsilon}) = u[\widehat{c}(a), f(a)R] - v[(w-a)R^2] \equiv \xi(a).$$

Then, $\xi(a) = \widehat{\mathcal{U}}_2'(a, p_{\varepsilon})$ is independent of any realizations of $\widetilde{p}$. Since $\xi'(a) > 0$, $\xi(\underline{a}) = 0$ and $a^{\star} > \underline{a}$ we have $\xi(a^{\star}) \geq 0$.

D.2 – Proof of statement (ii).

The strategy is to consider two independent DM. The problem of the first DM is given by $(\mathcal{P}_{\phi})$ and her optimal demand for annuities is denoted $a^{\star}$. The second DM faces the similar problem $(\mathcal{P}_{\psi})$ with $\psi \equiv T \circ \phi$ and where T is an increasing and concave function. Her optimal demand, denoted $a^{\star\star}$, is solution of:

$$\eta_{T\circ\phi}(a) = E(\psi'(\widehat{\mathcal{U}}(a,\widetilde{p}))\{\widetilde{p}(R_a - R)u_1'[\widehat{c}(a), f(a)R] - (1-\widetilde{p})R^2 v'[(w-a)R^2]\}) = 0,$$

where $\psi'(\widehat{\mathcal{U}}(a,\widetilde{p})) = T'(\phi(\widehat{\mathcal{U}}(a,\widetilde{p})))\phi'(\widehat{\mathcal{U}}(a,\widetilde{p}))$. Note that $\eta_{T\circ\phi}(a)$ is a decreasing function of a.

From Step 1 to Step 4 it supposed that $\widetilde{p}$ has only two realizations, p_1 and p_2, satisfying $1 \geq p_1 > p_2 \geq 0$, and whose occurrence probabilities are respectively q and $1-q$. The result obtained is generalized Step 5 for any distribution of $\widetilde{p}$.

Step 1 – $\widehat{\mathcal{U}}(a^\star, p_1) \geq \widehat{\mathcal{U}}(a^\star, p_2)$ and $\widehat{\mathcal{U}}(a^{\star\star}, p_1) \geq \widehat{\mathcal{U}}(a^{\star\star}, p_2)$.

Let $a^\sharp$ an optimum (i.e. $a^\sharp = a^\star$ or $a^{\star\star}$). Then, we have $\widehat{\mathcal{U}}(a^\sharp, p_1) - \widehat{\mathcal{U}}(a^\sharp, p_2) = (p_1 - p_2)\xi(a^\sharp)$. As we have proved Appendix D.1 that $\xi(a^\sharp) \geq 0$, we have $\widehat{\mathcal{U}}(a^\sharp, p_1) \geq \widehat{\mathcal{U}}(a^\sharp, p_2)$ for $a^\sharp = a^\star$ and $a^\sharp = a^{\star\star}$.

Step 2 – $\widehat{\mathcal{U}}(a^\star, p_1) - \widehat{\mathcal{U}}(a^{\star\star}, p_1)$ and $\widehat{\mathcal{U}}(a^\star, p_2) - \widehat{\mathcal{U}}(a^{\star\star}, p_2)$ have opposite signs.

Proceed by contradiction. Suppose first they are both positive. This implies that $E(\psi(\widehat{\mathcal{U}}(a^\star, \widetilde{p}))) > E(\psi(\widehat{\mathcal{U}}(a^{\star\star}, \widetilde{p})))$, which is not possible since $E(\psi(\widehat{\mathcal{U}}(a^{\star\star}, \widetilde{p})))$ is a maximum. Similarly, they can not be both negative because this would implies that $E(\phi(\widehat{\mathcal{U}}(a^\star, \widetilde{p})))$ is not a maximum.

Step 3 – $\widehat{\mathcal{U}}(a^\star, p_1) > \widehat{\mathcal{U}}(a^{\star\star}, p_1)$ and $\widehat{\mathcal{U}}(a^\star, p_2) < \widehat{\mathcal{U}}(a^{\star\star}, p_2)$.

As a preliminary, the definition of a maximum yields $E(\phi(\widehat{\mathcal{U}}(a^\star, \widetilde{p}))) > E(\phi(\widehat{\mathcal{U}}(a^{\star\star}, \widetilde{p})))$, which rewrites:

$$q[\phi(\widehat{\mathcal{U}}(a^{\star\star}, p_1)) - \phi(\widehat{\mathcal{U}}(a^\star, p_1))] < (1-q)[\phi(\widehat{\mathcal{U}}(a^\star, p_2)) - \phi(\widehat{\mathcal{U}}(a^{\star\star}, p_2))]. \tag{15}$$

Equivalently, $E(\psi(\widehat{\mathcal{U}}(a^{\star\star}, \widetilde{p}))) > E(\psi(\widehat{\mathcal{U}}(a^\star, \widetilde{p})))$ rewrites:

$$q[\psi(\widehat{\mathcal{U}}(a^{\star\star}, p_1)) - \psi(\widehat{\mathcal{U}}(a^\star, p_1))] > (1-q)[\psi(\widehat{\mathcal{U}}(a^\star, p_2)) - \psi(\widehat{\mathcal{U}}(a^{\star\star}, p_2))]. \tag{16}$$

Now proceed by contradiction by supposing that $\widehat{\mathcal{U}}(a^\star, p_1) < \widehat{\mathcal{U}}(a^{\star\star}, p_1)$. Using Step 1 and 2, this implies that $\widehat{\mathcal{U}}(a^{\star\star}, p_2) < \widehat{\mathcal{U}}(a^\star, p_2) < \widehat{\mathcal{U}}(a^\star, p_1) < \widehat{\mathcal{U}}(a^{\star\star}, p_1)$. Since ϕ and ψ are both increasing, dividing (16) by (15) yields the following inequalities:

$$\frac{\psi(\widehat{\mathcal{U}}(a^{\star\star}, p_1)) - \psi(\widehat{\mathcal{U}}(a^\star, p_1))}{\phi(\widehat{\mathcal{U}}(a^{\star\star}, p_1)) - \phi(\widehat{\mathcal{U}}(a^\star, p_1))} > \frac{\psi(\widehat{\mathcal{U}}(a^\star, p_2)) - \psi(\widehat{\mathcal{U}}(a^{\star\star}, p_2))}{\phi(\widehat{\mathcal{U}}(a^\star, p_2)) - \phi(\widehat{\mathcal{U}}(a^{\star\star}, p_2))} > 0. \tag{17}$$

Denote $y_1 = \phi(\widehat{\mathcal{U}}(a^{\star\star}, p_2))$, $y_2 = \phi(\widehat{\mathcal{U}}(a^\star, p_2))$, $y_3 = \phi(\widehat{\mathcal{U}}(a^\star, p_1))$, and $y_4 = \phi(\widehat{\mathcal{U}}(a^{\star\star}, p_1))$. Hence $y_1 < y_2 < y_3 < y_4$ and (17) rewrites as follows:

$$\frac{T(y_4) - T(y_3)}{y_4 - y_3} > \frac{T(y_2) - T(y_1)}{y_2 - y_1}. \tag{18}$$

This latter inequality is true if and only if T is convex. Since T is concave, conclude that $\widehat{\mathcal{U}}(a^\star, p_1) > \widehat{\mathcal{U}}(a^{\star\star}, p_1)$ and, using Step 2, that $\widehat{\mathcal{U}}(a^\star, p_2) < \widehat{\mathcal{U}}(a^{\star\star}, p_2)$.

Step 4 – $a^\star > a^{\star\star}$ for a binary distribution.

For an optimum a, we have $\widehat{\mathcal{U}}(a, p_1) - \widehat{\mathcal{U}}(a, p_2) = (p_1 - p_2)\xi(a)$ where $\xi(a) > 0$ and $\xi'(a) > 0$. Then, $\widehat{\mathcal{U}}(a, p_1) - \widehat{\mathcal{U}}(a, p_2)$ is an increasing function of a. According to Step 1 and 2, $\widehat{\mathcal{U}}(a^\star, p_1) - \widehat{\mathcal{U}}(a^\star, p_2) > \widehat{\mathcal{U}}(a^{\star\star}, p_1) - \widehat{\mathcal{U}}(a^{\star\star}, p_2) \geq 0$. Consequently, $a^\star > a^{\star\star}$.

Step 5 – $a^\star > a^{\star\star}$ for any distribution of $\widetilde{p}$.

We have previously shown that $\eta_\phi(a)$ and $\eta_{T\circ\phi}(a)$ are decreasing functions of a. Consequently, $(a^\star > a^{\star\star})$ if and only if $(E(g(\widetilde{p})) = 0 \Rightarrow E(h(\widetilde{p})) < 0)$ where $g(\widetilde{p}) = \phi'(\widehat{\mathcal{U}}(a^\star, \widetilde{p}))\{\widetilde{p}(R_a - R)u_1'[\widehat{c}(a^\star), f(a^\star)R] - (1-\widetilde{p})R^2 v'[(w-a^\star)R^2]\}$ and $h(\widetilde{p}) = T'(\phi(\widehat{\mathcal{U}}(a^\star, \widetilde{p})))\phi'(\widehat{\mathcal{U}}(a^\star, \widetilde{p}))\{\widetilde{p}(R_a - R)u_1'[\widehat{c}(a^\star), f(a^\star)R] - (1 - \widetilde{p})R^2 v'[(w - a^\star)R^2]\}$.

The diffidence theorem then applies (see Lemma 1 page 84 in Gollier [15]) and, thus, the result $a^\star > a^{\star\star}$ holds for any distribution of $\widetilde{p}$. Then, the demand for annuities decreases with ambiguity aversion.

D.3 – Proof of statement (iii).

The maxmin problem, denoted $(\mathcal{AA}_{Mm})$, writes:

$$\max_{a,x} \min_{p_\varepsilon \in \mathrm{Supp}(\widetilde{p})} \mathcal{U}(a, x, p_\varepsilon)$$

$$\text{s.t.} \begin{cases} \mathcal{U}(a, x, p_\varepsilon) = p_\varepsilon u[c(a, x), xR] + (1 - p_\varepsilon)v[(w - a)R^2], \\ c = c(a, x) = R_a a + R(w - a) - x. \end{cases} \qquad (\mathcal{AA}_{Mm})$$

First observe that the optimum $(a^\sharp, x^\sharp)$ is such that $x^\sharp = f(a^\sharp)$. Indeed, for any optimal value $p_\varepsilon^\sharp$, $(a^\sharp, x^\sharp)$ is determined by maximizing $\mathcal{U}(a, x, p_\varepsilon^\sharp)$ subject to (1) and (2). The FOC (4) of this expected utility problem $(\mathcal{P}_{EUp_\varepsilon^\sharp})$ is independent of $p_\varepsilon^\sharp$ and implies that $x^\sharp = f(a^\sharp)$.

Consequently, there are only three candidates for the optimum $a^\sharp$ according to the sign of $\xi(a^\sharp)$.

The first candidate corresponds to the case where $\xi(a^\sharp) = 0$. As this candidate satisfies (6) it corresponds to the pair $(\underline{a}, f(\underline{a}))$ which solves $(\mathcal{P}_{Mm})$ (see Proposition 1). The maxmin utility attainable is then $v[(w - \underline{a})R^2]$.

The second candidate $\breve{a}$ corresponds to the case where $\xi(a^\sharp) < 0$. As $\widehat{\mathcal{U}}(\breve{a}, p_\varepsilon) = p_\varepsilon \xi(\breve{a}) + v[(w - \breve{a})R^2] < v[(w - \breve{a})R^2]$, the maxmin utility attainable is lower than $v[(w - \breve{a})R^2]$. Importantly $\breve{a}$ is obviously the solution of the expected utility problem $(\mathcal{P}_{EU_{p_M}})$ where p_M is the upper bound of $\mathrm{Supp}(\widetilde{p})$, the support of $\widetilde{p}$. Then, according to Proposition 3 and Assumption 4, we have $\underline{a} < \breve{a}$ and, consequently, $v[(w - \breve{a})R^2] < v[(w - \underline{a})R^2]$. Since the maxmin utility attainable with $\breve{a}$ is strictly lower than the one attainable with $\underline{a}$, we have necessarily $\xi(a^\sharp) \geq 0$ at the optimum.

The third candidate $\breve{a}$ corresponds to the case where $\xi(a^\sharp) > 0$. Importantly $\breve{a}$ is the solution of the expected utility problem $(\mathcal{P}_{EU_{p_m}})$ where p_m is the lower bound of $\mathrm{Supp}(\widetilde{p})$. Then, according to Proposition 3 and Assumption 4, we have $\underline{a} < \breve{a}$. We now show that this third candidate cannot solves the problem $(\mathcal{AA}_{Mm})$ because if $\breve{a}$ solves the problem $(\mathcal{AA}_{Mm})$ we have necessarily $\underline{a} \geq \breve{a}$. Indeed, assume that $\breve{a}$ solves the problem $(\mathcal{AA}_{Mm})$. Then we have $\widehat{\mathcal{U}}(\underline{a}, p_m) \leq \widehat{\mathcal{U}}(\breve{a}, p_m)$. As $0 = \xi(\underline{a}) = \widehat{\mathcal{U}}_2'(\underline{a}, p_\varepsilon) < \xi(\breve{a}) = \widehat{\mathcal{U}}_2'(\breve{a}, p_\varepsilon)$, we obtain $\widehat{\mathcal{U}}(\underline{a}, 1) \leq \widehat{\mathcal{U}}(\breve{a}, 1)$, i.e. $u[\widehat{c}(\underline{a}), f(\underline{a})R] \leq u[\widehat{c}(\breve{a}), f(\breve{a})R]$. As, $u[\widehat{c}(a), f(a)R]$ is increasing function of a, we have $\underline{a} \geq \breve{a}$.

Consequently, we have $\xi(a^\sharp) = 0$ at the optimum and the optimal pair $(a^\sharp, x^\sharp)$ which solves the problem $(\mathcal{AA}_{Mm})$ is the pair $(\underline{a}, f(\underline{a}))$ exhibited Proposition 2 which solves $(\mathcal{P}_{Mm})$. Then, the demand for annuities is always negative when ambiguity aversion is infinite. $\square$

Appendix E – Derivation of equation (14).

There are two cases.

Suppose first that $u[c, xR] \geq v[(w - a)R^2]$. Then, it is straightforward that:

$$\lim_{\alpha \to +\infty} \phi^{-1}(E(\phi(\mathcal{U}(a, x, \widetilde{p})))) = \mathcal{U}(a, x, p_2).$$

Suppose now that $u[c, xR] < v[(w - a)R^2]$ and let us denote $\alpha = 1/\gamma$ and $\sigma =$

$v[(w-a)R^2]-u[c,xR]$. Then:

$$\lim_{\alpha\to+\infty}\phi^{-1}(E(\phi(\mathcal{U}(a,x,\widetilde{p})))) = \lim_{\gamma\to 0}[\mathcal{U}(a,x,p_2)-\gamma\ln(qe^{\frac{(p_1-p_2)\sigma}{\gamma}}+1-q)] = \mathcal{U}(a,x,p_1),$$

which implies (14). □

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